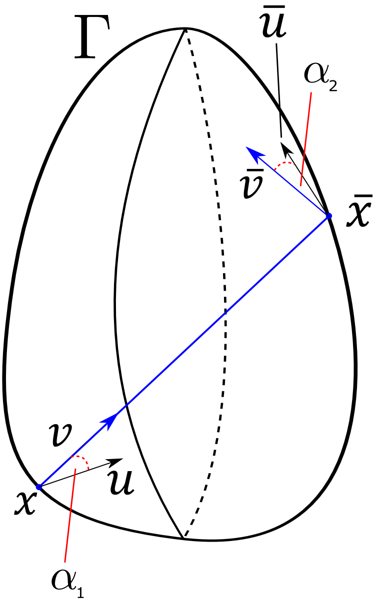
Arnold Diffusion in Multi-Dimensional Convex Billiards

Andrew Clarke, Dmitry Turaev

European Research Council
Established by the European Commission

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

- Γ is a real-analytic, closed and strictly convex hypersurface of $\mathbb{R}^d$ where $d \geq 3$
- Here: $x \in \Gamma$, $v \in T_x\mathbb{R}^d$ points inside Γ, and $u \in T_x\Gamma$ is the projection of v onto the tangent space to Γ at x
- Inside Γ: billiard dynamics
- On Γ: geodesic flow with respect to the induced metric



Assume:

- There is a hyperbolic closed geodesic γ on Γ
- There is a transverse homoclinic geodesic ξ to γ

Let $\{\alpha_n\}$ be the sequence of angles of reflection along a billiard trajectory.

Theorem (Diffusive Trajectories)

Generically, in the analytic topology, there exist trajectories of the billiard map for which $\alpha_n \to 0$ as $n \to \infty$.

Theorem (Oscillatory Trajectories)

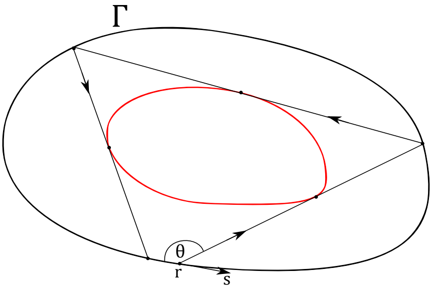
Generically, in the analytic topology, there exist trajectories of the billiard map for which

$$\limsup_{n\to\infty} \alpha_n > 0, \quad \liminf_{n\to\infty} \alpha_n = 0.$$

- Initial questions: Why is $d \geq 3$? Who cares? What is the C^ω topology for hypersurfaces? Why are the assumptions about the geodesic flow and the conclusions about the billiard map?

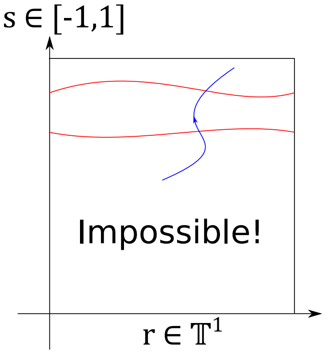
- Main ideas: description of diffusive trajectories, Arnold diffusion, normally hyperbolic invariant manifolds (NHIMs), scattering maps, shadowing of pseudo-orbits.

- Proof: hypersurface geometry, equations of the geodesic flow and billiard map, scaling of variables, existence of a NHIM for the billiard map, construction of an IFS of scattering maps and the inner map, destruction of invariant curves.



r = arclength parameter

$s = -\cos\theta$



- Lazutkin (1973): if $\Gamma \subset \mathbb{R}^2$ is strictly convex and sufficiently smooth, then there is a family of smooth caustics near the boundary
- Halpern (1977): example of a C^2-smooth boundary Γ with a trajectory that converges in finite time to a fixed point
- Mather (1982): if a billiard table that is at least C^2-smooth has at least one flat point then there exist trajectories that come arbitrarily close to being tangent with the boundary
- Gruber (1990): measure of the set of trajectories "asymptotically terminating on the boundary" is 0
- Berger, Gruber (1995): if $\Gamma \subset \mathbb{R}^d$ (where $d \geq 3$) then it has caustics if and only if it is an ellipsoid

- $\mathcal{V}$ = set of C^ω-functions $Q : \mathbb{R}^d \to \mathbb{R}$ such that the set

$$\Gamma = \{x \in \mathbb{R}^d : Q(x) = 0\}$$

  is a closed and strictly convex hypersurface of $\mathbb{R}^d$.

- Let $Q_1, Q_2 \in \mathcal{V}$ and $K \subset \mathbb{R}^d$ be compact. Then Q_1, Q_2 admit holomorphic extensions in a complex neighbourhood $\bar{K}$ of K. Q_1, Q_2 are close on K in the C^ω topology if the holomorphic extensions are uniformly close on $\bar{K}$.

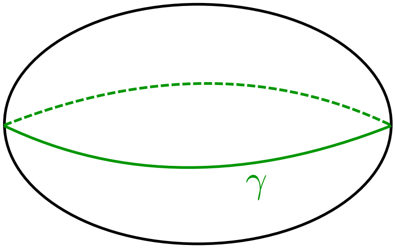
- A property is *generic* if it occurs on a residual set.

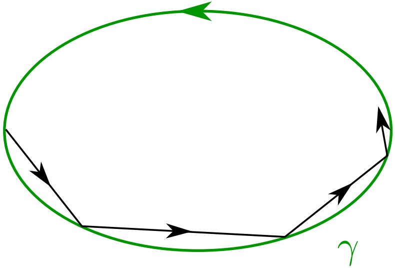
Bump functions are not analytic.

Trick (Broer and Tangerman 1986):

- Make open conditions to be satisfied by the perturbation (in a weaker topology, e.g. C^1)
- Show conditions are satisfied by $Q + \epsilon\psi$ for arbitrarily small values of ϵ where ψ is a bump function
- Approximate $Q + \epsilon\psi$ by a real-analytic family Q_ϵ where $Q_0 = Q$
- Since the conditions are open, they are satisfied by Q_ϵ for arbitrarily small ϵ

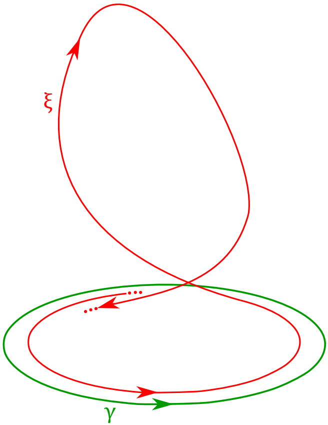
- Dynamical folkore: "The geodesic flow is the limit of the billiard map as the angle of reflection goes to 0"
- Example: an ellipsoid, where γ is a simple closed geodesic
- There are billiard trajectories in the same plane as γ
- In reality: the limiting flow follows geodesics, but the speed fluctuates with the curvature





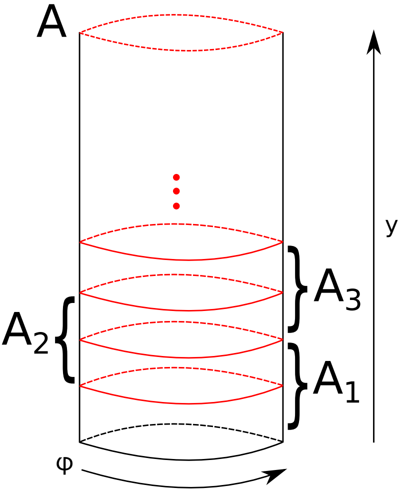
Genericity of assumptions on geodesic flow:

- Contreras (2010): on any closed manifold M there is a C^2-open and dense set of C^∞-Riemannian metrics for which the geodesic flow has a hyperbolic periodic orbit and a transverse homoclinic
- Knieper, Weiss (2002): C^∞-open and dense set of strictly convex RMs on $\mathbb{S}^2$
- C. (2019): C^ω-open and dense set of strictly convex surfaces in $\mathbb{R}^3$; and generically near an elliptic closed geodesic in $\mathbb{R}^d$

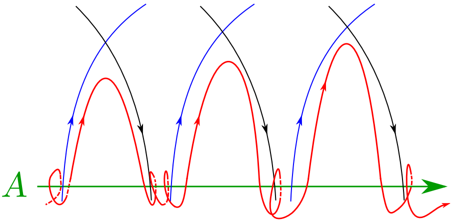


Sketch of the Proof:

- Our assumptions on the geodesic flow imply that it has a noncompact NHIC, consisting of the hyperbolic closed geodesic on an interval of energy levels;
- Moreover, its stable and unstable manifolds have a transverse homoclinic intersection;
- As the billiard map is approximated by the geodesic flow, it inherits this behaviour: there is a non-compact NHIC A for the billiard map with height component $\sim$ the reciprocal of the angle of reflection, and $W^s(A) \pitchfork W^u(A)$;
- Arnold diffusion methods imply there are trajectories drifting up along the cylinder.



- $\mathbb{T} \ni \varphi \approx$ angular variable on hyperbolic closed geodesic
- $[a, \infty) \ni y \approx$ reciprocal of angle of reflection



Let $\phi^t : M \to M$ be a smooth flow. A compact, ϕ^t-invariant set $A \subset M$ is a *normally hyperbolic invariant manifold* (NHIM) for ϕ^t if there exist $0 < \lambda < \mu^{-1} < 1$ and an invariant splitting

$$TM|_A = TA \oplus E^s \oplus E^u$$

such that:

- $\|D\phi^t v\| \leq \lambda^t \|v\|$ for all $v \in E^s$, $t \geq 0$
- $\|D\phi^{-t} v\| \leq \lambda^t \|v\|$ for all $v \in E^u$, $t \geq 0$
- $\|D\phi^t v\| \leq \mu^{|t|} \|v\|$ for all $v \in TA$, $t \in \mathbb{R}$

Fenichel (70s): NHIMs survive C^1-small perturbations of the flow
Hirsch, Pugh, Shub (70s): r-NHIMs survive C^r small perturbations as C^r-smooth NHIMs

A homoclinic cylinder is a manifold

$$B \subset (W^s(A) \pitchfork W^u(A)) \setminus A$$

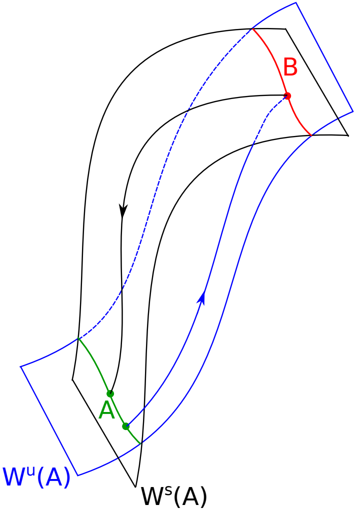
such that the holonomy maps

$$\pi^{s,u} : B \to A$$

are bijective. The scattering map $s : A \to A$ is

$$s = \pi^s \circ (\pi^u)^{-1}$$

[Delshams, de la Llave, Seara 2000]



- Take 8 independent homoclinic cylinders $B_1, \ldots, B_8$
- Let s_j denote the scattering map corresponding to the homoclinic cylinder B_j
- Construct an iterated function system $\{f, s_1, \ldots, s_8\}$ on the cylinder A by combining the 8 scattering maps and the billiard map
- Orbits of the IFS are called pseudo-orbits

[Gidea, de la Llave, Seara]: shadowing of infinite pseudo-orbits on compact cylinders

Theorem

Let $\{(\varphi_n, y_n)\}_{n=0}^{\infty}$ be a pseudo-orbit on the non-compact cylinder A. Then for any $\delta > 0$ there is an orbit $\{(x_m, u_m)\}_{m=0}^{\infty}$ of f and a sequence $\{m_n\}_{n=0}^{\infty}$ of natural numbers such that

$$d((\varphi_n, y_n), (x_{m_n}, u_{m_n})) < \delta$$

for each $n \in \mathbb{N}$.

- Moeckel (2002): inner map and one scattering map on a compact cylinder
- Gelfreich, Turaev (2017): inner map and n scattering maps on a compact cylinder

Theorem

Suppose the maps $f, s_1, \ldots, s_8$ have no common invariant essential curves on the non-compact cylinder A. Then there are trajectories $\{(\varphi_n, y_n)\}_{n=0}^{\infty}$ of the IFS such that $y_n \to \infty$ as $n \to \infty$, as well as trajectories where

$$\limsup_{n\to\infty} y_n = \infty, \quad \liminf_{n\to\infty} y_n < \infty.$$

Theorem

There is a residual set $\tilde{\mathcal{V}} \subset \mathcal{V}$ such that for each $Q \in \tilde{\mathcal{V}}$, the maps $f, s_1, \ldots, s_8$ corresponding to the billiard map in

$$\Gamma = \{x \in \mathbb{R}^d : Q(x) = 0\}$$

have no common invariant essential curves in the noncompact cylinder A.

On a hypersurface $\Gamma = \{x \in \mathbb{R}^d : Q(x) = 0\}$ we have, for $x \in \Gamma$, and $u \in T_x\Gamma$:

- Unit normal: $n(x) = -\frac{\nabla Q(x)}{\|\nabla Q(x)\|}$
- Curvature matrix: $C(x) = \|\nabla Q(x)\|^{-1} \left(\frac{\partial^2 Q}{\partial x^2}(x) \right)$
- Shape operator: $S(x)\, u = C(x)\, u - \langle C(x)\, u, n(x) \rangle n(x)$
- Normal curvature: $\kappa(x, u) = \langle S(x)\, u, u \rangle = \langle C(x)\, u, u \rangle$

Usually the geodesic flow is the flow of $H = \frac{1}{2}g(x)(u,u)$, but we use coordinates from $\mathbb{R}^d$ on a hypersurface $\{Q(x) = 0\}$. A curve $\gamma : [0,1] \to \Gamma$ is a geodesic if and only if

$$0 = \gamma''(t)^T = \gamma''(t) - \langle \gamma''(t), n(\gamma(t))\rangle n(\gamma(t)).$$

Since

$$\langle \gamma'(t), n(\gamma(t))\rangle \equiv 0$$

we have

$$0 = \frac{d}{dt}\langle \gamma'(t), n(\gamma(t))\rangle = \langle \gamma''(t), n(\gamma(t))\rangle - \kappa(x,u)$$

and so

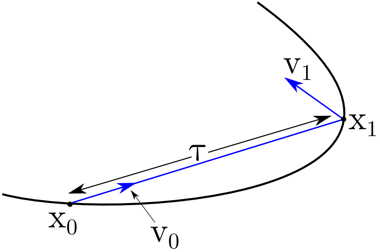
$$\gamma''(t) = \kappa(\gamma(t), \gamma'(t))\, n(\gamma(t)).$$

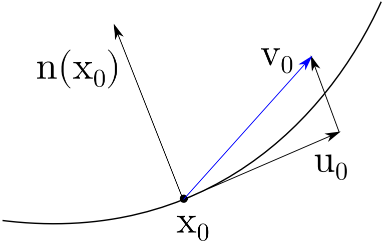
- Let $x_0 \in \Gamma$ and let v_0 be an inward-pointing unit vector at x_0
- Denote by $\tau = \tau(x_0, v_0)$ the flight time between collisions
- The billiard map is

$$f : \begin{cases} x_1 = x_0 + \tau v_0 \\ v_1 = v_0 - 2\langle v_0, n(x_1)\rangle n(x_1) \end{cases}$$

- There is a unique $u_0 \in T_x\Gamma$ such that $v_0 = u_0 + \sqrt{1 - u_0^2}\, n(x)$
- Let

$$M = \{(x, u) \in T\Gamma : \|u\| \leq 1\}$$





Billiard Map:

$$f(x,u) = \begin{cases} \bar{x} = x + \tau v \\ \bar{u} = v - \langle v, n(\bar{x}) \rangle n(\bar{x}) \end{cases}$$

where $v = u + \sqrt{1-u^2}n(x)$. Billiard map is exact symplectic:

$$f^*\lambda - \lambda = -d\tau$$

Geodesic Flow:

$$X(x,u) = \begin{cases} \dot{x} = u \\ \dot{u} = \kappa(x,u)n(x) \end{cases}$$

where X is the Hamiltonian vector field of

$$H(x,u) = \frac{u^2}{2} + \kappa(x,u)\frac{Q(x)}{\|\nabla Q(x)\|}$$

$$f = Id + \tau X + O(\tau^2) = \phi_X^\tau + O(\tau^2)$$

but

$$Q(\bar{x}) = 0 \implies \tau(x,u) = \frac{2\sqrt{1-u^2}}{\kappa(x,u)} + \cdots$$

Fix some small $\tau_* > 0$ and introduce new coordinates (x, w, z) where

$$w = \frac{u}{\|u\|}, \quad z = \tau_*^{-1} \frac{2\sqrt{1-u^2}}{\kappa(x,w)}.$$

Then

$$f = Id + \tau_* V + O(\tau_*^2) = \phi_V^{\tau_*} + O(\tau_*^2)$$

where

$$V(x,w,z) = \begin{cases} \dot{x} = zw \\ \dot{w} = z\kappa(x,w)n(x) \\ \dot{z} = -\frac{4}{3}z^2\kappa(x,w)^{-1}R(x,w) \end{cases}$$

with

$$R(x,w) = \sum_{i,j,k=1}^{d} \frac{\partial^3 Q}{\partial x_i \partial x_j \partial x_k}(x) w_i w_j w_k$$

Since

$$\frac{d}{dt}\kappa(x,w) = zR(x,w)$$

we have

$$\frac{d}{dt}\log z = \frac{\dot{z}}{z} = -\frac{4}{3}z\kappa(x,w)^{-1}R(x,w) = \frac{d}{dt}\log\kappa(x,w)^{-\frac{4}{3}}$$

and so

$$y^{-1} = z\kappa(x,w)^{\frac{4}{3}}$$

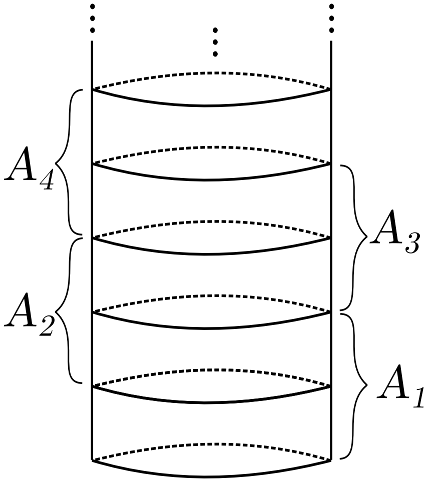
is preserved by V. The set

$$\bar{A}_* = \left\{ \left(\gamma(s), \gamma'(s), y^{-1}\kappa\left(\gamma(s), \gamma'(s)\right)^{-\frac{4}{3}}\right) : s \in \mathbb{T}, y \in \left[\frac{1}{2}, \frac{3}{2}\right] \right\}$$

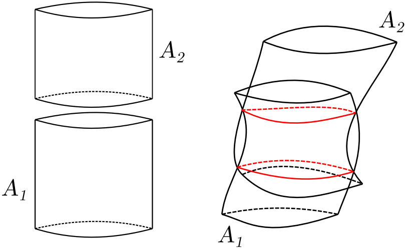
is a NHIC for V, and if $N = \lfloor \tau_*^{-1} \rfloor$ then

$$f^N = \phi_V^1 + O(\tau_*)$$

so Fenichel theory guarantees a NHIC A_* for f that is $O(\tau_*)$ close to $\bar{A}_*$.



- Choose $\{\tau_n\}_{n\in\mathbb{N}}$ so that the top of the cylinder $\bar{A}_1$ is the middle of the cylinder $\bar{A}_2$ etc
- Then $\tau_{n+1} = \left(\frac{2}{3}\right)^n \tau_1 \to 0$
- Perturbed cylinders A_n for f fit together to form a noncompact cylinder



With $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, we can write the cylinder as

$$\bar{A}_* = \left\{ \phi_V^{\theta l(y)} \left(\gamma(0), \gamma'(0), y^{-1} \kappa \left(\gamma(0), \gamma'(0) \right)^{-\frac{4}{3}} \right) : \theta \in \mathbb{T}, y \in \left[\frac{1}{2}, \frac{3}{2} \right] \right\}$$

where

$$l(y) = K^{-1} y^{-1}$$

is the period of a point on the cylinder in the energy level $y = const$, and

$$K^{-1} = \int_{s \in \mathbb{T}} \kappa(\gamma(s), \gamma'(s))^{-\frac{4}{3}} ds.$$

Replacing the variable y by Ky we get

$$V(\theta, y) = \begin{cases} \dot{\theta} = y \\ \dot{y} = 0. \end{cases}$$

The set

$$\bar{B}_* = \left\{ \phi_V^{\theta y^{-1}} \left(\xi(0), \xi'(0), K y^{-1} \kappa(\xi(0), \xi'(0))^{-\frac{4}{3}} \right) : \theta \in [0,1), y \in \left[\frac{K}{2}, \frac{3K}{2} \right] \right\}$$

is a homoclinic cylinder. The vector field V on $\bar{B}_*$ is

$$V(\theta, y) = \begin{cases} \dot{\theta} = y \\ \dot{y} = 0. \end{cases}$$

The holonomy maps are

$$\pi^s(\theta, y) = (\theta + a_+, y), \quad \pi^u(\theta, y) = (\theta + a_-, y)$$

and the scattering map is given by

$$s(\theta, y) = (\theta + \Delta, y)$$

where $\Delta = a_+ - a_-$.

The billiard map has equivalent objects A_*, B_*. The coordinates (θ, y) from $\bar{A}_*$ define coordinates on A_*. The restriction of the billiard map to A_* is

$$f|_{A_*} : (\theta, y) \mapsto (\theta + \tau_* y, y) + O(\tau_*^2)$$

and the scattering map is

$$s : (\theta, y) \mapsto (\theta + \Delta, y) + O(\tau_*).$$

We perturb the billiard map f by perturbing the boundary

$$\Gamma = \{x \in \mathbb{R}^d : Q(x) = 0\}.$$

via $Q \to Q_\epsilon = Q + \epsilon\psi$. The perturbed billiard map is given by

$$f_\epsilon(x, u) = f(x, u) + \epsilon\Omega\nabla H_{\mathrm{pert}} \circ f(x, u) + O(\epsilon^2)$$

where

$$H_{\mathrm{pert}}(\bar{x}, \bar{u}) = 2\frac{\psi(\bar{x})}{\|\nabla Q(\bar{x})\|}\sqrt{1 - \bar{u}^2} + \Big[2\Theta(\bar{x}, \bar{u})\langle C(\bar{x})v, \bar{v}\rangle + \\ + 2\|\nabla Q(\bar{x})\|^{-1}\langle\nabla\psi(\bar{x}), \bar{v}\rangle\Big]\frac{Q(\bar{x})}{\|\nabla Q(\bar{x})\|}$$

with $(\bar{x}, \bar{u}) = f(x, u)$, $v = \bar{u} - \sqrt{1 - \bar{u}^2}n(\bar{x})$, $\bar{v} = \bar{u} + \sqrt{1 - \bar{u}^2}n(\bar{x})$, and

$$\Theta(\bar{x}, \bar{u}) = -\frac{\psi(\bar{x})}{\|\nabla Q(\bar{x})\|\sqrt{1 - \bar{u}^2}}.$$

The perturbed scattering map must also look like

$$s_\epsilon = s + \epsilon \Omega \nabla J \circ s + O(\epsilon^2).$$

Assume the perturbation ψ is localised near $\pi \circ f(B_*)$. Then the Hamiltonian J of the perturbation of the scattering map is

$$\begin{aligned} J = \lim_{N_\pm \to \infty} \Bigg[& \sum_{j=0}^{N_- - 1} \left(H_{\mathrm{pert}} \circ f^{-j} \circ (\pi^u)^{-1} \circ s^{-1} - H_{\mathrm{pert}} \circ f^{-j} \circ s^{-1} \right) + \\ & + \sum_{j=1}^{N_+} \left(H_{\mathrm{pert}} \circ f^{j} \circ (\pi^s)^{-1} - H_{\mathrm{pert}} \circ f^{j} \right) \Bigg] \\ = H_{\mathrm{pert}} \circ f \circ (\pi^s)^{-1} & \end{aligned}$$

[Delshams, de la Llave, Seara 2008]

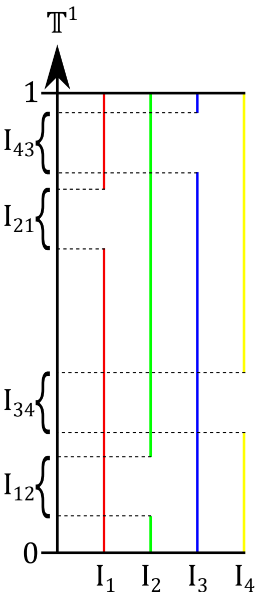
Take 8 homoclinic cylinders $B_1, \ldots, B_8$ and let s_j denote the corresponding scattering maps. How can we destroy invariant curves of the IFS $\{f, s_1, \ldots, s_8\}$ in A?

$$f|_A : (\theta, y) \mapsto (\theta + \tau_* y, y) + O(\tau_*^2)$$

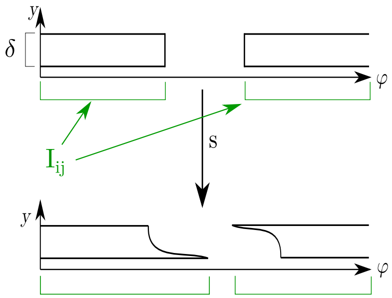
Birkhoff's theorem $\implies$ all invariant essential curves of f in A are graphs of Lipschitz functions. Denote by L an upper bound of Lipschitz constants of invariant curves of f in A

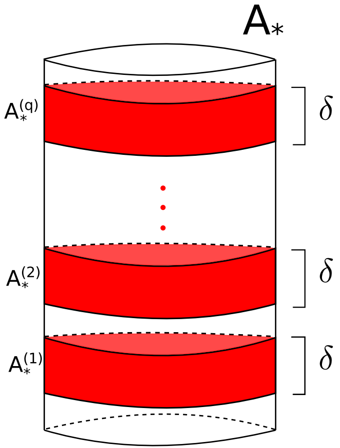
The property of $s_1, \ldots, s_8$ having no L-Lipschitz common invariant curves in the compact subcylinder A_* is open; we must show that it is dense

The perturbations ψ must be defined so that the scattering maps are periodic in the angular coordinate on A_*



$$s_\epsilon = s + \epsilon \Omega \nabla J \circ s + O(\epsilon^2)$$





Perturb Q by adding 8 locally supported functions:

$$\tilde{Q}_\epsilon = Q + \epsilon_1 \sum_{j=1}^{4} \psi_j + \epsilon_2 \sum_{j=5}^{8} \psi_j$$

where $\epsilon = (\epsilon_1, \epsilon_2)$. Suppose the terms of order ϵ in the expansion of the scattering maps satisfy certain conditions: if $s_j(\varphi, y, \epsilon) = (\Psi_j(\varphi, y, \epsilon), Y_j(\varphi, y, \epsilon))$ then for $j = 1, 2$ (resp. 3,4) and whenever $\Psi_j(\varphi, y, \epsilon) \in I_j$ we have

$$\frac{\partial Y_j}{\partial \epsilon_1}(\varphi, y, \epsilon) > 2L \left| \frac{\partial \Psi_j}{\partial \epsilon_1}(\varphi, y, \epsilon) \right| > 2L$$

$$\frac{\partial Y_j}{\partial \epsilon_2}(\varphi, y, \epsilon) < -2L \left| \frac{\partial \Psi_j}{\partial \epsilon_2}(\varphi, y, \epsilon) \right| < -2L$$

Lemma

Then the set of parameters $\epsilon = (\epsilon_1, \epsilon_2)$ for which the scattering maps $s_1, \ldots, s_8$ have an L-Lipschitz common invariant curve in $A_*^{(p)}$ has Lebesgue measure 0.

Take two values ϵ_1 and ϵ_2 of ϵ, and suppose the scattering maps have an L-Lipschitz common invariant curve $\mathcal{L}_1$ at ϵ_1, and $\mathcal{L}_2$ at ϵ_2. Write

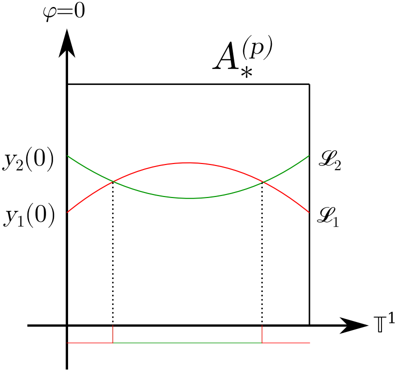
$$\mathcal{L}_1 = \mathrm{graph}(y_1)$$

and

$$\mathcal{L}_2 = \mathrm{graph}(y_2).$$

We show that

$$\|\epsilon_1 - \epsilon_2\| \leq \tau_*^{-2} |y_1(0) - y_2(0)|.$$



An example of the perturbations we require is

$$\psi_j(\bar{x}) = C_{j,1} \exp\left(C_{j,2}\left\langle \xi_j'(0), \bar{x} \right\rangle\right)$$

where for $j = 1, 2$ the positive constant $C_{j,1}$ is sufficiently large, and

$$C_{j,2} = -1 + \min_{(x,w)\in T^1\Gamma}\left[-2LK^{-1}\kappa(x,w)^{\frac{4}{3}}\right].$$