

Symbolic dynamics and oscillatory motions in the 3 body problem

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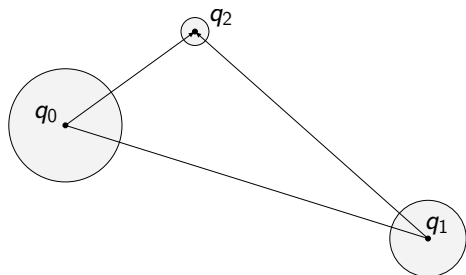
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The planar three body problem

- Three co-planar bodies of masses $m_0, m_1, m_2 > 0$ under Newtonian gravitational force:

$$\frac{d^2 q_i}{dt^2} = \sum_{j=0, j \neq i}^2 m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}, \quad q_0, q_1, q_2 \in \mathbb{R}^2$$

- Two strongly related problems:
 - Chaotic motions
 - Final motions



Chaotic motions: symbolic dynamics

- Take either $X = \{1, \dots, M\}$ or $X = \mathbb{N}$ and $X^{\mathbb{Z}}$.
- Shift $\sigma : X^{\mathbb{Z}} \rightarrow X^{\mathbb{Z}}$ defined as

$$(\sigma\omega)_k = \omega_{k+1}$$

- Can we embed this dynamics in the 3BP (Smale horseshoe)?
- We want to construct hyperbolic sets whose dynamics is conjugated to the shift.

Chazy (1922): Final motions for the 3 body problem

Call r_i the mutual distances between bodies.

Final motions: Possible behaviors of the 3BP when $t \rightarrow \pm\infty$.

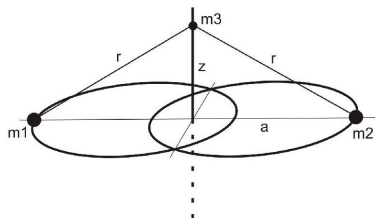
- Hyperbolic (H^\pm): $|r_i| \rightarrow \infty$, $|\dot{r}_i| \rightarrow c_i > 0$.
- Parabolic (P^\pm): $|r_i| \rightarrow \infty$, $|\dot{r}_i| \rightarrow 0$.
- Bounded (B^\pm): $\sup_{t \geq t_0} |r_i| < \infty$.
- Hyperbolic-Parabolic (HP_k^\pm): $|r_i| \rightarrow \infty \forall i$, $|\dot{r}_k| \rightarrow 0$, $|\dot{r}_i| \rightarrow c_i > 0$, $i \neq k$.
- Hyperbolic-Elliptic (HE_k^\pm): $|r_i| \rightarrow \infty$, $|\dot{r}_i| \rightarrow c_i > 0$, $i \neq k$, $\sup_{t \geq t_0} |r_k| < \pm\infty$.
- Parabolic-Elliptic (PE_k^\pm): $|r_i| \rightarrow \infty$, $|\dot{r}_i| \rightarrow 0$, $i \neq k$, $\sup_{t \geq t_0} |r_k| < \infty$.
- Oscillatory (OS^\pm): $\limsup_{t \rightarrow \pm\infty} \sup_i |r_i| = \infty$, $\liminf_{t \rightarrow \pm\infty} \sup_i |r_i| < \infty$.

Final motions for the 3BP

- In the limit $m_1, m_2 \rightarrow 0$: Two uncoupled two 2BP.
- Only $H, P, HP_k, HE_k, PE_k, B$ (All except oscillatory motions!).
→ Past and future final motions must coincide.
- Questions by Chazy (1922) for the 3BP:
 - Do oscillatory motions exist?
 - Can one combine different past and future final motions?
- Long literature on oscillatory motions and on chaotic motions for the 3BP. But:
 - Most for the Restricted 3BP: $m_2 = 0$ so q_0, q_1 perform a 2BP.
 - Quite strict assumptions on the masses of the bodies.

Oscillatory motions: First result

- **Sitnikov** (1960) considered the Restricted Spatial Elliptic 3BP:
 - $m_1 = m_2 = 1/2$
 - q_1, q_2 move on ellipses of small eccentricity.
 - q_3 ($m_3 = 0$) moves on the (invariant) vertical axis.



He obtained:

- Oscillatory motions
- Free combination of past and future final motions.

Moser (1973): New proof of Sitnikov results via symbolic dynamics.

Oscillatory motions: Past results

- **Restr. Planar Circular 3BP**: q_0, q_1 (masses μ and $1 - \mu$, $\mu \in (0, \frac{1}{2}]$) perform circular motion and q_2 is coplanar:
 - Simó and Llibre (1980): Oscillatory motions for $0 < \mu \ll 1$.
 - G.-Martín-Seara (2016): Oscillatory motions for **all masses**: $\mu \in (0, \frac{1}{2}]$.
- Other approaches, other results for the Restricted 3BP:
Kaloshin-Galante, G.-Martín-Sabbagh-Seara, Xia, Seara-Zhang...
- Oscillatory motions for the **full 3BP** ($m_2 > 0$):
 - Alexeev (1969) for a Sitnikov-like model: $m_0 = m_1 \gg m_2$.
 - Moeckel (2007): For a “large” (non-generic) choice of masses. The proof relies on the passage close to triple collision. He obtains semiconjugation with the Smale horseshoe.

Abundance of the Final motions

- Measure of each $X^- \cap Y^+$ in phase space?
- It is known for each $X^- \cap Y^+$ whether it has positive or zero measure except for $OS^- \cap OS^+$.
- **(Wide open) Conjecture** (Kolmogorov, Alexeev): Lebesgue measure of $OS^- \cap OS^+$ is zero.
- Kaloshin–Gorodetski (2011): The Hausdorff dimension of oscillatory motions is maximal for both the Sitnikov problem and the RPC3BP.
- Recall that most existence results deal with Restricted models/narrow ranges of parameters!

Main result: Final motions

Theorem (G.–Martín–Paradela–Seara)

Consider the three body problem with masses $m_0, m_1, m_2 > 0$ such that $m_0 \neq m_1$. Then,

$$X^- \cap Y^+ \neq \emptyset \quad \text{with} \quad X, Y = OS, B, PE_2, HE_2.$$

In particular, $OS^- \cap OS^+ \neq \emptyset$.

- We can combine (almost) all possible negative energy final motions.
- The bodies of masses m_0 and m_1 perform (approximately) circular motions. That is, $|q_0 - q_1|$ is approximately constant.
- The third body may have radically different behaviors: oscillatory, bounded, hyperbolic or parabolic.

Main result: Chaotic motions

Theorem (G.–Martín–Paradela–Seara)

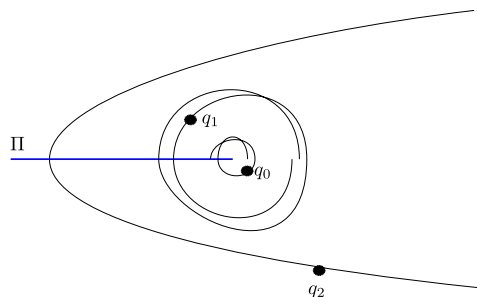
Consider the three body problem with masses $m_0, m_1, m_2 > 0$ such that $m_0 \neq m_1$ and denote by Φ_t its flow. Then, there exists a section Π transverse to Φ_t such that the induced Poincaré map

$$\mathcal{P} : \mathcal{U} = \dot{\mathcal{U}} \subset \Pi \rightarrow \Pi$$

has an invariant set \mathcal{X} which is homeomorphic to $\mathbb{N}^{\mathbb{Z}}$ such that $\mathcal{P}|_{\mathcal{X}}$ is topologically conjugated to the shift.

- Positive topological entropy.
- \mathcal{X} is a hyperbolic set once for the symplectically reduced 3BP.
- Previous results also in other regions of phase space (but for quite strict mass choices): Bolotin-McKay, Bolotin, Marco-Niederman, Arioli, Wilczak-Zgliczynski, Capinski,...

Main result: Chaotic motions



- q_0 and q_1 evolve at a bounded distance from their center of mass.
- q_2 makes excursions close to a parabolic motion.

- Let N_k be the number of complete revolutions of q_0, q_1 between two consecutive passages of q_2 through Π .
- There exist integers $L \gg 1$ such that for any $\omega \in \mathbb{N}^{\mathbb{Z}}$, there exists a solution of the 3BP such that $N_k = L + \omega_k$.

Moser's approach for the Sitnikov problem

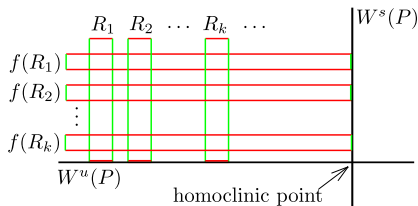
- Sitnikov Problem \rightarrow 2 dimensional Poincaré map
- It has a fixed point at infinity P .
- **Step 1:** Check that P has stable and unstable invariant manifolds

Not obvious! Its linearization is not hyperbolic but parabolic, i.e. linearization=Identity (McGehee 1972).

- **Step 2:** The invariant manifolds intersect transversally (Perturbative techniques: Melnikov Method).
- If P were hyperbolic: Smale Theorem would lead to a Smale horseshoe.

Moser's approach for the Sitnikov problem

- **Step 3:** Parabolic Lambda lemma for the local dynamics.



- **Step 4:** Isolating blocks (plus cone fields) \rightarrow Smale horseshoe.
- **Key point:** The model can be reduced to a **2D area preserving map**.
- Same happens for the RPC3BP and the Alexeev model!

How can one implement Moser approach for the 3BP?

- Reduction by the classical first integrals + Poincaré map:
4 dimensional symplectic map

Instead of a fixed point at infinity: a disk of fixed points $\mathbb{D} \subset \mathbb{R}^2$.

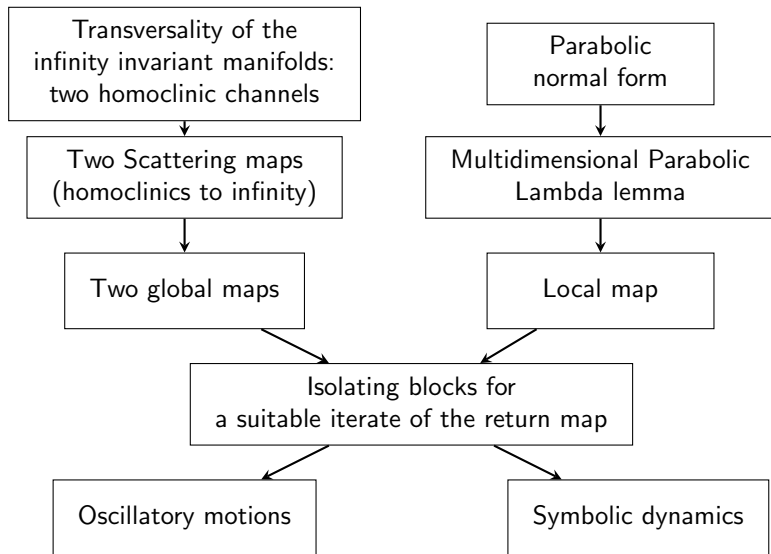
There are “center” directions: harder to build hyperbolic sets.

- We want results for all values of the masses $m_0, m_1, m_2 > 0$
($m_0 \neq m_1$).

Transversality of the invariant manifolds of infinity?

We need a close to integrable regime.

Moser Approach: Scheme



A different nearly integrable regime

- To prove transversality between the invariant manifolds of infinity we must be in a perturbative regime.
- Regime: $|q_2 - q_0|, |q_2 - q_1| \gg |q_1 - q_0|$
- At first order, the third body only “sees” one body at the center of mass of the other two:

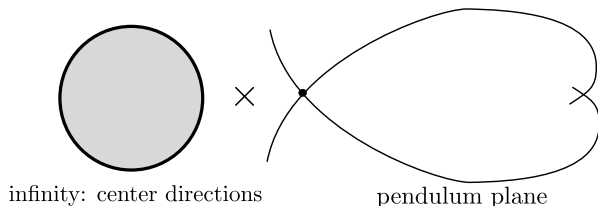
$H = \text{Two uncoupled Kepler problems} + \text{Small perturbation.}$

- Two time scales:
 - q_0 and q_1 on ellipses with fast rotation.
 - q_2 on a parabola with slow motion.

Transversality between the invariant manifolds

- The transversality between the invariant manifolds is exponentially small (in some directions) with respect to the distance of the third body with respect to the other two.
(Exponentially small splitting of separatrices)
- Why? Fast rotation coupled with slow motion on the invariant manifolds.
- Classical perturbative methods (Melnikov Theory) cannot be applied.
- Following Lazutkin (1984): Deep analysis of the analytic extension of the perturbed invariant manifolds in complex domains.
- 3BP has a rather high dimension compared to most of the models analyzed up to now.

How to construct the isolating blocks



- We need a 4 dimensional isolating block structure for \mathcal{P} .
- Pendulum–plane: Proceed as Moser for the Sitnikov problem
- How do we construct an isolating block for the “center” directions?
- \mathbb{D} filled by fixed points: Center directions are **extremely degenerate!**

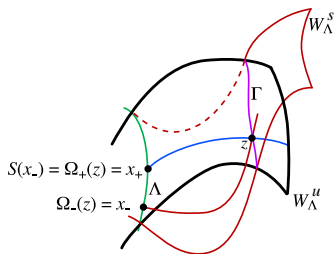
The scattering map (after Delshams – de la Llave – Seara)

- Normally hyperbolic (parabolic) invariant manifold Λ
- Its stable and unstable manifolds intersect along a homoclinic channel Γ .
- Scattering map associated to the homoclinic channel Γ

$$S : \Lambda \rightarrow \Lambda, \quad x_+ = S(x_-)$$

when

$$\emptyset \neq W^s(x_+) \cap W^u(x_-) \in \Gamma$$



- Delshams – de la Llave – Seara: Scattering map is symplectic.

Two scattering maps

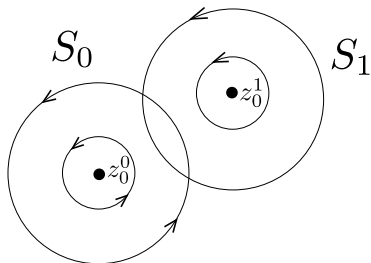
- In \mathbb{D} we can define two scattering maps given by two homoclinic channels

$$S_i : \mathbb{D}_i \subset \mathbb{D} \rightarrow \mathbb{C}.$$

- There exists $z_0^0, z_0^1 \in \mathbb{D}$ such that

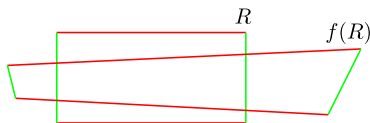
$$z_0^i = S_i(z_0^i)$$

- z_0^0, z_0^1 are points in \mathbb{D} with transverse homoclinic orbits.
- They are **elliptic fixed points** by the scattering maps.
- Close to them, S_i are close to integrable twist maps.



Isolating block for (a suitable iterate of) the scattering map

- There exists $M > 0$ such that $S_1^M \circ S_0$ has an isolating block.



- Tools: Birkhoff normal form, KAM Theorem...
- The Scattering maps encodes heteroclinic orbits to infinity.
- We want an isolating block for the “true” map!

The return map

- Return map from a neighborhood of \mathbb{D} to itself.
- Return map is the composition of:
 - Global map: Along the invariant manifolds (essentially given by transversality between the invariant manifolds and the scattering map).
 - Local map: In a neighborhood of the disk \mathbb{D} at infinity.
- Analysis of the local map: One has to control the dynamics close to infinity by a Parabolic Lambda Lemma.

A Multidimensional Parabolic Lambda Lemma

- In suitable coordinates the behavior close to \mathbb{D} and $p + q > 0$:

$$\dot{q} = q((q + p)^3 + \mathcal{O}_4(q, p))$$

$$\dot{p} = -p((q + p)^3 + \mathcal{O}_4(q, p))$$

$$\dot{t} = 1$$

$$\dot{z} = \mathcal{O}_6(q, p).$$

where $\mathbb{D} = \{q = p = 0\}$.

- We need a deep analysis of the dynamics close to \mathbb{D} .
- It should not spoil the hyperbolicity created by the scattering map!
- We follow the approach by Kaloshin&Gorodetski.

An isolating block for (a suitable iterate of) the return map

- Applying the return map a large number of times one can construct the isolating block using:
 - The techniques by Moser in the “pendulum” plane.
 - The isolating block of the scattering map in the “degenerate” directions.

