ON SMALL BREATHERS OF NONLINEAR KLEIN-GORDON EQUATIONS VIA EXPONENTIALLY SMALL HOMOCLINIC SPLITTING

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Abstract. Breathers are nontrivial time-periodic and spatially localized solutions of nonlinear dispersive partial differential equations (PDEs). Families of breathers have been found for certain integrable PDEs but are believed to be rare in non-integrable ones such as nonlinear Klein-Gordon equations.

In this paper we consider semilinear Klein-Gordon equations and prove that single bump small amplitude breathers do not exist for generic analytic odd nonlinearities.

Breathers with small amplitude can exist only when its temporal frequency is close to be resonant with the Klein-Gordon dispersion relation. For these frequencies, we identify the leading order term in the exponentially small (with respect to the small amplitude) obstruction to the existence of such small breathers in terms of the so-called Stokes constant. We also construct generalized breathers, which are periodic in time and spatially localized solutions up to exponentially small tails.

We rely on the spatial dynamics approach where breathers can be seen as homoclinic orbits. The birth of such small homoclinics is analyzed via a singular perturbation setting where a Bogdanov-Takens bifurcation is coupled to infinitely many rapidly oscillatory directions. The leading order term of the exponentially small splitting between the stable/unstable invariant manifolds is obtained through a careful analysis of the analytic continuation of their parameterizations. This requires the study of another limit equation in the complexified evolution variable, the so-called *inner equation*.

Contents

1. Introduction

Breathers are nontrivial time-periodic and spatially localized solutions of nonlinear dispersive partial differential equations (PDEs). This kind of solutions play an important role in physical applications and the interest in their existence or breakdown gives rise to a fundamental problem in the study of the dynamics of such PDEs.

So far breathers have been constructed mostly for completely integrable PDEs. As far as the authors know, the *sine-Gordon equation*

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = 0, \tag{sG}$$

is one of the first PDEs found to admit a family of breathers (see [1]), which is given explicitly by

$$u_\omega(x,t) = 4\arctan\left(\frac{m}{\omega}\frac{\sin(\omega t)}{\cosh(mx)}\right), \quad m,\omega > 0,\ m^2+\omega^2 = 1. \tag{1.1}$$

They are viewed as the locked states of a kink and an anti-kink. Along with spatial and temporal translation, the breathers form a 3-dim surface in the infinite dimensional phase space of (sG).

1.1. Existence/non-existence of small breathers of nonlinear Klein-Gordon equations.

The sine-Gordon equation (sG) is a particular case of the family of *nonlinear Klein-Gordon equations* in one space dimension. In this paper, through a bifurcation approach in a singular perturbation framework, we study the existence/non-existence of *small* breathers of a class of nonlinear Klein-Gordon equations

$$\partial_t^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0, \tag{1.2}$$

where the nonlinearity f satisfies

$$f(u) \text{ is a real-analytic odd function near 0 and } f(u) = \mathcal{O}(u^5). \tag{A}$$

While their signs are natural restrictions, the coefficients 1 and $\frac{1}{3}$ in the above equation are not. In fact, given any nonlinear Klein-Gordon equation $(\partial_T^2 v - \partial_X^2)v + F(v) = 0$ with a smooth real valued odd function $F(v)$ with $F'(0) > 0$ and $F'''(0) < 0$, it is always possible to rescale $v(X,T) = Au(aX, aT)$ so that $u(x,t)$ satisfies (1.2).

Let $\omega > 0$ denote the temporal frequency of a possible breather $u(x,t)$ of (1.2). A solution $u(x,t)$ of (1.2) is a breather of temporal frequency ω if $u(x,t)$ is $\frac{2\pi}{\omega}$-periodic in t and

$$\lim_{x\to\pm\infty} u(x,\cdot) = 0$$

in some appropriate metric. Any real valued function $\frac{2\pi}{\omega}$-periodic in t can be expressed as a Fourier series

$$u(t) = \sum_{n=-\infty}^{+\infty}\left(-\frac{i}{2}\right)u_n e^{in\omega t}, \quad u_{-n} = -\overline{u_n}, \tag{1.3}$$

where the factor $-\frac{i}{2}$ is purely for the technical convenience when the problem is reduced to functions odd in t represented in Fourier sine series. We denote

$$\Pi_n[u] = u_n = \frac{i\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} u(t)e^{-in\omega t}dt, \qquad \|u\|_{\ell_1} = \sum_{n=-\infty}^{+\infty}|u_n| = \sum_{n=-\infty}^{+\infty}|\Pi_n[u]|. \tag{1.4}$$

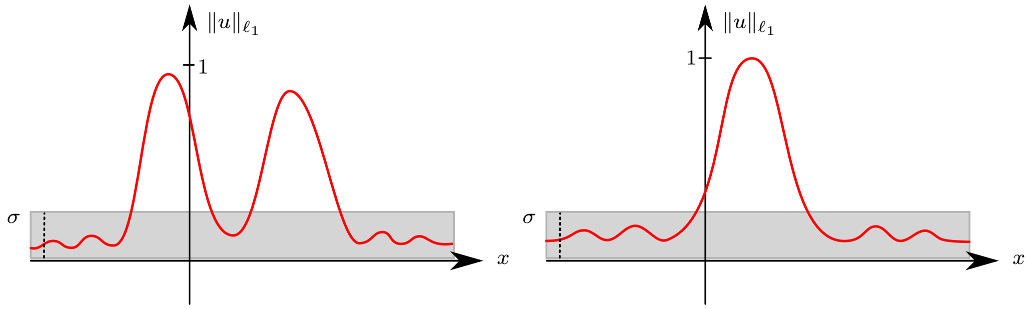

FIGURE 1. Multi-bump (left) and single-bump (right) functions according to Definition 1.1.

Sometimes, with slight abuse of the notation, we also use $\Pi_n[u]$ to denote the mode $-\frac{i}{2}u_n e^{in\omega t}$. The above norm $\|\cdot\|_{\ell_1}$ is invariant under rescaling in t. As we shall also take the advantage of the energy conservation of (1.2) as an evolution problem with the dynamic variable x, Sobolev norms like $\|\cdot\|_{H_t^k\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)}$ will be involved as well. Our first main theorem, Theorem 1.2 below, is on small breathers of a single bump in x (see Figure 1).

Definition 1.1. *Let $\sigma \in (0,1)$ and $\omega > 0$. We say that a $\frac{2\pi}{\omega}$-periodic-in-t function $u(x,t)$ is σ-multi-bump in x in the ℓ_1 norm (or in some other norm such as $H_t^1\big((-\frac{\pi}{\omega},\frac{\pi}{\omega})\big)$) if there exist $x_1 < x_2 < x_3 < x_4 < x_5$ such that*

$$\max\{\|u(x_{j_1},\cdot)\|_{\ell_1} \mid j_1 \in 1,3,5\} \le \sigma \min\{\|u(x_{j_2},\cdot)\|_{\ell_1} \mid j_2 \in 2,4\}.$$

A function $u(x,t)$ is said to be σ-single-bump if it is not σ-multi-bump.

The following theorem is the main result of this paper. It is a corollary of the much more detailed Theorem 1.3 proved through a spatial dynamics approach.

Theorem 1.2. *Assume $f(u)$ satisfies hypothesis* (A)*, then the following hold.*

(1) *There exists $C_{\rm in} \in \mathbb{C}$, which depends on $f(\cdot)$ analytically, such that if $C_{\rm in} \neq 0$, then for any $\sigma \in (0,1)$, there exists $\rho^* > 0$ such that there does not exist any solution $u(x,t)$ to* (1.2) *which:*
   (a) *is $\frac{2\pi}{\omega}$-periodic in t for some $\omega > 0$,*
   (b) *is σ-single-bump in the ℓ_1 norm in the sense of Definition 1.1,*
   (c) *satisfies that, as $|x| \to +\infty$,*

$$\|u(x,\cdot)\|_{H_t^1\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} + \|\partial_x u(x,\cdot)\|_{L_t^2\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} \to 0, \tag{1.5}$$

   (d) *satisfies*

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{\ell_1} < \min\{1, \rho^*\omega^{\frac{1}{2}}\}. \tag{1.6}$$

(2) *There exist $M, \varepsilon_0 > 0$, such that for any*

$$\omega = \sqrt{\frac{1}{k(k+\varepsilon^2)}}, \qquad k \ge 1,$$

and $\varepsilon \in (0,\varepsilon_0)$, there exist infinitely many $\frac{2\pi}{\omega}$-periodic-in-t solutions $u(x,t)$ (referred to as generalized breathers) such that

$$\Big|u(x,t) - \frac{2\sqrt{2k}\varepsilon\omega}{\cosh\sqrt{k}\varepsilon\omega x}\sin k\omega t\Big| \le M\Big(\frac{\varepsilon^3}{k^{\frac{3}{2}}\cosh\sqrt{k}\varepsilon\omega x} + \frac{\sqrt{k}}{\varepsilon}e^{-\frac{\sqrt{2k}\pi}{\varepsilon}}\Big), \quad \forall\, x,t \in \mathbb{R}.$$

Moreover, if $C_{\rm in} \neq 0$, then it also holds

$$\lim\inf_{|x|\to\infty}\left(\|u\|_{H_t^1(-\frac{\pi}{\omega},\frac{\pi}{\omega})} + \left\|\partial_x u\right\|_{L_t^2(-\frac{\pi}{\omega},\frac{\pi}{\omega})}\right) \ge \frac{\sqrt{k}}{M}e^{-\frac{\sqrt{2k}\pi}{\varepsilon}}.$$

Here the constant $C_{\rm in}$ is often referred to as the *Stokes constant* in the literature, which is the coefficient of the leading order term in an associated exponentially small splitting phenomena (See Subsection 2.3 for more discussions on $C_{\rm in}$).It depends on the full jet of the real analytic nonlinearity $f(u)$ and it is not given by a closed formula (see Section 2.3). However it depends analytically on f in the sense of Lemma 3.5 below. Namely, for any $f(u,c)$ analytic in both u and $c \in \mathbb{C}^N$, $C_{\rm in}$ is also analytic in c. While apparently $C_{\rm in} = 0$ for (sG), as it possesses the single-bump arbitrarily small breathers (1.1), the analyticity of $C_{\rm in}$ implies that it should be nonzero for generic f. In fact, this is rigorously proved in Appendix C when (1.2) is analyzed close to (sG) through a leading order perturbation analysis based on some calculations derived in [18].

Note moreover, that $C_{\rm in}$ depends only on f. In particular, it does not depend on the frequency of the breather ω. This implies that just *one condition* ($C_{\rm in} \neq 0$) rules out the existence of single-bump small amplitude breathers *of any frequency.*

It is also worth pointing out that, for a function $\frac{2\pi}{\omega}$-periodic in t, while the norm $\|\cdot\|_{\ell_1}$ is at the same level as $\|\cdot\|_{L^\infty}$ in scaling, the smallness in the theorem (see (1.6)) is measured uniformly in ω in terms of $\omega^{-\frac{1}{2}}\|u(x,\cdot)\|_{\ell_1}$ comparable to $\|\cdot\|_{L^2_t}$ in scaling. As this norm is weaker than $\|\cdot\|_{H^1_t}$, clearly the above theorem also implies the nonexistence of single-bump localized solutions, small in the energy norm (as in (1.5)), under the assumption $C_{\rm in} \neq 0$.

Theorem 1.2 is only concerned with small single-bump-in-x breathers. Clearly if $u(x,t)$ is σ-multi-bump, then it is also multi-bump for any $\sigma' \in (\sigma, 1)$. Hence the constant ρ_* of the smallness increases as σ increases. Even if $C_{\rm in} \neq 0$, it does not rule out possible small amplitude periodic-in-t solutions which decay as $|x| \to \infty$ but with multiple bumps.

While small single-bump breathers are not expected for (1.2) with most $f(u)$ satisfying (A), statement (2) of Theorem 1.2 indicates that the more generic phenomenon is the existence of small breathers with exponentially small, but non-vanishing, tails for ω slightly smaller than each $\frac{1}{k}$. This is consistent with the fact that small (sG) breathers (1.1) have periods slightly greater than $2k\pi$. They actually form submanifolds of finite codimension. These small single-bump breather-like solutions to (1.2) are the superposition of a small exponentially localized-in-x wave of the order $\mathcal{O}(\varepsilon k^{-\frac{1}{2}} e^{-\varepsilon k^{-\frac{1}{2}}|x|})$ and an L^∞_{xt} perturbation up to the order $\mathcal{O}(k^{\frac{1}{2}}\varepsilon^{-1} e^{-\frac{\sqrt{2k}\pi}{\varepsilon}})$. In the non-degenerate case of $C_{\rm in} \neq 0$, the infimum of the tails of such generalized breathers is also bounded below by this exponential order differing by a factor of ε (only due to a simplification of the norms), see Theorem 1.3 for the optimal statement. Recalling that the breathers of (sG) form a 3-dim manifold in the infinite dimensional space of solutions, the generalized breathers actually form a family of finite codimension. See Theorem 1.3 and Theorem 2.1 and remarks thereafter for more details.

While the (sG) breathers (1.1) are obtained based on the complete integrability of (sG), in general the existence of breathers for nonlinear wave equations is rare[1] (see [18, 60]). Hence it is a fundamental question whether the existence of breathers is a special phenomenon due to the integrability or it occurs more generally. The importance of breathers also lies in that they may serve as building blocks organizing the infinite dimensional dynamics of the underlying evolutionary PDE. In a recent paper [12], Chen, Liu, and Lu proved the soliton resolution of (sG) based on the integrable theory. Namely, in certain weighted Sobolev norm, solutions to (sG) decay (at an algebraic rate) to a finite superposition of kinks, anti-kinks, and breathers, where breathers are the only spatially localized class. Therefore breather type structures could play a crucial role in the asymptotic dynamics of the nonlinear Klein-Gordon equations. In particular, unlike relative equilibria such as kinks, standing waves, *etc.*, breathers may be of arbitrarily small amplitude and energy and thus give rise to obstacles to possible nonlinear dispersive decay or scattering of small solutions. (Small amplitude breathers become large in certain weighted norms adopted in some literatures, e.g. [32, 15, 12] *etc.*)

For non-integrable Klein-Gordon equations, the existence of (small amplitude) breathers is a completely different (and much harder) problem due to the lack of effective tools such as inverse scattering methods. In the seminal work [58] from 1987, Kruskal and Segur used formal asymptotic expansions to justify the nonexistence of small $\mathcal{O}(\varepsilon)$ amplitude breathers in a class of nonlinear Klein-Gordon equations for close to resonant frequencies. The obstacle to solving for breathers is exponentially small in $\varepsilon \ll 1$. For the last 30 years, as far as the authors know, no rigorous justification of their leading order exponentially small asymptotics had been given. Theorem 1.2 (and Theorem 1.3 below) not only provides a proof of the Kruskal and Segur's exponentially small leading order expansion (for generic odd analytic nonlinearities) but also extends the non-existence of small amplitude breathers to all frequencies and constructs generalized breathers.

[1]On the contrary breathers are more likely to exist in Hamiltonian systems on lattices, see for instance [44, 43, 70, 54, 55].

In [37], the authors proved that small breathers odd in x do not exist for (1.2). The oddness is, however, contrary to the well-known examples – the (sG) breathers (1.1) are even in x. In the generic case of Theorem 1.2 where (1.2) does not have small breathers, the asymptotic behavior of small solutions in the energy space $H_x^1(\mathbb{R}) \times L_x^2(\mathbb{R})$ is a natural but intriguing question. The nonlinear Klein-Gordon equation (1.2) is quite different from the one studied in [61] (see also [10]) where slow radiation was proved when an extra potential term $V(x)u$ creates an isolated oscillatory eigenvalue of the linear problem. In contrast, (1.2) does not contain such a term and its small breathers/generalized breathers have temporal frequencies slightly less than 1, which is the end point of the continuous spectrum of (1.2) linearized at 0.[2]

In [18, 19, 11], Denzler and Birnir-McKean-Weinstein studied the rigidity of breathers, namely, the persistence of (families of) breathers (1.1) when (sG) is perturbed as

$$\partial_t^2 u - \partial_x^2 u + \sin(u) = \varepsilon \Delta(u) + \mathcal{O}(\varepsilon^2),$$

where Δ is an analytic function in a small neighborhood of $u = 0$. They proved that breathers corresponding to infinitely many amplitudes $m = \sqrt{1-\omega^2}$ persist only if $\Delta(u)$ results from a trivial rescaling of (sG). In [35], (sG) was also singled out as the only 1-dim nonlinear Klein-Gordon equation admitting breathers in certain form (see also [45]). Even though for small amplitude, (1.2) might also be viewed as close to (sG), our approach is rather different from doing so.

In [56], temporally periodic and spatially decaying solutions were found for the nonlinear Klein-Gordon equation with cubic nonlinearity, i.e. $f(u) = 0$, for $x \in \mathbb{R}^3$. These solutions are close to some stationary solutions (not necessarily small) with $\mathcal{O}(1/|x|)$ spatial decay. Such decay, which is too slow for the solutions to be in the energy space, is due to the 3-dim Helmholtz equation, whose solutions would only be in L^∞ and oscillate if $x \in \mathbb{R}^1$. Hence these solutions are more analogous to the generalized breathers constructed in [59] in 1-dim.

Even though the generalized breathers $u(x,t)$ obtained in Theorem 1.2 are not in the energy space $H_x^1(\mathbb{R}) \times L_x^2(\mathbb{R})$, conceptually they do shed some light on the dynamics of (1.2). Assume $k = 1$ for simplicity. Let $\gamma \in C_0^\infty(\mathbb{R}, \mathbb{R})$ be a cut-off function satisfying $\gamma(s) = 1$ for $|s| \le 1$ and $u(x,t)$ be a sufficiently smooth generalized breather (see Theorem 1.3 and Remark 2.2). Consider the solution $\tilde{u}(x,t)$ to (1.2) with initial value $\gamma(\frac{1}{\varepsilon^3} e^{-\frac{2\sqrt{2}\pi}{\varepsilon}} x)\big(u(x,t_0), \partial_t u(x,t_0)\big)$ for some t_0. Roughly its $H_x^1 \times L_x^2$ norm is of the order $\mathcal{O}(\sqrt{\varepsilon})$. The propagation speed of (1.2) being equal to 1 implies that $\tilde{u}$ is periodic in t for $|x|, t \le \mathcal{O}(\varepsilon^{-3} e^{\frac{2\sqrt{2}\pi}{\varepsilon}})$. Hence, this exponential time scale has to be relevant in studying the asymptotic dynamics of small energy solutions of (1.2). Generalized breathers or traveling modulated pulse solutions of (1.2) with tails bounded above by certain exponentially small order had also been obtained in [28, 41].

1.2. **Breathers and generalized breathers via spatial dynamics.** The spatial dynamics method (see, e.g. [36, 69]) is often an effective approach in constructing certain coherent structures for nonlinear PDEs where a spatial variable x plays a distinct role. In such a framework, the desired solutions are sought as special solutions in an evolutionary system where this x is treated as the dynamic variable.

Fix a temporal frequency $\omega > 0$. Considering x as the evolutionary variable, given any $\frac{2\pi}{\omega}$-periodic-in-t initial values $\big(u(0,\cdot), \partial_x u(0,\cdot)\big)$, the nonlinear Klein-Gordon equation (1.2) defines a well-posed dynamical system (evolving in x, with ω as a system parameter) in appropriate spaces of $\frac{2\pi}{\omega}$-periodic-in-t functions. This dynamical system has a Hamiltonian structure with the standard symplectic structure and the Hamiltonian energy

$$\mathbf{H} = \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} \left(\frac{1}{2}(\partial_x u)^2 + \frac{1}{2}(\partial_t u)^2 - \frac{1}{2}u^2 + \frac{1}{12}u^4 + F(u) \right) dt,$$

where $F(u) = \int_0^u f(s) ds$. In this system the trivial state 0 is stationary, and breathers correspond to orbits homoclinic to 0, namely, orbits which converge to 0 as both $x \to \pm\infty$. They usually belong to the intersection between the stable and unstable manifolds of 0. In the spatial dynamics framework of (1.2), the dimension of the hyperbolic eigenspace of 0 is finite and increases by 1 as the frequency ω decreases through $\frac{1}{k}$.

[2] Breathers have also been proven not to exist for some generalized KdV equations and the Benjamin-Ono equation, see [49, 50].

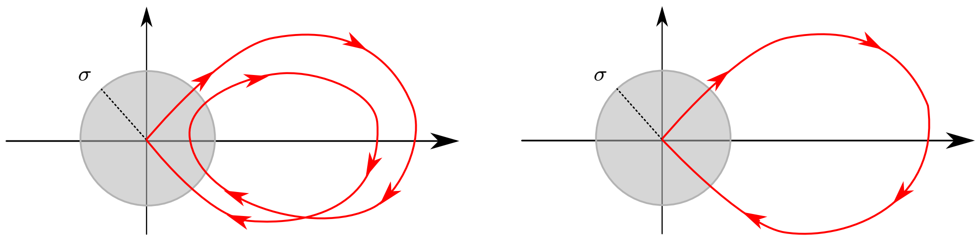

FIGURE 2. Multi-bump (left) and single-bump (right) solutions in the spatial dynamics framework.

We divide $\omega \in \mathbb{R}^+$ into two primary classes

$$I_k(\varepsilon_0) = \left[\sqrt{\frac{1}{k(k+\varepsilon_0^2)}}, \frac{1}{k}\right), \; k \in \mathbb{N}, \quad \textit{and} \quad J_k(\varepsilon_0) = \left[\frac{1}{k+1}, \sqrt{\frac{1}{k(k+\varepsilon_0^2)}}\right), \; k \in \mathbb{N}\cup\{0\}, \tag{1.7}$$

where $0 < \varepsilon_0 \leq 1/2$ is a parameter to be determined later. Note that $J_0(\varepsilon_0) = [1, \infty)$ and $(0,\infty) = (\cup_{k\in\mathbb{N}} I_k) \bigcup (\cup_{k\geq 0} J_k)$.

Roughly speaking, when $\omega \in J_k(\varepsilon_0)$, the hyperbolicity of the linearized (1.2) at 0 is strong enough to prevent the existence of small homoclinic orbits. In contrast to the case of $J_k(\varepsilon_0)$, when ω decreases through $\frac{1}{k}$ and enters $I_k(\varepsilon_0)$, the linearized (1.2) is only weakly hyperbolic in the newly generated hyperbolic directions and small homoclinic orbits may appear through a homoclinic bifurcation. This is actually consistent with the fact that the periods of small (sG) breathers (1.1) are close to $2k\pi$.

The following theorem rephrases (and implies) Theorem 1.2 in terms of invariant manifolds and is obtained through a careful analysis of the spatial dynamics of (1.2) near 0. In the intervals $I_k(\varepsilon_0)$ it requires the study of the exponentially small splitting between the stable and unstable manifolds of 0.

Note that the notion of single-bump and multibump breathers (see Definition 1.1) gets translated to homoclinics as seen in Figure 2.

Theorem 1.3. *Assume $f(u)$ satisfies hypothesis* (A). *Then the following statements hold.*

(1) *There exists $\rho_1^* > 0$ such that for any $\varepsilon_0 \in (0, 1/2]$, $\omega \in J_k(\varepsilon_0)$, $k \in \mathbb{N}\cup\{0\}$, if $u(x,t)$ is a $\frac{2\pi}{\omega}$-periodic-in-t solution to* (1.2) *satisfying* (1.5) *as $x \to +\infty$ or $-\infty$, then*

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{\ell_1} \geq \rho_1^* \min\{1, \varepsilon_0 \omega^{\frac{1}{2}}\}.$$

(2) *There exist $\varepsilon_0, M > 0$ such that for*

$$\omega = \sqrt{\frac{1}{k(k+\varepsilon^2)}} \in I_k(\varepsilon_0), \quad \forall\, k \geq 1, \tag{1.8}$$

there exist $\frac{2\pi}{\omega}$-periodic and odd in t solutions $u^\star_{\mathrm{wk}}(x,t)$, $\star = s, u$, to (1.2), *only containing Fourier modes $n \in k\mathbb{Z}$ with odd $\frac{n}{k}$ in* (1.3), *such that*

(a) *For $x \geq 0$ for $\star = s$ and $x \leq 0$ for $\star = u$, they can be approximated as*

$$\left\| \left(1 - \frac{1}{(k\omega)^2}\partial_t^2\right) \left(\begin{pmatrix} u^\star_{\mathrm{wk}}(x,t) \\ \frac{\partial_x u^\star_{\mathrm{wk}}(x,t)}{\sqrt{k}\varepsilon\omega} \end{pmatrix} - \sqrt{k}\varepsilon\omega \begin{pmatrix} v^h(\varepsilon\sqrt{k}\omega x) \\ (v^h)'(\varepsilon\sqrt{k}\omega x) \end{pmatrix} \sin k\omega t \right) \right\|_{\ell_1} \leq M k^{-\frac{3}{2}} \varepsilon^3 v^h(\varepsilon\sqrt{k}\omega x), \tag{1.9}$$

where $v^h(y) = \frac{2\sqrt{2}}{\cosh y}$;

(b) *They also satisfy $\Pi_k\left[\partial_x u^\star_{\mathrm{wk}}(0,\cdot)\right] = 0$, $\star = s, u$, and*

$$\left\| \left(|-\partial_t^2 - 1|^{\frac{1}{2}}(u^u_{\mathrm{wk}} - u^s_{\mathrm{wk}}) + i\partial_x(u^u_{\mathrm{wk}} - u^s_{\mathrm{wk}})\right)(0,t) - 4\sqrt{2}C_{\mathrm{in}} e^{-\frac{\sqrt{2k}\pi}{\varepsilon}} \sin 3k\omega t \right\|_{\ell_1} \leq \frac{M e^{-\frac{\sqrt{2k}\pi}{\varepsilon}}}{\frac{1}{2}\log k - \log\varepsilon},$$

where C_{in} is the Stokes constant determined by f as given in Theorem 1.2.

(c) *There always exist infinitely many $\frac{2\pi}{\omega}$-periodic-in-t solutions $u(x,t)$ such that*

$$\left\| |-\partial_t^2 - 1|^{\frac{1}{2}}(u - u^\star_{\mathrm{wk}})(x,\cdot) \right\|_{L_t^2(-\frac{\pi}{\omega},\frac{\pi}{\omega})} + \left\| \partial_x(u - u^\star_{\mathrm{wk}})(x,\cdot) \right\|_{L_t^2(-\frac{\pi}{\omega},\frac{\pi}{\omega})} \leq M k^{\frac{1}{2}} e^{-\frac{\sqrt{2k}\pi}{\varepsilon}},$$

for all $x \geq 0$ with $\star = s$ and $x \leq 0$ with $\star = u$.

(3) *Suppose $C_{\rm in} \neq 0$, then for any $\sigma \in (0,1)$, there exist $\varepsilon_0, M, \rho_2^* > 0$, where ε_0 and M also satisfy the properties in (2), such that, if ω satisfies* (1.8), *then*
   (a) *those generalized solutions $u(x,t)$ in (2c) also satisfy*
$$\big\||-\partial_t^2 - 1|^{\frac{1}{2}}(u - u_{\rm wk}^\star)(x,\cdot)\big\|_{L_t^2(-\frac{\pi}{\omega},\frac{\pi}{\omega})} + \big\|\partial_x(u - u_{\rm wk}^\star)(x,\cdot)\big\|_{L_t^2(-\frac{\pi}{\omega},\frac{\pi}{\omega})} \geq \tfrac{1}{M}k^{\frac{1}{2}}e^{-\frac{\sqrt{2k}\pi}{\varepsilon}},$$
   for all $x \geq 0$ with $\star = s$ and $x \leq 0$ with $\star = u$.
   (b) *and if $u(x,t)$ is a $\frac{2\pi}{\omega}$-periodic-in-t solution to* (1.2) *satisfying* (1.5) *as $|x| \to \infty$ and*
$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{\ell_1} \leq \rho_2^*\sqrt{\omega},$$
   then $u(x,t)$ is σ-multi-bump in the ℓ_1 norm.

For frequencies $\omega \in J_k(\varepsilon_0)$, statement (1), which is proved in Section 8, implies that all orbits on both the stable and unstable manifolds of 0 leave a small neighborhood of 0 (worth noting that ρ_1^* is uniform in ω). Therefore there are no small orbits homoclinic-in-x to 0 for such $\omega \in J_k(\varepsilon_0)$.

In the other case of $\omega \in I_k(\varepsilon_0)$, statement (2a) indicates that there exist special small solutions $u_{\rm wk}^\star(x,t)$, $\star = s, u$, of amplitude $\mathcal{O}(\varepsilon k^{-\frac{1}{2}})$ on the $(2k+1)$-dimensional stable/unstable manifolds of 0, both of which are well approximated with errors of the order of $\mathcal{O}(\varepsilon^3 k^{-\frac{3}{2}})$ by the exponentially localized $v^h(x)$ after some rescaling. The scale $\mathcal{O}(\varepsilon k^{-\frac{1}{2}})$ of their amplitudes and of their spatial variable x is due to the cubic nonlinearity and the scale of the weakest hyperbolic eigenvalues.

The most significant result of the whole paper is statement (2b), which gives the precise leading $\mathcal{O}(e^{-\frac{\sqrt{2k}\pi}{\varepsilon}})$ order term of the splitting between $u_{\rm wk}^u$ and $u_{\rm wk}^s$, with the leading coefficient $C_{\rm in} \in \mathbb{C}$ often referred to as the Stokes constant. The relative scale between u and $\partial_x u$ is consistent with the quadratic part of the Hamiltonian $\mathbf{H}$, where $|-\partial_t^2 - 1|$ is somewhat degenerate of the order $\mathcal{O}(k^{-1}\varepsilon^2)$ when applied to the k-th Fourier mode $e^{ik\omega t}$. Intuitively, the exponentially small upper bounds may be viewed as coming from oscillatory integrals involving analytic functions and can be derived from normal form transformations. Here we stress that the precise exponentially small leading order approximation is obtained for this problem which has *infinitely many oscillatory directions* (more explanations below).

In addition to the derivation of $C_{\rm in}$ from the invariant stable foliations in Section 5.3 (see also the heuristic ideas explained in Section 2.2), we also give another perspective based on Borel summation of divergent series in Section 2.3 and conjecture a possible algorithm to compute $C_{\rm in}$.

If $C_{\rm in} \neq 0$, one may expect the non-existence of (at least single-bump) small homoclinic orbits, which is statement (3b). The proof is given in Section 9. Observe that, combining (1) and (3b), we obtain statement (1) of Theorem 1.2. In particular, for $\omega \in I_k(\varepsilon_0)$, the non-existence is proved in a neighborhood of order $\mathcal{O}(k^{-\frac{1}{2}})$ with a constant uniform in ω, which is much greater than the size $\mathcal{O}(\varepsilon k^{-\frac{1}{2}})$ of the candidates of the homoclinic orbits.

Even if single-bump breathers may not exist, namely, the stable and unstable manifolds of 0 may miss each other by an exponentially small distance, we prove that the center-stable and center-unstable manifolds of 0 intersect transversally and produce an invariant submanifold of finite codimension homoclinic to the center manifold, consisting of generalized breathers of the nonlinear Klein-Gordon equation (1.2). Statements (2c) and (3a) provide more detailed estimates on the generalized breathers than Theorem 1.2.

As $u_{\rm wk}^\star$, $\star = u, s$, are special solutions of high regularity, the norm in (1.9) on their approximations actually could be refined to be H_t^n basically for any $n \geq 0$. In contrast, since the set of generalized breathers is of finite codimension in the phase space of the spatial dynamics, the norms in (2d) and (3a) arising from the quadratic part of the Hamiltonian $\mathbf{H}$ are not expected to be improved.

1.3. **Framework of the proof and further discussions.** We fix the frequency $\omega > 0$, let $\tau = \omega t$, and first look for solutions to the nonlinear Klein-Gordon equation (1.2) which are 2π-periodic in τ and decay to 0 as either $x \to +\infty$ or $x \to -\infty$. In the (x,τ) coordinates, equation (1.2) takes the form

$$\omega^2\partial_\tau^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0. \tag{1.10}$$

Denote

$$g(u) = \frac{1}{3}u^3 + f(u). \tag{1.11}$$

In terms of the Fourier series expansion (1.3) (see also (1.4)), we obtain

$$(\partial_x^2 + n^2\omega^2 - 1)u_n = -\Pi_n\left[g(u)\right], \qquad n \in \mathbb{Z}. \tag{1.12}$$

The eigenvalues of the linearization of (1.10) at 0,

$$\partial_x^2 u - \omega^2 \partial_\tau^2 u - u = 0,$$

are $\pm\nu_n$, where

$$\nu_n = \sqrt{1 - n^2\omega^2},$$

and their eigenfunctions can be calculated using the Fourier modes.

Assume $\omega \in [\frac{1}{k+1}, \frac{1}{k})$ for some $k \geq 0$. For $0 \leq |n| \leq k$, the eigenvalues $\pm\nu_n$ are hyperbolic, while the center eigenvalues $\pm\nu_n = \pm i\vartheta_n$, $\vartheta_n = \sqrt{n^2\omega^2 - 1}$, correspond to $|n| \geq k+1$.

If $\omega \in J_k(\varepsilon_0)$ (see (1.7)), where $\varepsilon_0 \in (0, \frac{1}{2}]$, then the smallest hyperbolic eigenvalues satisfy

$$\nu_k > \tfrac{\varepsilon_0}{\sqrt{k+\varepsilon_0^2}} > \min\{1, \tfrac{\varepsilon_0}{2\sqrt{k}}\}.$$

Based on the general local invariant manifold theory (see, e.g. [14]) and this spectral gap along with the cubic nonlinearity of wave type equation (1.10), one expects that the local stable/unstable manifolds are close to the stable/unstable subspaces in a neighborhood of radius of the order $\mathcal{O}\big(\min\{1, \frac{\varepsilon_0}{2\sqrt{k}}\}\big)$. Therefore they cannot intersect in such a small neighborhood to produce homoclinic orbits. This argument is carried out uniformly in k and ω in Section 8 and statement (1) of Theorem 1.3 follows consequently.

For $\omega \in I_k(\varepsilon_0)$, where $k \geq 1$ and $\varepsilon_0 \in (0, \frac{1}{2}]$, ν_k can be arbitrarily small. The different scales in x in these weakly hyperbolic directions and the other much faster directions make the local dynamics of (1.10) near 0 a singular perturbation problem. To be more precise, let

$$\varepsilon = \sqrt{\frac{1}{k}\left(\frac{1}{\omega^2} - k^2\right)} \in (0, \varepsilon_0) \tag{1.13}$$

and consider the following scaling of the amplitude and x,

$$u = \varepsilon\sqrt{k}\omega v \text{ and } y = \varepsilon\sqrt{k}\omega x. \tag{1.14}$$

Thus $u(x,\tau)$ satisfies (1.2) if, and only if, $v(y,\tau)$ satisfies

$$\partial_y^2 v - \frac{1}{\varepsilon^2 k}\partial_\tau^2 v - \frac{1}{\varepsilon^2 k\omega^2} v + \frac{1}{3}v^3 + \frac{1}{\varepsilon^3 k^{\frac{3}{2}}\omega^3} f\left(\varepsilon\sqrt{k}\omega v\right) = 0, \tag{1.15}$$

which is a Hamiltonian PDE in the dynamical variable y with the Hamiltonian

$$\mathcal{H}(v, \partial_y v) = \int_{\mathbb{T}} \left(\frac{(\partial_y v)^2}{2} + \frac{(\partial_\tau v)^2}{2\varepsilon^2 k} - \frac{v^2}{2\varepsilon^2 k\omega^2} + \frac{v^4}{12} + \frac{F(\varepsilon\sqrt{k}\omega v)}{\varepsilon^4 k^2\omega^4} \right) d\tau, \tag{1.16}$$

where F is the real analytic function such that $F'(z) = f(z)$ and $F(z) = \mathcal{O}(z^6)$. Using the projection Π_n in (1.4) and denoting $\cdot = d/dy$, we obtain (see (1.12)),

$$\ddot{v}_n = -\frac{(n^2\omega^2 - 1)}{\varepsilon^2 k\omega^2} v_n - \frac{1}{\varepsilon^3 k^{\frac{3}{2}}\omega^3}\Pi_n\left[g(\varepsilon\sqrt{k}\omega v)\right], \ n \in \mathbb{Z}. \tag{1.17}$$

By (1.13),

$$\lambda_n = \sqrt{\frac{1}{k}\left|n^2 - \frac{1}{\omega^2}\right|} \geq \frac{1}{2}, \quad \text{for each } \ |n| \neq k.$$

Using this notation, (1.17) becomes

$$\begin{cases} \ddot{v}_n = \dfrac{\lambda_n^2}{\varepsilon^2} v_n - \dfrac{1}{\varepsilon^3 k^{\frac{3}{2}}\omega^3}\Pi_n\left[g(\varepsilon\sqrt{k}\omega v)\right], & |n| < k, \\ \ddot{v}_{\pm k} = v_{\pm k} - \dfrac{1}{\varepsilon^3 k^{\frac{3}{2}}\omega^3}\Pi_{\pm k}\left[g(\varepsilon\sqrt{k}\omega v)\right], & \\ \ddot{v}_n = -\dfrac{\lambda_n^2}{\varepsilon^2} v_n - \dfrac{1}{\varepsilon^3 k^{\frac{3}{2}}\omega^3}\Pi_n\left[g(\varepsilon\sqrt{k}\omega v)\right], & |n| > k. \end{cases} \tag{1.18}$$

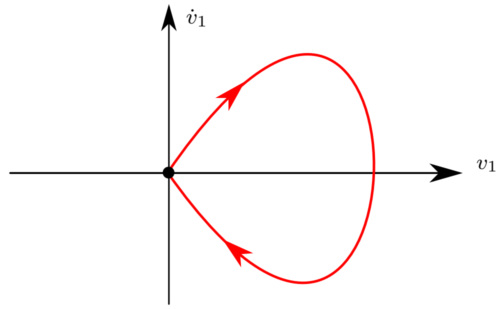

FIGURE 3. Real positive homoclinic (1.22) to 0 of the Duffing equation (1.21) in the critical manifold $\mathcal{M}$.

Since $v_{-n} = -\overline{v_n}$, actually we need to consider $n \geq 0$ only. Notice that $(\varepsilon k^{\frac{1}{2}}\omega)^{-3} g(\varepsilon\sqrt{k}\omega v) = \mathcal{O}(|v|^3)$ is smooth with bounds uniform in $\varepsilon\sqrt{k}\omega$. The stable and unstable invariant manifolds $W^s(0)$ and $W^u(0)$ of $2k+1$ real dimensions correspond to solutions v^s and v^u of (1.18) satisfying the asymptotic conditions (see also Corollary 8.4)

$$\lim_{y\to+\infty} v_n^s(y) = \lim_{y\to+\infty} \dot v_n^s(y) = \lim_{y\to-\infty} v_n^u(y) = \lim_{y\to-\infty} \dot v_n^u(y) = 0, \qquad \text{for all } \ n \in \mathbb{Z}. \tag{1.19}$$

The singular perturbation problem (1.18) can be written as

$$\begin{cases} \varepsilon \dot v_n = \lambda_n w_n \\ \varepsilon \dot w_n = \lambda_n v_n - \lambda_n^{-1}\varepsilon^{-1} k^{-\frac{3}{2}}\omega^{-3}\Pi_n\left[g(\varepsilon\sqrt{k}v)\right], \quad |n| < k, \\ \varepsilon \dot w_n = -\lambda_n v_n - \lambda_n^{-1}\varepsilon^{-1} k^{-\frac{3}{2}}\omega^{-3}\Pi_n\left[g(\varepsilon\sqrt{k}\omega v)\right], \ |n| > k, \\ \ddot v_{\pm k} = v_{\pm k} - (\varepsilon\sqrt{k}\omega)^{-3}\Pi_{\pm k}\left[g(\varepsilon\sqrt{k}\omega v)\right]. \end{cases} \tag{1.20}$$

The singular limit of this system as $\varepsilon \to 0$ defines a critical manifold

$$\mathcal{M} = \{(v, w) \mid v_n = w_n = 0 \,\text{for all}\, n \neq \pm k\}$$

of real dimension 4 due to $v_{-n} = -\overline{v_n}$. The limiting dynamics on $\mathcal{M}$ is given by the Duffing equation

$$\ddot v_k = v_k - \frac{1}{3}\Pi_k\left[\left(\mathrm{Im}(v_k e^{ik\tau})\right)^3\right] = v_k - \frac{1}{4}|v_k|^2 v_k \tag{1.21}$$

which is integrable. It is known that in (1.21) the 2-dim stable and unstable manifolds of 0 coincide. In particular, it has a unique real homoclinic orbit to 0 satisfying $v_k > 0$, which is given by

$$v_k = v^h(y) = \frac{2\sqrt{2}}{\cosh(y)}. \tag{1.22}$$

(see Figure 3).

Outside the singular limit problem (1.21) defined on the 4-dim $\mathcal{M}$, if the fast directions in the full singular perturbation system (1.20) had been *hyperbolic*, then a 4-dim slow manifold close to $\mathcal{M}$ would persist for small $\varepsilon > 0$. The Hamiltonian structure and the symmetry of the problem would lead to persistent homoclinic orbits on the 4-dim slow manifold, which would give rise to small breathers. This mechanism indeed happens in the construction of some special solutions in some nonlinear PDEs (more of the elliptic type and not necessarily in a singular perturbation framework, see, e.g. [48]). However, starting with a wave type nonlinear PDE like (1.10), the fast dynamics in the singular perturbation problems, resulted from the spatial dynamics framework, are mostly *elliptic/oscillatory*. In these cases, slow manifolds usually do *not* persist for small $\varepsilon > 0$ and we are forced to analyze the splitting in the whole infinite dimensional phase space. One may also try directly applying the Melnikov method to analyze the persistence of homoclinic orbits (1.22) for (1.20) for small $\varepsilon > 0$. However, in this problem, the Melnikov function turns out to be very degenerate and does not give the correct asymptotic distance between the perturbed stable and unstable manifolds of 0. Consequently, it is not useful to analyze the existence of homoclinic orbits for small $\varepsilon > 0$.

In this paper we prove that "typically" $W^s(0)$ and $W^u(0)$ of (1.20) do not intersect to produce single-bump homoclinic orbits for $0 < \varepsilon \ll 1$. More importantly, we give the exact exponentially small (with respect to $0 < \varepsilon \ll 1$) *leading order approximation* of their splitting difference by studying another singular limit obtained after allowing y to become imaginary.

The Duffing equation (1.21) has a one-parameter family of homoclinic orbits to 0 with $v^\theta = v^h e^{i\theta}$, $\theta \in \mathbb{T}$. All the results given in this paper also hold for these other homoclinic orbits due to the translation invariance in t of equation (1.15).

The core of the proof of most of the results in Theorem 1.3 (statements (2) and (3a)) is for the case $k = 1$, $\omega \in I_1(\varepsilon_0)$, and under the oddness assumption in t, carried out from Section 2 to Section 7. More detailed results in this case can be found in Theorem 2.1. Finally the reduction of the general case to this special one and the completion of the proof are in Sections 8 and 9.

Generalized breathers as well as some other similar types of solutions had been obtained through this spatial dynamics framework, but often with only upper bound estimates on the tails, instead of their precise orders or lower bounds. In [41], Lu derived generalized breathers with tails bounded by $\mathcal{O}(e^{-\frac{c}{\varepsilon}})$ for some $c > 0$. In order to obtain such estimates, partial normal form transformations as in [51] were applied roughly $\mathcal{O}(\frac{1}{\varepsilon})$ times to reduce the error terms to be exponentially small. In a sequence of papers, Groves and Schneider considered small amplitude (order $\mathcal{O}(\varepsilon)$) modulating pulse solutions to a class of semilinear [28] and quasilinear [29, 30] reversible wave equations. These are solutions consisting of pulse-like spatially localized envelopes advancing in the laboratory frame and modulating an underlying wave-train of a fixed wave number $\xi_0 > 0$, which are time-periodic in a moving frame of reference. They would become breathers if $\xi_0 = 0$. For quasilinear reversible wave equations, Groves and Schneider constructed solutions $u(x, t)$ of this type with tails bounded by $\mathcal{O}(e^{-\frac{c}{\sqrt{\varepsilon}}})$ but only defined for $|x| \leq \mathcal{O}(e^{\frac{c}{\sqrt{\varepsilon}}})$. The finite length of the domain in x was mainly due to difficulties arising in quasilinear PDEs. In the semilinear case, such solutions could be derived globally in $x, t \in \mathbb{R}$ with the same estimates on the tails. The upper bounds of the modulating pulse solutions were also obtained by making the error terms small through consecutive applications of partial normal forms, e.g. as in [34]. Unlike [41] and the current paper, their derivation of the intersection of the center-stable and center-unstable manifolds, which leads to the modulating pulse solutions, was based on the reversibility and thus those solutions are also symmetric about time reversion. Even tough (1.2) is also reversible, in the current paper, as in [59, 41], this intersection (of finite codimension) is proved based on the positivity of the conserved energy in the center direction, where the reversibility plays very little role.

1.4. **Birth of small homoclinics via "eigenvalue collision": Exponentially small splitting of separatrices.** On the one hand, breather solutions are rare for nonlinear Klein-Gordon as well as other dispersive PDEs. So far most of breathers have been found for completely integrable PDEs. On the other hand, the above discussion indicates that the collision of eigenvalues at 0 as certain parameters vary, even if typically does not generate small breathers, it does generate small breathers with much smaller tails.

In general, suppose one seeks certain special solutions to an N-dimensional system, $N \leq \infty$, which can be reformulated into small homoclinic orbits to the steady state 0 in another system (P_α) involving a parameter α. Assume the following happens in (P_α).

(a) *"Eigenvalue collision"* at $\alpha = 0$. Namely, in a neighborhood of $0 \in \mathbb{C}$, there are exactly two eigenvalues $\pm\lambda(\alpha) \sim \pm\sqrt{\alpha}$ (modulo symmetries, but counting the algebraic multiplicity) of the linearization of (P_α) at 0. As α varies, they move towards 0 from the imaginary axis $i\mathbb{R}$ and then move into the real axis $\mathbb{R}$ after coinciding at 0 when $\alpha = 0$.

(b) The *normal form* of the local nonlinear system (P_α) near 0 projected to the 2-dimensional eigenspace $\mathcal{M}$ associated to $\pm\lambda(\alpha)$ is equivalent to

$$\ddot{u} - \lambda(\alpha)^2 u + u^m = 0, \quad m \geq 2,$$

where the "+" sign matters only when m is odd. Apparently this normal form system has one or two small homoclinic orbits of amplitude $\mathcal{O}\big(\lambda(\alpha)^{\frac{2}{m-1}}\big)$ for $0 < \alpha \ll 1$.

If the whole system (P_α) is 2-dimensional, it is actually one type of the Bogdanov-Takens bifurcations. If, in addition, there is a conserved quantity, then a small homoclinic orbit exists for all small $\alpha > 0$.

When (P_α) is of higher dimensions, then the dynamics in the directions transversal to $\mathcal{M}$ is at a fast scale and thus (P_α) is a typical singular perturbation system.

If the fast dynamic is hyperbolic, then by the standard normally hyperbolic invariant manifold theory, a 2-dimensional slow manifold $\mathcal{M}_\alpha$ persists for $0 < \alpha \ll 1$ and the question of the existence of small homoclinic orbit can be reduced to $\mathcal{M}_\alpha$.

However, if there are fast elliptic/oscillatory directions, then there does not necessarily exist a slow manifold and one cannot reduce (P_α) to 2 dimensions. Without such reduction, one is forced to find small homoclinic orbits as the intersection of the 1-dimensional stable and unstable manifolds $W^{u,s}(0)$ of 0 close to $\mathcal{M}$ in the N-dimensional phase space of (P_α), but this is highly unlikely due to the too many fast oscillatory dimensions. Homoclinic orbits are generated via such eigenvalue collision mechanism only in some very lucky/rare systems, such as the completely integrable (sG) where the family of breathers actually can be extended to large amplitudes.

In a generic system, while $W^{u,s}(0)$ do not intersect, the existence of a conserved quantity whose Hessian is positive definite in the center direction of the linearized (P_α) at 0 often ensures the transversal intersection of the center-stable and center-unstable manifolds $W^{cs,cu}(0)$. This intersection yields a finite co-dimensional tube homoclinic to the center manifold. The splitting distance between $W^{u,s}(0)$ determines how close this homoclinic tube is to 0, corresponding to how small the tails of the generalized breathers of the nonlinear Klein-Gordon equations can be. Regarding the splitting distance, the strong averaging effect of the fast oscillations actually makes $W^{u,s}(0)$ very close to each other – usually $\mathcal{O}(\alpha^n)$ if (P_α) has finite smoothness and $\mathcal{O}(e^{-\frac{1}{\alpha^\delta}})$ if (P_α) analytic. A leading order approximation such as the one obtained in Theorem 1.3 provides accurate information of the splitting and also sheds light for the future study of scattering maps [16] induced by the homoclinic tube and multi-bump homoclinic orbits.

To summarize, the mechanism of eigenvalue collision leading to a Bogdanov-Takens type bifurcation embedded in a normally elliptic singular perturbation problem is primarily responsible for the birth of small homoclinic orbits/generalized breathers for the nonlinear Klein-Gordon equation (1.10). It yields exact breathers in some very special cases such as the completely integrable (sG).

This coupling between the eigenvalue collision mechanism and fast oscillatory directions also occurs in some local bifurcations of Hamiltonian/Reversible/Volume preserving systems, such as the Hopf-zero or Hamiltonian Hopf bifurcations ([21, 4, 23, 5, 6]), and it is fundamental in the construction of solitary and traveling waves in PDEs and lattices [2, 20, 63, 39, 62, 40, 64, 65, 53]. However, in most of the papers the fast oscillatory directions are finite (often two) instead of infinite as is the case in the present paper.

In fact, this mechanism leads to what is usually called *exponentially small splitting of separatrices*, a phenomenon that usually arises in analytic systems with two time scales with fast oscillations and slow hyperbolic dynamics with a homoclinic loop (also called separatrix) as in the setting explained above. Other settings are the resonances of nearly integrable Hamiltonian systems and close to the identity area preserving maps. Analysis of such phenomena is fundamental in the construction of unstable behaviors in these models such as Arnold diffusion or chaotic dynamics.

The study of the exponentially small splitting of separatrices goes back to the seminal paper by Lazutkin [38] which dealt with the standard map. His strategy can be described as follows:

(1) The singular limit (that is (1.21) in the current paper) has a homoclinic orbit whose time parameterization is analytic in a strip containing the real line and has singularities in the complex plane.
(2) One can look for parameterizations of the perturbed invariant manifolfs which are close to the unperturbed homoclinic. They can be extended to complex values of the parameter which are close to the singularities of the unperturbed homoclinic with smallest imaginary part.
(3) One analyzes the difference between the perturbed invariant manifolds close to these singularities. To this end, one has to look for the leading order of the perturbed invariant manifolds in these complex domains. Then, one is encountered with two different situations:
    (i) In some problems, the perturbed invariant manifolds are well approximated by the unperturbed homoclinic also near its singularities. In this case, one can show that the classical Melnikov method gives the first order of the difference between these manifolds.
    (ii) In most of the problems, like the problem at hand, the unperturbed homoclinic is not a good approximation of the perturbed invariant manifolds in these complex domains. Therefore, one must look for new first order approximations. These first orders are solutions of the so-called *inner equation*, which is a limit equation independent of the perturbative parameter.
    The analysis of this inner equation gives the asympotic formula for the difference between the

invariant manifolds. In particular, it provides the Stokes constant C_{in} appearing in Theorems 1.2 and 1.3.

(4) The last step is to translate the analysis in the complex domain to the real parameterizations of the invariant manifolds.

In the last decades these strategy or similar ones relying on analytic continuation of the parameterization of the invariant manifolds to rigorously validate the Melnikov method (setting 3(i) above) has been applied to different problems [33, 57, 17, 25, 8]. Other methods using direct series expansions can be found in [22, 67, 68].

Several works using the analytic continuation of the parameterization of the invariant manifolds and an inner equation can be found in [26, 24, 46, 47] for near identity maps, [27, 52, 3, 7] for Hamiltonian systems with fast periodic forcing and [9, 4, 5, 6, 21] for local bifurcations.

This is the strategy that we follow in the present paper. It is explained heuristically in more detail in Sections 2.1 and 2.2. Exponentially small splitting has also been studied by other methods such as continuous averaging [66].

2. Analysis of the first bifurcation ($k = 1$) with oddness assumption in t

We devote this section to analyze the stable and unstable manifolds of $v = 0$ and their splitting for equation (1.15) with $k = 1$ and $\omega \in I_1(\varepsilon_0)$ (see (1.7)). function $f(\tau)$ as in (2.2), we consider the associated ℓ_1-norm of its Fourier coefficients To make the function space setting precise, recall the norm $\|\cdot\|_{\ell_1}$ defined in (1.4) which is simply the ℓ_1 norm of the Fourier coefficients in τ. Since

$$\|f_1 f_2\|_{\ell_1} \leq \|f_1\|_{\ell_1} \|f_2\|_{\ell_1},$$

treating y as the evolution variable, the local-in-y well-posedness of (1.15) with $(v, \partial_y v)(y, \cdot) \in \mathbf{X}$ where

$$\mathbf{X} := \{(v, w) \mid v, w \text{ are } 2\pi\text{-periodic in } \tau \text{ and } \|(v, w)\|_{\mathbf{X}} := \|v\|_{\ell_1} + \|(1 + |\partial_\tau|)^{-1} w\|_{\ell_1} < \infty\} \tag{2.1}$$

follows from a standard procedure. Here the operator $|\partial_\tau|$ is simply the multiplication of $|n|$ to the n-th Fourier modes for each n. For some results where the conservation of energy is used, we also consider the energy space $H^1_\tau \times L^2_\tau$ which is a dense subspace of $\mathbf{X}$ where (1.15) is also well-posed.

Due to the oddness assumptions on f, the subspace

$$\mathbf{X}_o = \{(v, w) \in \mathbf{X} \mid v, w \text{ are odd in } \tau\} \subset \mathbf{X}$$

of 2π-periodic odd functions of τ is invariant under the flow of (1.15), so we first restrict the analysis to this subspace.

For such odd functions (of real values) of τ, the Fourier series (1.3) turns out to be

$$v(t) = \sum_{n=-\infty}^{+\infty} \left(-\frac{i}{2}\right) v_n e^{in\tau} = \sum_{n\geq 1} v_n \sin(n\tau), \quad \tau = \omega t, \quad \Pi_n[v] = v_n \in \mathbb{R}, \ n \in \mathbb{N}. \tag{2.2}$$

With a slight abuse of notation, sometimes we may also use $\Pi_n[u]$ to denote the n-th mode $u_n \sin n\tau$. Later, in Section 9, we extend the analysis to the general setting.

As explained in Section 1.3, we refer to the analysis in the setting of $k = 1$ and $\omega \in I_1(\varepsilon_0)$ as the *first bifurcation*. Indeed, for $\omega \in I_1(\varepsilon_0)$, the linearization around $v = 0$ possesses (in the odd-in-t functions space) a pair of weak hyperbolic eigenvalues and all the other eigenvalues are elliptic. In particular the stable and unstable manifolds, $W^s(0)$ and $W^u(0)$, are one dimensional.

We shall give an asymptotic formula for the splitting between $W^u(0)$ and $W^s(0)$ in the cross section

$$\Sigma = \{(v, \partial_y v) \in \mathbf{X}_o : \Pi_1 [\partial_y v] = 0\}, \tag{2.3}$$

(see Figure 4).

Theorem 2.1. *Consider the equation* (1.15) *(equivalently* (1.18) *or* (1.20)*) for $k = 1$ and $\omega \in I_1(\varepsilon_0)$ defined as in* (1.8)*. Then, there exist a constant $C_{\text{in}} \in \mathbb{C}$ such that for any fixed $y_0 > 0$ there exists $\varepsilon_0, M > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_0$, the following statements hold.*

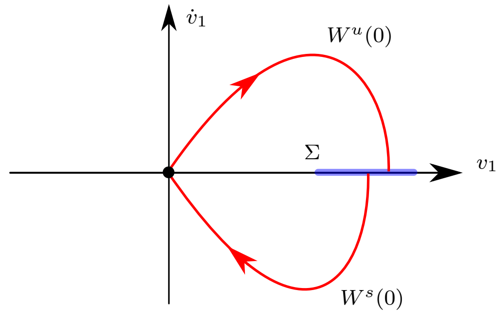

FIGURE 4. The (infinite dimensional) transverse section Σ (see (2.3)) where we measure the distance between the perturbed manifolds $W^u(0)$ and $W^s(0)$.

(1) *The invariant manifolds $W^u(0)$ and $W^s(0)$ of (1.15) in $\mathbf{X}_o$ correspond to solutions $v^u(y,\tau)$ and $v^s(y,\tau)$ of (1.18) satisfying (1.19), which are real-analytic in y, τ, 2π-periodic in τ, and satisfy $\Pi_1[\partial_y v^{u,s}](0) = 0$, respectively. Moreover, $\Pi_{2l}[v^{u,s}] \equiv 0$, for every $l \in \mathbb{N}$ and*

$$\left\|\partial_\tau^2\big(v^u(y,\tau) - v^h(y)\sin\tau\big)\right\|_{\ell_1} + \left\|\partial_\tau^2\partial_y\big(v^u(y,\tau) - v^h(y)\sin\tau\big)\right\|_{\ell_1} \le M\varepsilon^2 v^h(y) \quad \text{for } y \le y_0,$$
$$\left\|\partial_\tau^2\big(v^s(y,\tau) - v^h(y)\sin\tau\big)\right\|_{\ell_1} + \left\|\partial_\tau^2\partial_y\big(v^s(y,\tau) - v^h(y)\sin\tau\big)\right\|_{\ell_1} \le M\varepsilon^2 v^h(y) \quad \text{for } y \ge -y_0,$$

where v^h is the homoclinic orbit given in (1.22).

(2) *At $y = 0$, the splitting satisfies*

$$\left\|\Big(|-\partial_\tau^2 - \omega^{-2}|^{\frac12}(v^u - v^s) + i\varepsilon\partial_y(v^u - v^s)\Big)(0,\tau) - \frac{4\sqrt2}{\varepsilon}e^{-\frac{\pi\sqrt2}{\varepsilon}}C_{\text{in}}\sin(3\tau)\right\|_{\ell_1} \le \frac{Me^{-\frac{\pi\sqrt2}{\varepsilon}}}{\varepsilon\log(\varepsilon^{-1})}, \tag{2.4}$$

(3) *If $C_{\text{in}} \ne 0$, the invariant manifolds $W^u(0)$ and $W^s(0)$ do not intersect the first time they reach Σ (see Figure 4).*

(4) *Fix $\rho > 0$. There exists an open and dense set (in C^0 topology) of analytic functions $f : B_\rho(0) \to \mathbb{C}$ such that for these f the constant C_{in} satisfies $C_{\text{in}} \ne 0$.*

(5) *Let $W \subset \Sigma$ be the intersection near $(v^h(0)\sin\tau, 0)$ of the center-stable manifold $W^{cs}(0)$ and center-unstable manifold $W^{cu}(0)$ when they intersect the hyperplane Σ for the first time.*

  (a) *Let*

$$\mathcal{N} = \{(v,\partial_y v) \mid \varepsilon^{-1}\big\||-\partial_\tau^2 - \omega^{-2}|^{\frac12}\big(v - v^\star(0)\big)\big\|_{L^2} + \|\partial_y v - \partial_y v^\star(0)\|_{L^2}$$
$$\le M\big(\varepsilon^{-1}\big\||-\partial_\tau^2 - \omega^{-2}|^{\frac12}\big(v^u(0) - v^s(0)\big)\big\|_{L^2} + \|\partial_y v^u(0) - \partial_y v^s(0)\|_{L^2}\big),\ \star = u,s\}$$

  then $W \cap \mathcal{N} \ne \emptyset$ and the Hamiltonian $\mathcal{H}$ satisfies

$$\tfrac{1}{M}\inf_W \mathcal{H} \le \big(\varepsilon^{-2}\big\||-\partial_\tau^2 - \omega^{-2}|^{\frac12}\big(v^u(0) - v^s(0)\big)\big\|_{L^2}^2 + \|\partial_y v^u(0) - \partial_y v^s(0)\|_{L^2}^2\big) \le M\inf_W \mathcal{H}.$$

  (b) *Each $(v,\partial_y v) \in W$ corresponds to a single bump homoclinic orbit $\big(v(y,\tau), \partial_y v(y,\tau)\big)$ to $W^c(0)$, i.e $(v,\partial_y v)$ is asymptotic to two orbits $\big(v_c^\pm(y), \partial_y v_c^\pm(y)\big)$ in the center manifold as $y \to \pm\infty$. Moreover, $(v,\partial_y v)$ satisfies*

$$\tfrac{1}{M}\mathcal{H}(v,\partial_y v) \le \varepsilon^{-2}\big\||-\partial_\tau^2 - \omega^{-2}|^{\frac12}\big(v(y) - v^\star(y)\big)\big\|_{L^2}^2 + \|\partial_y v(y) - \partial_y v^\star(y)\|_{L^2}^2 \le M\mathcal{H}(v,\partial_y v), \tag{2.5}$$

  for $y \ge -y_0$ with $\star = s$ and $y \le y_0$ with $\star = u$.

  (c) *If $v^u = v^s$, then it is homoclinic orbit to 0, otherwise the intersection $W^{cs}(0) \cap W^{cu}(0)$, which is codimension 2 in $\mathbf{X}_o$, is transversal in $\mathcal{N}$.*

Remark 2.2. *In the case $v^u \ne v^s$, the transversality of the intersection of the codimension 1 $W^{cs}(0)$ and $W^{cu}(0)$ implies that a dense subset of W consists of smooth functions.*

The metric $\varepsilon^{-2}\big\||-\partial_\tau^2 - \omega^{-2}|^{\frac12}v\big\|_{L^2}^2 + \|\partial_y v\|_{L^2}^2$ corresponds to the quadratic part of the Hamiltonian $\mathcal{H}$ defined in (1.16). We highlight that Theorem 2.1 is concerned with the distance between the invariant manifolds at the first crossing with the transversal section Σ. This does not exclude intersections at further crossings and thus existence of multibump breathers. See figures 2 and 4.

Theorem 2.1 implies several of the statements given in Theorem 1.3 for $k = 1$ and $\omega \in I_1(\varepsilon_0)$.

Indeed, Theorem 2.1 proves statements 2(a),(b),(c) in Theorem 1.3 for $k = 1$ and $\omega \in I_1(\varepsilon_0)$ which deal with the one-dimensional stable and unstable manifolds in the odd setting.

Statement (5) in Theorem 2.1 implies the statements 2(c) and 3(a) in Theorem 1.3 for $k = 1$ and $\omega \in I_1(\varepsilon_0)$ which deal with generalized breathers with exponentially small tails. Indeed, Theorem 2.1(5) implies the existence of a family of orbits homoclinic to the center manifold with exponentially small energy. They correspond to breather like solutions $u(x,t)$ of (1.2) which are $\frac{2\pi}{\omega}$-periodic in t and decaying in x like $\mathcal{O}(\varepsilon e^{-\varepsilon|x|})$ subject to perturbations whose $L^\infty_x(H^1_t \times L^2_t)$ norm is bounded by $\mathcal{O}(\frac{1}{\varepsilon^2} e^{-\frac{\sqrt{2}\pi}{\varepsilon}})$.

2.1. **Heuristics of the proof of Theorem 2.1; exponentially small bounds.** Looking at formula (2.4) one can see that the distance between the one dimensional invariant manifolds $W^u(0)$ and $W^s(0)$ is exponentially small in ε. In this section we give some intuition why the distance between the one dimensional invariant manifolds $W^u(0)$ and $W^s(0)$ is exponentially small in ε and which are the steps needed to obtain upper bounds on this distance. Later, in Section 2.2, we will show how to obtain the asymptotic formula (2.4) for this splitting. A complete description of the proof of Theorem 2.1 can be found in Section 3.

Since the invariant manifolds are one-dimensional, one can parameterize them as solutions of the second order equation (1.18) for $k = 1$, which satisfy

$$\begin{aligned}\ddot{v}_1 &= v_1 - \frac{v_1^3}{4} + \mathcal{O}_{\ell_1}(\widetilde{\Pi}v) + \mathcal{O}(\varepsilon^2)\\ \ddot{v}_n &= -\frac{\lambda_n^2}{\varepsilon^2} v_n + \mathcal{O}_{\ell_1}(v^3),\ n \geq 2.\end{aligned}$$

where we have introduced the following notation, which is also used in the forthcoming Sections 3–7,

$$\widetilde{\Pi}(v) = v - \Pi_1(v)\sin\tau = \sum_{n\geq 2} v_n \sin(n\tau). \tag{2.6}$$

Imposing decay at infinity (as $y \to +\infty$ for $W^s(0)$, as $y \to -\infty$ for $W^u(0)$) and $\partial_y v_1^{u,s}(0) = 0$, Item 1 of Theorem 2.1 looks natural: the distance between the perturbed and unperturbed manifolds $(v_1, \widetilde{\Pi}v) = (v^h, 0)$ is of the same order as the perturbation (notice the singular character of the model and the different size of each component of the vector field). These estimates can be proven through a fixed point argument by using the standard Perron Method.

Even if the perturbed invariant manifolds are $\mathcal{O}(\varepsilon^2)$ close to the unperturbed ones, the singular character of the model makes their difference beyond all orders in ε, in fact exponentially small. Let us give some heuristic ideas of why this phenomenon happens. We have chosen parameterizations such that $\partial_y v_1^{u,s}(0) = 0$. Moreover, as the system is Hamiltonian, both manifolds belong to the energy level of the saddle-center critical point $v = 0$. Therefore, the difference $v_1^u - v_1^s$ at $y = 0$ can be recovered from the differences projected to the rest of directions, namely $\widetilde{\Pi}(v^u - v^s)$ and $\widetilde{\Pi}(\partial_y v^u - \partial_y v^s)$. Thus, we focus on measuring these differences. Let us write the equations for these components as a first order equation for $n \geq 3$ (recall that $\Pi_{2l} v = 0$ for $l \geq 0$),

$$\begin{aligned}\dot{v}_n &= w_n\\ \dot{w}_n &= -\frac{\lambda_n^2}{\varepsilon^2} v_n + \frac{1}{\varepsilon^3\omega^3}\Pi_n\left[g(\varepsilon\omega v)\right].\end{aligned}$$

As the parameterizations of both invariant manifolds satisfy the same equation, their difference

$$(\Delta_n, \Xi_n) \triangleq (v_n^u - v_n^s, \partial_y v_n^u - \partial_y v_n^s)$$

satisfies a linear equation for $n \geq 3$,

$$\begin{aligned}\dot{\Delta}_n &= \Xi_n\\ \dot{\Xi}_n &= -\frac{\lambda_n^2}{\varepsilon^2}\Delta_n + \Pi_n\left[M(v^u, v^s)\Delta\right].\end{aligned}$$

Since the last term is much smaller than the oscillating one, to give a heuristic idea of the phenomenon taking place, let us assume that $M = 0$. Then, the system becomes a linear system of constant coefficients which we can diagonalize by taking

$$\begin{aligned} \Gamma_n &= \lambda_n \Delta_n + i\varepsilon \Xi_n \\ \Theta_n &= \lambda_n \Delta_n - i\varepsilon \Xi_n \end{aligned} \tag{2.7}$$

to obtain

$$\begin{aligned} \dot{\Gamma}_n &= -i\frac{\lambda_n}{\varepsilon}\Gamma_n \\ \dot{\Theta}_n &= i\frac{\lambda_n}{\varepsilon}\Theta_n. \end{aligned}$$

The solutions of this system can be easily computed as

$$\begin{aligned} \Gamma_n(y) &= e^{-i\frac{\lambda_n}{\varepsilon}(y-y^+)}\Gamma_n(y^+) \\ \Theta_n(y) &= e^{i\frac{\lambda_n}{\varepsilon}(y-y^-)}\Theta_n(y^-) \end{aligned}$$

for any points $y^\pm$.

By the definition of (Γ_n, Θ_n) in (2.7), one has

$$\begin{aligned} \Gamma_n(y^+) &= \lambda_n(v_n^u(y^+) - v_n^s(y^+)) + i\varepsilon(\partial_y v_n^u(y^+) - \partial_y v_n^s(y^+)) \\ \Theta_n(y^-) &= \lambda_n(v_n^u(y^-) - v_n^s(y^-)) - i\varepsilon(\partial_y v_n^u(y^-) - \partial_y v_n^s(y^-)). \end{aligned}$$

The main observation here is that if we are able to extend the stable and unstable manifolds $v_n^{u,s}(y)$ to some complex values $y^\pm = \pm i\sigma$, $\sigma > 0$, one obtains the following estimates for $y \in \mathbb{R}$ near $y = 0$,

$$\begin{aligned} |\Gamma_n(y)| &\le e^{-\frac{\lambda_n \sigma}{\varepsilon}} |\Gamma_n(i\sigma)| \\ |\Theta_n(y)| &\le e^{-\frac{\lambda_n \sigma}{\varepsilon}} |\Theta_n(-i\sigma)|, \end{aligned}$$

which are exponentially small in ε and strongly depend on the size of the unstable/stable solutions at the complex points $y^\pm = \pm i\sigma$.

For the nonlinear system, we will find the solutions

$$v_n^s(y) \text{ for } \Re y \ge 0, \qquad v_n^u(y) \text{ for } \Re y \le 0$$

as perturbations of the singular solution $v_1 = v^h(y)$, $v_n = 0$, $n \ge 2$, where $v^h(y)$ is the unperturbed homoclinic solution (1.22). As this function has poles of order one at the points $y^\pm = \pm i\pi/2$, it is natural to expect that the optimal value for σ is $\sigma = \pi/2 - \kappa\varepsilon$ for a suitable $\kappa > 0$.

In fact, in Section 4 below, we will see that

$$|v^{u,s}(y^\pm)| \lesssim \frac{1}{\varepsilon}, \text{ at } y_\pm = \pm i\sigma,$$

and consequently $|\Gamma_n(y^+)| \lesssim \frac{1}{\varepsilon}$, $|\Theta_n(y^-)| \lesssim \frac{1}{\varepsilon}$. Therefore, one expects that for $y \in \mathbb{R}$ close to $y = 0$,

$$|\Gamma_n(y)| \lesssim \frac{1}{\varepsilon} e^{-\frac{\lambda_n \pi}{2\varepsilon}}, \qquad |\Theta_n(y)| \lesssim \frac{1}{\varepsilon} e^{-\frac{\lambda_n \pi}{2\varepsilon}}.$$

As $\lambda_n \ge \lambda_3 = 2\sqrt{2} + \mathcal{O}(\varepsilon)$ for $n \ge 3$, one obtains an upper bound for the difference

$$|\Gamma_n(y)| \lesssim \frac{1}{\varepsilon} e^{-\frac{\sqrt{2}\pi}{\varepsilon}}, \qquad |\Theta_n(y)| \lesssim \frac{1}{\varepsilon} e^{-\frac{\sqrt{2}\pi}{\varepsilon}}$$

and similar bounds are satisfied by $\Delta_n(y) = v_n^u(y) - v_n^s(y)$.

Observe that these bounds fit with the estimates (2.4) given in Theorem 2.1. However, Theorem 2.1 gives certainly more information since it provides an asymptotic formula for Δ.

To obtain an asymptotic formula of the difference between $v^u - v^s$ instead of an upper bound, one needs a finer analysis of these functions at (neighborhoods of) the points $y^\pm = \pm i(\pi/2 - \kappa\varepsilon)$. At such points the homoclinic orbit (1.22) is not a good approximation of the perturbed invariant manifolds. We then need to obtain a new first order approximation of these manifolds. This will be the strategy to proof Theorem 2.1. Its heuristics are given in the next section.

2.2. **Strategy of the proof of Theorem 2.1; exponentially small asymptotics.** As we haven seen in Subsection 2.1, to obtain an asymptotic formula of $v^s - v^u$, one needs a deeper study of these functions near $y^{\pm} = \pm i(\pi/2 - \kappa\varepsilon)$ for some $\kappa > 0$.

In Theorem 3.1 below we show that

$$v^{s,u}(y,\tau) = v^h(y)\sin\tau + \mathcal{O}_{\ell_1}\left(\frac{\varepsilon^2}{|y^2+\frac{\pi^2}{4}|^3}\right) = \frac{2\sqrt{2}}{\cosh y}\sin\tau + \mathcal{O}_{\ell_1}\left(\frac{\varepsilon^2}{|y^2+\frac{\pi^2}{4}|^3}\right). \tag{2.8}$$

Therefore, for real values of y, the invariant manifolds are ε^2-perturbations of the unperturbed homoclinic orbit, but, when $y \mp \frac{i\pi}{2} = \mathcal{O}(\varepsilon)$ we have that both the homoclinic and error term become of the same size $\mathcal{O}(\frac{1}{\varepsilon})$ and therefore $v^{s,u}(y,\tau)$ are not well approximated by the homoclinic solution $v^h(y)\sin\tau$ anymore. Thus, we look for suitable leading orders of $v^{s,u}(y,\tau)$ for y such that $y \mp \frac{i\pi}{2} = \mathcal{O}(\varepsilon)$.

We focus on the singularity $y = i\pi/2$ (the same analysis can be performed near the singularity $y = -i\pi/2$ analogously). We proceed as follows. We perform the singular change to the *inner variable*

$$z = \varepsilon^{-1}\left(y - i\frac{\pi}{2}\right)$$

and the scaling

$$\phi(z,\tau) = \varepsilon v\left(i\frac{\pi}{2} + \varepsilon z, \tau\right).$$

From (1.15), one can deduce the equation satisfied by $\phi(z,\tau)$,

$$\partial_z^2\phi - \partial_\tau^2\phi - \frac{1}{\omega^2}\phi + \frac{1}{3}\phi^3 + \frac{1}{\omega^3}f(\omega\phi) = 0, \qquad \omega = \frac{1}{\sqrt{1+\varepsilon^2}}.$$

(1.10). However, notice that now the spatial variable is $z = x - i\dfrac{\pi}{2\varepsilon}$. The first order of this equation corresponds to the regular limit $\varepsilon = 0$, which gives the so-called *inner equation*

$$\partial_z^2\phi^0 - \partial_\tau^2\phi^0 - \phi^0 + \frac{1}{3}(\phi^0)^3 + f(\phi^0) = 0.$$

The estimates (2.8) show, that, after these changes, the stable/unstable manifolds behave as

$$\phi^{s,u}(z,\tau) = -\frac{2\sqrt{2}i}{z}\sin\tau + \mathcal{O}_{\ell_1}\left(\frac{1}{z^3},\varepsilon\right).$$

Therefore, it is natural to look for solutions of the inner equation which "match" these asymptotics. This is done in Theorem 3.3 below where we obtain and analyze two solutions $\phi^{0,u}$, $\phi^{0,s}$ of the inner equation which are the first order of the unstable/stable manifolds "close to the singularity" $y = i\pi/2$. They are of the form

$$\begin{aligned}\phi^{0,s}(z,\tau) &= -\frac{2\sqrt{2}i}{z}\sin\tau + \psi^s(z,\tau), \text{ for } \Re z > 0\\ \phi^{0,u}(z,\tau) &= -\frac{2\sqrt{2}i}{z}\sin\tau + \psi^u(z,\tau), \text{ for } \Re z < 0\end{aligned} \tag{2.9}$$

with $\psi^{s,u} = \mathcal{O}(\frac{1}{z^3})$ in suitable complex domains satisfying $|z| > \kappa$ and containing the negative imaginary axis $\Im z < -\kappa$ (recall that $z = \varepsilon^{-1}(y - \frac{i\pi}{2})$ and therefore $y = 0$ lies on this negative imaginary axis).

Moreover, in Theorem 3.3 we provide a formula for the difference of these two solutions which reads

$$\phi^{0,u}(-ir,\tau) - \phi^{0,s}(-ir,\tau) = e^{-2\sqrt{2}r}\left(C_{\rm in}\sin(3\tau) + \mathcal{O}_{\ell_1}\left(\frac{1}{r}\right)\right) \qquad \text{as} \qquad r \to +\infty. \tag{2.10}$$

Let us give here a (very) heuristic idea of the origin of this result.

Writing a solution of the inner equation as $\phi^0 = \sum \phi_n^0 \sin(n\tau)$ we obtain

$$\begin{aligned}\frac{d^2}{dz^2}\phi_1^0 - \frac{1}{4}(\phi_1^0)^3 &= F_1\left(\phi^0\right)\\ \frac{d^2}{dz^2}\phi_n^0 + \mu_n^2\phi_n^0 &= F_n\left(\phi^0\right),\ n \geq 3.\end{aligned} \tag{2.11}$$

where $' = d/dz$, $\mu_n = \sqrt{n^2-1}$, and F_n contain higher order terms.

Let us assume that $F_n = 0$ to give an heuristic idea of the process. First, let us make the change $z = -ir$ and write system (2.11) as a first order system through the change

$$\Psi_c(r) = \left(\phi_1^0(-ir), -i\frac{d}{dz}\phi_1^0(-ir)\right), \quad \Psi_{n,\pm} = -i\frac{d}{dz}\phi_n^0(-ir) \pm \sqrt{n^2-1}\phi_n^0(-ir), \quad \Psi^{\pm} = (\Psi_{2l-1,\pm})_{l=2}^{\infty},$$

which gives

$$\begin{aligned} \frac{d}{dr}\Psi_c^1 &= \Psi_c^2 & \qquad \frac{d}{dr}\Psi_{n,-} &= -\sqrt{n^2-1}\Psi_{n,-} \\ \frac{d}{dr}\Psi_c^2 &= \frac{1}{4}\left(\Psi_c^1\right)^3 & \qquad \frac{d}{dr}\Psi_{n,+} &= \sqrt{n^2-1}\Psi_{n,+}. \end{aligned}$$

Observe that $\Psi = 0$ is a critical point with a center manifold W^c given by $\Psi^+ = \Psi^- = 0$, and a center-sable manifold W^{cs} given by $\Psi^+ = 0$. Moreover, W^{cs} possesses the classical stable foliation. Indeed, given a point $\Psi = (\Psi^c, \Psi^-, 0) \in W^{cs}$ there exists a point $\Psi_b = (\Psi^c, 0, 0) \in W^c$ such that $|\Phi_r(\Psi) - \Phi_r(\Psi_b)| \leq \mathcal{O}(e^{-2r})$, as $2 \in (0, \sqrt{3^2-1})$, where Φ_r is the flow on W^{cs} which is well defined for $r \geq 0$. The points whose trajectories are asympotic to a given $\Psi_b \in W^c$ form a leaf of the foliation.

These foliation allows us to give an asymptotic formula for $\phi^{0,s}(z) - \phi^{0,u}(z)$:

- The first observation is that our solutions $\phi^{0,s}(z)$, $\phi^{0,u}(z)$, when restricted to the negative imaginary axis away from 0 and written in these coordinates, correspond to $\Psi^{u,s}(r) = (\Psi_c^{u,s}, \Psi_-^{u,s}, \Psi_+^{u,s})(r)$ satisfying
$$\lim_{r\to+\infty} \Psi^{s,u}(r) = 0.$$
Therefore, they belong to W^{cs} and, in this simplified model, should have the "unstable coordinate" $\Psi_+^{u,s}(r) \equiv 0$.
- The second observation is that we know, by (2.9), that
$$|\Psi^u(r) - \Psi^s(r)| \leq \mathcal{O}_{\ell_1}\left(\frac{1}{r^3}\right), \text{ as } r \to +\infty,$$
which implies that they should have the same "central coordinate" $\Psi_c^u(r) = \Psi_c^s(r)$ and therefore they belong to the same leaf in the stable foliation. One can see this fact using the linearized fundamental solutions in the central coordinates which give: $\Psi_c^u(r) - \Psi_c^s(r) \sim c_1 r^{-2} + c_2 r^3$ and the decay of this difference immediately gives $c_1 = c_2 = 0$.
- Now that we know that $\Psi^{u,s}(r) = (\Psi_c(r), \Psi_-^{u,s}(r), 0)$ we only need to compute the difference in the stable coordinate $\beta_-(r) = \Psi_-^u(r) - \Psi_-^s(r)$ which satisfies:
$$\frac{d}{dr}\beta_- = A\beta_-, \quad A = \operatorname{diag}(-\sqrt{n^2-1})$$
and this immediately implies that
$$\beta_-(r) = e^{(r-r_0)A}\beta_-(r_0) = e^{-2\sqrt{2}(r-r_0)}\beta_{3,-}(r_0)\sin(3\tau) + \mathcal{O}_{\ell_1}\left(e^{-3r}\right).$$
Calling $C = e^{2\sqrt{2}r_0}\beta_{3,-}(r_0)$ we have
$$\lim_{r\to+\infty} e^{2\sqrt{2}r}\beta_-(r) - C\sin 3\tau = 0.$$

Using these ideas, in Theorem 3.3 below, we incorporate the dismissed higher order terms (see (2.11)) and give a complete proof of the asymptotic formula for the difference between the solutions of the inner equation. Note that the constant C above corresponds to the constant C_{in} in (2.10).

Once we know how to compute the difference between the inner solutions $\phi^{0,u} - \phi^{0,s}$, we must show that this difference gives indeed a first order of the difference between the perturbed invariant manifolds. That is, we must estimate the function $(\phi^u - \phi^s) - (\phi^{0,u} - \phi^{0,s})$ in some appropriate complex domain. To this end, we first have to show that the solutions of the inner equation $\phi^{0,u}(z)$, $\phi^{0,s}(z)$, when written in the original variables $y = i\frac{\pi}{2} + \varepsilon z$, are good approximations of the stable and unstable solutions v^u, v^s for y satisfying $y \mp \frac{i\pi}{2} = \mathcal{O}(\varepsilon)$. Such analysis is done in Theorem 3.6.

From such estimates, applying the ideas in Section 2.1, we obtain smaller exponentially small errors at $y = 0$. This shows that the difference of $\phi^{0,u} - \phi^{0,s}$ provides the main term of the exponentially small distance between v^u and v^s.

The first order of this distance is given by the Stokes constant C_{in}, which is analyzed in the next section.

2.3. **The Stokes constant.** Generically the constant C_{in} appearing in Theorem 2.1 (see also (2.10)) does not vanish (and therefore the difference between the invariant manifolds does not vanish either). To see this, consider a toy model in the form of (2.11),

$$\gamma_3'' + \mu_3^2\gamma_3 = g(z) = \sum_{l=3}^{\infty} \frac{a_l}{z^l},$$

where the power series on the above righthand side is convergent on a disk. The same proof as in Section 5 yields two solutions

$$\gamma^*(z) = \mathcal{O}(z^{-3}), \quad z \in D^{*,in}_{\theta,\kappa}, \quad * = u, s,$$

where $D^{*,in}_{\theta,\kappa}$ are sectorial complex domains with vertex at $z = \infty$ which are defined in (3.10) below. However, in general, $\gamma^{u,s}(z)$ cannot be extended to be analytic functions defined in a neighborhood of ∞. In fact, one may obtain the same formal asymptotic expansions for both $\gamma^{u,s}(z)$ as $|z| \to \infty$

$$\gamma^{u,s}(z) = \sum_{l=3}^{\infty} \frac{\gamma_l}{z^l}, \quad \gamma_l = \sum_{j=0}^{[\frac{l-3}{2}]} (-1)^j \mu_3^{-2(j+1)} \frac{(l-1)!}{(l-2j-1)!} a_{l-2j},$$

where $[a]$ denote the smallest integer no greater than a. It is easy to see that this is generally a divergent series and thus $\gamma^u \neq \gamma^s$ in general. We also observe that this series is in the Gevrey-1 class.

There is another way to illustrate $\gamma^s \neq \gamma^u$ in general. In fact their decay at infinity implies

$$\gamma^u(z) = \frac{1}{2i\mu_3}\int_{-\infty}^{z} e^{-i\mu_3(s-z)} g(s)ds - \frac{1}{2i\mu_3}\int_{-\infty}^{z} e^{i\mu_3(s-z)} g(s)ds,$$
$$\gamma^s(z) = \frac{1}{2i\mu_3}\int_{+\infty}^{z} e^{-i\mu_3(s-z)} g(s)ds - \frac{1}{2i\mu_3}\int_{+\infty}^{z} e^{i\mu_3(s-z)} g(s)ds.$$

For $\kappa > 0$, let $B_\kappa \subset \mathbb{C}$ be the disk centered at 0 with radius κ and S be the path going from $-\infty$ to $-\kappa$ along the negative real axis, then to κ along the lower half of ∂B_κ, then to $+\infty$ along the real axis. By the Cauchy integral theorem we obtain

$$\begin{aligned}\gamma^u(-i\kappa) - \gamma^s(-i\kappa) =& \frac{1}{2i\mu_3}\int_S e^{-i\mu_3(s+i\kappa)} g(s)ds - \frac{1}{2i\mu_3}\int_S e^{i\mu_3(s+i\kappa)} g(s)ds \\ =& -\frac{e^{-\mu_3\kappa}}{2i\mu_3}\oint_{\partial B_\kappa} e^{i\mu_3 s} g(s)ds = -\pi e^{-\mu_3\kappa}\sum_{l=3}^{\infty}\frac{i^{l-1}\mu_3^{l-2}}{(l-1)!}a_l.\end{aligned}$$

The above right side are related to the Borel transformation of g evaluated at $i\mu_3$ and gives the difference between γ^u and γ^s.

This means that in the derivation of the stable/unstable solutions $\phi^{0,*}$, $* = u, s$, through the Lyapunov-Perron approach, a nonzero splitting appears even after the first iteration. In fact, it can be proved by the Borel-Laplace summation theory, that $\phi^{0,u}$ and $\phi^{0,s}$, while analytic on their own domains and non-equal in the intersection of the domains, share the same formal series as $z \to \infty$ in suitable sectors

$$\phi^{0,*} \sim \sum_{j=3}^{\infty} \frac{b_j}{z^j},$$

which is generally divergent, but belongs to the Gevrey-1 class, namely

$$\sup_j \left(\frac{|b_j|}{j!}\right)^{\frac{1}{j}} < \infty.$$

Moreover, the right sides F_n of (2.11) are also associated to former series

$$F_n\left(\frac{1}{z}, \psi^{u,s}\right) \sim \sum_{j=3}^{\infty} \frac{\beta_{n,j}}{z^j},$$

in the Gevrey-1 class. Due to its length, we skip the details of that argument as it has little impact on the main result of this paper.

On the other hand, the above considerations motivate us to make the following conjecture.

Conjecture. The constant $C_{\rm in}$ in the leading order approximation of the stable/unstable solutions can be expressed as

$$C_{\rm in} = -\pi \sum_{l=3}^{\infty} \frac{i^{l-1}\mu_3^{l-2}}{(l-1)!}\beta_{3,l}.$$

Even though the formula of the splitting constant $C_{\rm in}$ in this conjecture is still very complicated, if proved, it would give an algorithm to compute $C_{\rm in}$ which may be implemented by numerical computations. The proof of this conjecture is beyond this paper.

3. Description of the Proof of Theorem 2.1

We describe the main steps of the proof of Theorem 2.1 where $k = 1$ and the odd symmetry of functions in τ is assumed.

3.1. **Estimates of the invariant manifolds in complex domains.** In order to compute the distance between the perturbed invariant manifolds $W^s(0)$ and $W^u(0)$ in Σ, we consider their suitable parameterizations. Since the invariant manifolds $W^u(0)$ and $W^s(0)$ are one dimensional, they are the images of solutions v^u and v^s of (1.18) with the asymptotic conditions

$$\lim_{y\to+\infty} v^s(y,\tau) = \lim_{y\to-\infty} v^u(y,\tau) = 0, \qquad \text{for all } \tau\in\mathbb{T}.$$

We write equation (1.18) as

$$\begin{cases} \ddot v_1 = v_1 - \dfrac{v_1^3}{4} + \left(-\dfrac{1}{\varepsilon^3\omega^3}\Pi_1\left[g(\varepsilon\omega v)\right] + \dfrac{v_1^3}{4}\right), \\ \ddot v_n = -\dfrac{\lambda_n^2}{\varepsilon^2} v_n - \dfrac{1}{\varepsilon^3\omega^3}\Pi_n\left[g(\varepsilon\omega v)\right], \; n\ge 2, \end{cases}$$

where Π_n is the Fourier projection given by (2.2) and g is given by (1.11).

We study the solutions v^u, v^s as perturbations of the homoclinic orbit $v^h(y)\sin\tau$ given by (1.22), which satisfies $\ddot v^h = v^h - (v^h)^3/4$. Thus, we set

$$\xi(y,\tau) = v(y,\tau) - v^h(y)\sin\tau = \sum_{n\ge1}\xi_n(y)\sin(n\tau),$$

whose Fourier coefficients satisfy

$$\begin{cases} \ddot\xi_1 = \xi_1 - \dfrac{3(v^h)^2\xi_1}{4} - \dfrac{3v^h\xi_1^2}{4} - \dfrac{\xi_1^3}{4} + \left(-\dfrac{1}{\varepsilon^3\omega^3}\Pi_1(g(\varepsilon\omega(\xi+v^h\sin\tau))) + \dfrac{(\xi_1+v^h)^3}{4}\right), \\ \ddot\xi_n = -\dfrac{\lambda_n^2}{\varepsilon^2}\xi_n - \dfrac{1}{\varepsilon^3\omega^3}\Pi_n(g(\varepsilon\omega(\xi+v^h\sin\tau))), \qquad n\ge 2. \end{cases}$$

Define the operators

$$\mathcal{L}(\xi) = \left(\ddot\xi_1 - \xi_1 + \frac{3(v^h)^2\xi_1}{4}\right)\sin\tau + \sum_{n\ge2}\left(\ddot\xi_n + \frac{\lambda_n^2}{\varepsilon^2}\xi_n\right)\sin(n\tau), \tag{3.1}$$

$$\mathcal{F}(\xi) = -\frac{1}{\varepsilon^3\omega^3} g(\varepsilon\omega(\xi+v^h\sin\tau)) + \left(\frac{(\xi_1+v^h)^3}{4} - \frac{3v^h\xi_1^2}{4} - \frac{\xi_1^3}{4}\right)\sin\tau. \tag{3.2}$$

To obtain solutions $v^\star$, $\star = u,s$, of (1.15) satisfying (1.19) is equivalent to find solutions $\xi^\star$ of the functional equation

$$\mathcal{L}(\xi) = \mathcal{F}(\xi), \tag{3.3}$$

satisfying

$$\lim_{y\to-\infty}\xi^u(y,\tau) = \lim_{y\to\infty}\xi^s(y,\tau) = 0, \qquad \text{for all} \quad \tau\in\mathbb{T}. \tag{3.4}$$

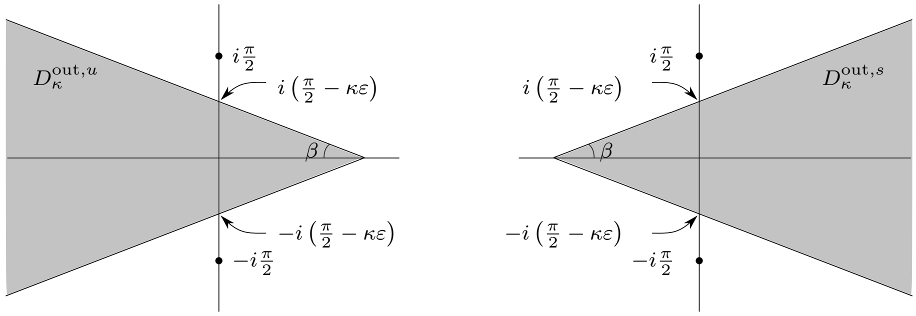

FIGURE 5. Outer domains $D_\kappa^{\mathrm{out},u}$ and $D_\kappa^{\mathrm{out},s}$.

We analyze these parameterizations in the following complex sectorial domains, usually called *outer domains*,

$$
\begin{aligned}
D_\kappa^{\mathrm{out},u} &= \left\{ y\in\mathbb{C}; |\mathrm{Im}(y)| \le -\tan\beta\,\mathrm{Re}(y) + \frac{\pi}{2} - \kappa\varepsilon \right\} \\
D_\kappa^{\mathrm{out},s} &= \left\{ y\in\mathbb{C}; -y\in D_\kappa^{\mathrm{out},u} \right\},
\end{aligned}
\tag{3.5}
$$

where $0<\beta<\pi/4$ is a fixed angle independent of ε and $\kappa\ge 1$ (see Figure 5). Observe that $D_\kappa^{\mathrm{out},*}$, $*=u,s$, reach domains at a $\kappa\varepsilon$–distance of the singularities $y=\pm i\pi/2$ of v^h (see Section 2.1).

Next theorem proves the existence and estimates of the functions ξ^u,ξ^s. It is proven in Section 4.

Theorem 3.1 (Outer). *Consider the equation* (1.15) *with* $k=1$. *There exist* $\kappa_0\ge 1$ *big enough and* $\varepsilon_0>0$ *small enough, such that, for each* $0<\varepsilon\le\varepsilon_0$ *and* $\kappa\ge\kappa_0$, *the invariant manifolds* $W^\star(0)\subset\mathbf{X}$ *of* (1.15), $\star=u,s$, *are parameterized as solutions of equation* (1.15) *by*

$$
v^\star(y,\tau) = v^h(y)\sin\tau + \xi^\star(y,\tau),\ y\in D_\kappa^{\mathrm{out},\star},\ \tau\in\mathbb{T},
$$

where v^h *is given by* (1.22) *and* $\xi^\star : D_\kappa^{\mathrm{out},\star}\times\mathbb{T}\to\mathbb{C}$ *are functions real-analytic in the variable* y *such that*

1. *They satisfy the asymptotic condition* (3.4), $\partial_y\Pi_1[\xi^\star](0)=0$ *and* $\Pi_{2l}[\xi^\star](y)\equiv 0$ *for* $l\in\mathbb{N}$.
2. *There exists a constant* $M_1>0$ *independent of* ε *and* κ, *such that*

$$
\begin{aligned}
\|\xi^\star\|_{\ell_1}(y) &\le \frac{M_1\varepsilon^2}{|\cosh(y)|} \qquad \text{for} \quad y\in D_\kappa^{\mathrm{out},\star}\cap\{|\mathrm{Re}(y)|>1\} \\
\|\xi^\star\|_{\ell_1}(y) &\le \frac{M_1\varepsilon^2}{|y^2+\pi^2/4|^3} \qquad \text{for} \quad y\in D_\kappa^{\mathrm{out},\star}\cap\{|\mathrm{Re}(y)|\le 1\}
\end{aligned}
$$

Moreover, the derivatives of $\xi^\star$ *can be bounded as*

1. *For* $y\in D_\kappa^{\mathrm{out},\star}\cap\{|\mathrm{Re}(y)|>1\}$,

$$
\left\|\partial_\tau^2\xi^\star\right\|_{\ell_1}(y),\ \|\partial_\tau^2\partial_y\xi^\star\|(y) \le \frac{M_1\varepsilon^2}{|\cosh(y)|}.
$$

2. *For* $y\in D_\kappa^{\mathrm{out},\star}\cap\{|\mathrm{Re}(y)|\le 1\}$,

$$
\left\|\partial_\tau^2\xi^\star\right\|_{\ell_1}(y) \le \frac{M_1\varepsilon^2}{|y^2+\pi^2/4|^3} \quad \text{and} \quad \left\|\partial_\tau^2\partial_y\xi^\star\right\|_{\ell_1}(y) \le \frac{M_1\varepsilon^2}{|y^2+\pi^2/4|^4}.
$$

Remark 3.2. *While the 1-dim stable and unstable manifolds of the equilibrium* 0 *is determined by their exponential asymptotic behavior as* $y\to\pm\infty$ *where the freedom of translation in* y *is fixed by* $\partial_y\Pi_1[\xi^{u,s}]=0$, *it is extremely important that the precise order of the error* $\xi^{u,s}=\mathcal{O}(\frac{\varepsilon^2}{|y^2+\frac{\pi^2}{4}|^3})$ *is obtained near the singularity* $y=\pm\frac{\pi}{2}i$. *This does not only allows one to identify the correct scaling leading to the limit of the so-called inner equation in the next subsection, but also uniquely fix the solutions of the inner equation optimally approximating* $v^{u,s}$.

3.2. **Analysis close to the singularities.** Notice that the parameterizations $v^\star(y,\tau)$ of $W^\star(0)$, $\star = u, s$ given by Theorem 3.1, are ε^2-close to the homoclinic orbit $v^h(y)\sin(\tau)$ for $y \in \mathbb{R} \cap D_\kappa^{\mathrm{out},\star}$. Nevertheless, at distance $\mathcal{O}(\varepsilon)$ of the poles $y = \pm i\pi/2$ of v^h, we have that $v^h \sim \varepsilon^{-1}$ has comparable size to the error $\xi^\star \sim \varepsilon^{-1}$.

To obtain a first order approximation of the invariant manifolds at distance $\mathcal{O}(\varepsilon)$ of the poles $y = \pm i\pi/2$ we proceed as follows. We focus on the singularity $y = i\pi/2$ since similar results can be proved near the singularity $y = -i\pi/2$ analogously. Consider the *inner variable*

$$z = \varepsilon^{-1}\left(y - i\frac{\pi}{2}\right) \tag{3.6}$$

and the scaling

$$\phi(z,\tau) = \varepsilon v\left(i\frac{\pi}{2} + \varepsilon z, \tau\right). \tag{3.7}$$

Writing equation (1.15) for $\phi(z,\tau)$ and recalling $\omega = (1+\varepsilon^2)^{-\frac{1}{2}}$, we obtain

$$\partial_z^2\phi - \partial_\tau^2\phi - \frac{1}{\omega^2}\phi + \frac{1}{3}\phi^3 + \frac{1}{\omega^3}f(\omega\phi) = 0. \tag{3.8}$$

This equation coincides with the original Klein-Gordon equation (1.10)[3]. However, notice that now the evolution variable is $z = x - i\dfrac{\pi}{2\varepsilon}$.

The first order of (3.8) corresponds to the regular limit $\varepsilon = 0$, which gives the so-called *inner equation*

$$\partial_z^2\phi^0 - \partial_\tau^2\phi^0 - \phi^0 + \frac{1}{3}(\phi^0)^3 + f(\phi^0) = 0. \tag{3.9}$$

We are interested in identifying certain solutions of (3.9) with the first order of the outer solutions $v^{u,s}(y,\tau)$ given in Theorem 3.1 near the pole $y = i\pi/2$. Therefore, we look for solutions $\phi^{0,*}(z,\tau)$, $* = u, s$, of (3.9) which have the same expansion as $\phi^{u,s}(z,\tau) = \varepsilon v^{u,s}\left(i\left(\frac{\pi}{2} + \varepsilon z\right), \tau\right)$. Near the pole $y = i\pi/2$, by Theorem 3.1 we have

$$v^{u,s}(y,\tau) = v^h(y)\sin\tau + \mathcal{O}\left(\frac{\varepsilon^2}{(y - i\pi/2)^3}\right) = \frac{-2\sqrt{2}i}{y - i\pi/2}\sin\tau + \mathcal{O}(y - i\pi/2) + \mathcal{O}\left(\frac{\varepsilon^2}{(y - i\pi/2)^3}\right)$$

which, in the inner variables (3.6) and (3.7), corresponds to

$$\phi^{u,s}(z,\tau) = \frac{-2\sqrt{2}i}{z}\sin\tau + \mathcal{O}(\varepsilon^2 z) + \mathcal{O}\left(\frac{1}{z^3}\right).$$

Taking into account the change of variables (3.6) and the shape of the outer domains (3.5), this asymptotic condition must hold for $\operatorname{Im} z < 0$ and $\operatorname{Re} z < 0$ for ϕ^u and for $\operatorname{Im} z < 0$ and $\operatorname{Re} z > 0$ for ϕ^s.

More precisely we consider the *inner domains*

$$\begin{aligned} D_{\theta,\kappa}^{u,\mathrm{in}} &= \{z \in \mathbb{C}; |\operatorname{Im}(z)| > \tan\theta \operatorname{Re}(z) + \kappa\}, \\ D_{\theta,\kappa}^{s,\mathrm{in}} &= \{z \in \mathbb{C}; -z \in D_{\theta,\kappa}^{u,\mathrm{in}}\}, \end{aligned} \tag{3.10}$$

for $0 < \theta < \pi/2$ and $\kappa > 0$ (see Figure 6), and we look for solutions of the inner equation of the form

$$\phi^{0,*}(z,\tau) = \frac{-2\sqrt{2}i}{z}\sin\tau + \psi^*(z,\tau), \quad \text{with} \quad \psi^* = \mathcal{O}\left(\frac{1}{z^3}\right), \quad \text{for} \quad (z,\tau) \in D_{\theta,\kappa}^{*,\mathrm{in}} \times \mathbb{T}, * = u, s.$$

Now, we present the results concerning the existence of these solutions $\phi^{0,\star}$ of (3.8), $\star = u, s$. Moreover, we provide an asymptotic expression for the difference $\phi^{0,u}(z,\tau) - \phi^{0,s}(z,\tau)$ as $\operatorname{Im}(z) \to -\infty$, which will be crucial to compute the first order of the difference $v^u - v^s$. The following Theorem will be proved in Section 5.

Theorem 3.3 (Inner). *Let $\theta > 0$ be fixed. There exists $\kappa_0 \geq 1$ big enough such that, for each $\kappa \geq \kappa_0$,*

[3]Warning: It is the original one for $\psi = \omega\phi$

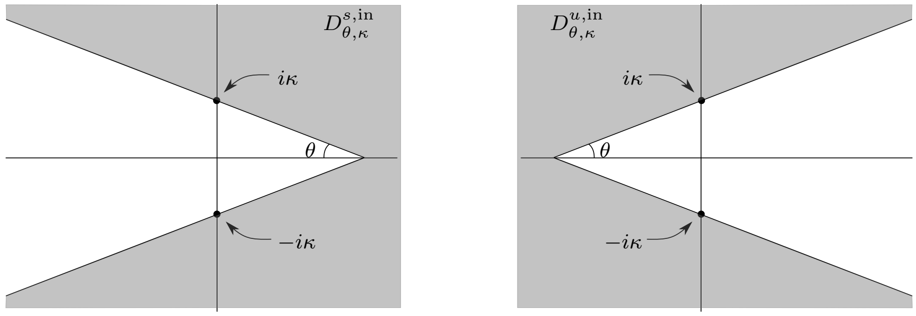

FIGURE 6. Inner domains $D^{s,\mathrm{in}}_{\theta,\kappa}$ and $D^{u,\mathrm{in}}_{\theta,\kappa}$.

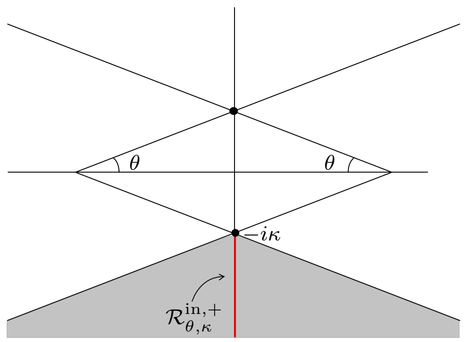

FIGURE 7. Domain $\mathcal{R}^{\mathrm{in},+}_{\theta,\kappa}$.

(1) *Equation* (3.9) *has two solutions* $\phi^{0,\star} : D^{\star,\mathrm{in}}_{\theta,\kappa} \times \mathbb{T} \to \mathbb{C}$, $\star = u, s$, *given by*

$$\phi^{0,\star}(z,\tau) = -\frac{2\sqrt{2}i}{z}\sin\tau + \psi^{\star}(z,\tau), \tag{3.11}$$

which are analytic in the variable z. *Moreover,* $\Pi_{2l}\left[\phi^{0,\star}\right] \equiv 0$ *for every* $l \in \mathbb{N}$, *and there exists a constant* $M_2 > 0$ *independent of* κ *such that, for every* $z \in D^{\star,\mathrm{in}}_{\theta,\kappa}$ *and* $z' \in D^{\star,\mathrm{in}}_{2\theta,4\kappa}$

$$\left\|\partial^2_\tau \psi^\star\right\|_{\ell_1}(z) \leq \frac{M_2}{|z|^3}, \quad \left\|\partial^2_\tau \partial_z \psi^\star\right\|_{\ell_1}(z') \leq \frac{M_2}{|z'|^4}.$$

(2) *The difference* $\Delta\phi^0(z,\tau) = \phi^{0,u}(z,\tau) - \phi^{0,s}(z,\tau)$ *is given by (see Figure 7),*

$$\Delta\phi^0(z,\tau) = e^{-i\mu_3 z}\left(C_{\mathrm{in}}\sin(3\tau) + \chi(z,\tau)\right), \quad z \in \mathcal{R}^{\mathrm{in},+}_{\theta,\kappa} = D^{u,\mathrm{in}}_{\theta,\kappa} \cap D^{s,\mathrm{in}}_{\theta,\kappa} \cap \{z;\ \mathrm{Re}(z) = 0, \mathrm{Im}(z) < 0\} \tag{3.12}$$

where $\mu_3 = 2\sqrt{2}$, C_{in} *is a constant, and* χ *is analytic in* z *and satisfies that, for* $z \in \mathcal{R}^{\mathrm{in},+}_{\theta,\kappa}$,

$$\|\partial_\tau \chi\|_{\ell_1}(z) \leq \frac{M_2}{|z|} \quad \text{and} \quad \|\partial_z \chi\|_{\ell_1}(z) \leq \frac{M_2}{|z|^2}.$$

Remark 3.4. *It is interesting to see that the stable and unstable solutions* $\phi^{0,\star}$, $\star = u, s$, *are identified by the* $\mathcal{O}(|z|^{-1})$ *decay as* $Rez \to \pm\infty$, *where the same Lyapunov-Perron approach works. The freedom of translation in* z, *which causes a variation of the order* $\mathcal{O}(|z|^{-2})$ *is fixed by the* $\mathcal{O}(|z|^{-3})$ *restriction of the error terms. The splitting* $\Delta\phi^0$ *between* $\phi^{0,u}$ *and* $\phi^{0,s}$ *would turn out to be the principal part of the splitting between* v^u *and* v^s. *The leading order form of* $\Delta\phi^0$ *can be understood in two different perspectives. On the one hand, it is related to the Borel summation of divergent power series and the readers are referred to Subsection 2.3 for related discussions and our conjecture on how to compute* C_{in}, *which is generally none zero. On the other hand, along the real direction of* z, *the inner equation* (3.9) *is hyperbolic in the PDE sense and oscillatory. However, when we view it along the imaginary axis, it becomes strongly hyperbolic in the dynamical systems sense (and ill-posed) and elliptic in the PDE sense. All the originally oscillatory directions become hyperbolic in the dynamical systems sense and thus in particular the stable manifolds become infinite*

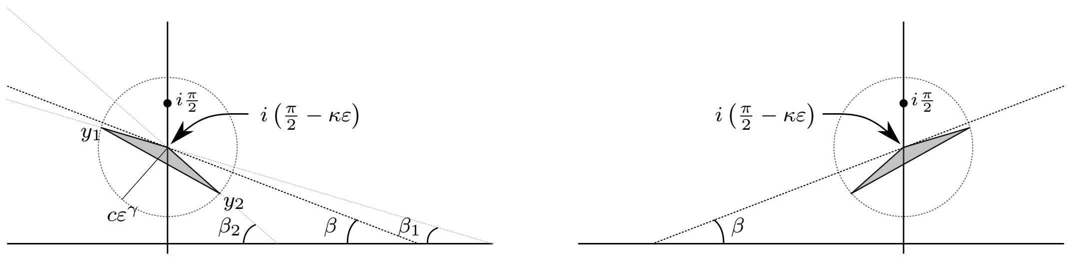

FIGURE 8. Matching domains $D^{\mathrm{mch},u}_{+,\kappa}$ (on the left) and $D^{\mathrm{mch},s}_{+,\kappa}$ (on the right).

dimensional containing $\phi^{0,\star}$. The splitting $\Delta\phi^0$ is dominated by the weakest exponential decay rate and the Stokes constant C_{in} basically comes from the difference between the weakest stable coordinates of $\phi^{0,u}$ and $\phi^{0,s}$.

The following lemma states that the Stokes constant C_{in} depends on the nonlinearity analytically.

Lemma 3.5. *Suppose $f(u,c)$ is a real analytic odd function of u satisfying $f(u) = \mathcal{O}(|u|^5)$ which also depends on a complex parameter c analytically, then $C_{\mathrm{in}}(c)$ is also analytic in c.*

Proof. We use the same notations as in Theorem 3.3 where $\phi^{0,\star}(z,\tau,c)$, $\star = u,s$, denote the unstable/stable solution of (3.9). By carefully tracking the dependence on c in the construction of $\phi^{0,\star}$ based on the Lyapunov-Perron integral equation, one can prove that $\phi^{0,\star}$ are analytic in c. For $s \in (-\infty,-\kappa)$, let

$$\widetilde{C}(s) = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-\mu_3 s}\Delta\phi^0(is,\tau,c)\sin 3\tau d\tau,$$

which is analytic in c. From (3.12) we have $C_{\mathrm{in}} = \lim_{s\to-\infty}\widetilde{C}(s)$ and thus the analyticity of C_{in} in c. □

In Appendix C we show that for a generic analytic odd f (in the $\mathcal{C}^0$ topology) the constant C_{in} is not-zero. We achieve this result by using the analyticity of C_{in} with respect to f and proving that $C_{\mathrm{in}} \neq 0$ for f in a suitable open set in the neighborhood of the f associated to the integrable Sine-Gordon equation (sG), for which $C_{\mathrm{in}} = 0$.

Our next step is to prove that the solutions of the inner equation obtained in Theorem 3.3 are good approximations of the parameterizations $v^*(y,\tau)$, $* = u,s$, obtained in Theorem 3.1 near the pole $y = i\pi/2$. To prove this fact we introduce the following *matching domains*.

Take $0<\gamma<1$, $0<\beta_1<\beta<\beta_2<\pi/4$ constants independent of ε and κ. Then, we consider the points $y_j \in \mathbb{C}$, $j=1,2$ satisfying

(1) $\mathrm{Im}(y_j) = -\tan\beta_j\,\mathrm{Re}(y_j) + \pi/2 - \kappa\varepsilon$;
(2) $|y_j - i(\pi/2-\kappa\varepsilon)| = \varepsilon^\gamma$;
(3) $\mathrm{Re}(y_1) < 0$ and $\mathrm{Re}(y_2) > 0$.
(4) $e^{5(\pi-\beta_1)} - e^{-5\beta_2} \neq 0$.

Note that $\mathrm{Im}(y_2) < \frac{\pi}{2} - \kappa\varepsilon < \mathrm{Im}(y_1)$. Then, consider the following *matching domains* (see Figure 8),

$$\begin{aligned} D^{\mathrm{mch},u}_{+,\kappa} &= \{y\in\mathbb{C};\ \mathrm{Im}(y) \le -\tan\beta_1\,\mathrm{Re}(y) + \pi/2-\kappa\varepsilon,\ \mathrm{Im}(y)\le -\tan\beta_2\,\mathrm{Re}(y)+\pi/2-\kappa\varepsilon,\\ &\qquad \mathrm{Im}(y) \ge \mathrm{Im}(y_1) - \tan\left(\frac{\beta_1+\beta_2}{2}\right)(\mathrm{Re}(y)-\mathrm{Re}(y_1))\Big\},\\ D^{\mathrm{mch},s}_{+,\kappa} &= \Big\{y\in\mathbb{C}; -\overline{y}\in D^{\mathrm{mch},u}_{+,\kappa}\Big\}. \end{aligned} \tag{3.13}$$

Notice that there exist constants $M_1, M_2 > 0$ independent of ε and κ such that

$$M_1\varepsilon^\gamma \le |y_j - i\pi/2| \le M_2\varepsilon^\gamma, \qquad j=1,2,$$
$$M_1\kappa\varepsilon \le |y - i\pi/2| \le M_2\varepsilon^\gamma, \qquad \text{for} \quad y\in D^{\mathrm{mch},u}_{+,\kappa}.$$

In terms of the inner variable z (see (3.6)), the matching domains are given by

$$\mathcal{D}^{\mathrm{mch},\star}_{+,\kappa} = \{z\in\mathbb{C}; \varepsilon z + i\pi/2 \in D^{\mathrm{mch},\star}_{+,\kappa}\},\ \star = u,s.$$

Notice that,

$$M_1\varepsilon^{\gamma-1}\leq |z_j|\leq M_2\varepsilon^{\gamma-1},\qquad j=1,2,$$
$$M_1\kappa\leq |z|\leq M_2\varepsilon^{\gamma-1}\qquad \text{for},\quad z\in\mathcal{D}^{\mathrm{mch},u}_{+,\kappa}.$$

where z_1 and z_2 are the vertices of the inner domain y_1 and y_2, respectively, expressed in the inner variable.

Next theorem estimates the difference in the matching domains (3.13) between the functions ϕ^*, $*=u,s$ in (3.7) and the functions $\phi^{0,*}$, $*=u,s$, given by Theorem 3.3. The theorem is proven in Section 6.

Theorem 3.6 (Matching)**.** *Fix $\gamma\in(1/3,1)$. Let $\phi^\star(z,\tau)=\varepsilon v^\star(i\pi/2+\varepsilon z,\tau)$, $\star=u,s$, where $v^\star$ is the parameterization obtained in Theorem 3.1. Then, there exist $\varepsilon_0,\delta_0>0$ sufficiently small such that, for each $0<\varepsilon\leq\varepsilon_0$ and κ satisfying $\kappa\varepsilon^{1-\gamma}+\frac{|\log\varepsilon|}{\kappa^2}\leq\delta_0$, and $z\in\mathcal{D}^{\mathrm{mch},\star}_{+,\kappa}$,*

$$\phi^\star(z,\tau)=\phi^{0,\star}(z,\tau)+\varphi^\star(z,\tau),$$

where $\phi^{0,\star}$ is the solution of the inner equation (3.9) obtained in Theorem 3.3, and $\varphi^\star$ satisfies that for $(z,\tau)\in\mathcal{D}^{\mathrm{mch},\star}_{+,\kappa}$

$$\|\partial_\tau^2\varphi^\star\|_{\ell_1}(z)\leq\frac{M_3(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1})|\log\varepsilon|}{|z|^2}\quad\text{and}\quad\|\partial_\tau^2\partial_z\varphi^\star\|_{\ell_1}(z)\leq\frac{M_3(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1})|\log\varepsilon|}{\kappa|z|^2},$$

where $M_3>0$ is a constant independent of ε and κ.

Remark 3.7. *Notice that $\gamma=1/2$ minimizes the size of $\|\varphi\|_{\ell_1,2}$ in Theorem 3.6. In this case,*

$$\|\partial_\tau^2\varphi\|_{\ell_1,2}\leq M|\log\varepsilon|\varepsilon^{1/2}|z|^{-2}.$$

Remark 3.8. *The idea to obtain the above matching estimate is that y_1 and y_2 are connected by a segment with nontrivial slope in the complex plane, where the linear part of the problem becomes somewhat elliptic in the 1-dim variable z (in the PDE sense) except in the direction of the mode $\sin\tau$. Therefore $\varphi^\star$, $\star=u,s$, is nicely determined by the values at y_1 and y_2 which simply come from the asymptotic form $\phi^{0,\star}$ and $\varphi^\star$. The order $\mathcal{O}(|z|^{-2})$ is largely determined by the mode $\sin\tau$.*

3.3. **The distance between the invariant manifolds.** Our next step is to give an asymptotic formula for the difference

$$\Delta(y,\tau)=v^u(y,\tau)-v^s(y,\tau)=\xi^u(y,\tau)-\xi^s(y,\tau),\tag{3.14}$$

where $\xi^{u,s}$ are the functions obtained in Theorem 3.1 (recall that $\Pi_{2l}[\Delta v]=0$ for every $l\geq 0$), in the domain (see Figure 9).

$$\mathcal{R}_\kappa=D^{\mathrm{out},u}_\kappa\cap D^{\mathrm{out},s}_\kappa\cap i\mathbb{R},$$

Next lemma shows that the difference Δ satisfies a linear equation.

Lemma 3.9. *The function Δ introduced in (3.14) satisfies the linear equation*

$$\mathcal{L}(\Delta)=\Pi_1\left[\eta_1(y,\tau)\Pi_1[\Delta]\sin\tau+\eta_2(y,\tau)\widetilde{\Pi}[\Delta]\right]\sin\tau+\widetilde{\Pi}[\eta_3(y,\tau)\Delta],$$

where $\mathcal{L}$ is the operator given in (3.1) and $\eta_j:\mathcal{R}_\kappa\times\mathbb{T}\to\mathbb{C}$, $j=1,2,3$, are functions analytic in y. Moreover, there exists a constant $M>0$ independent of κ and ε such that

$$\|\eta_1\|_{\ell_1}(y)\leq\frac{M\varepsilon^2}{|y^2+\pi^2/4|^4},\quad\text{and}\quad\|\eta_2\|_{\ell_1}(y),\|\eta_3\|_{\ell_1}(y)\leq\frac{M}{|y^2+\pi^2/4|^2}.$$

Proof. From (3.3) and Theorem 3.1, we have that

$$\mathcal{L}(\Delta)=\mathcal{F}(\xi^u)-\mathcal{F}(\xi^s),$$

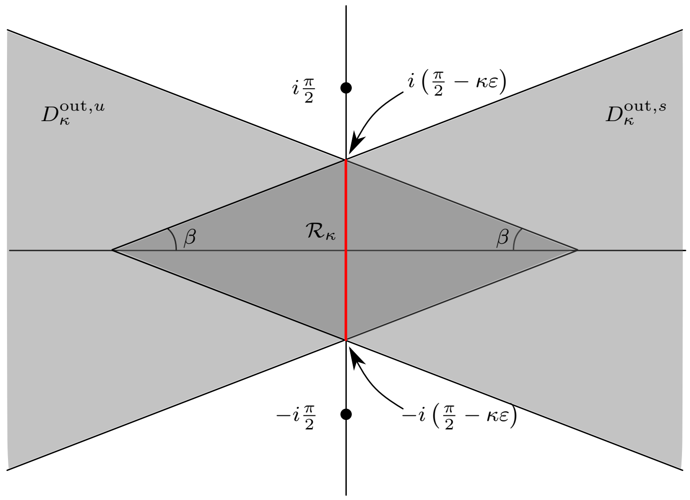

FIGURE 9. Domain $\mathcal{R}_\kappa$.

where $\mathcal{F}$ is the operator given in (3.2). Using the expression of $\mathcal{F}$ (see also (4.5)), we obtain that

$$\begin{aligned}
\mathcal{F}(\xi^u)-\mathcal{F}(\xi^s) = &-\frac{1}{\varepsilon^3\omega^3}\widetilde{\Pi}\left[g(\varepsilon\omega(\xi^u+v^h\sin\tau))-g(\varepsilon\omega(\xi^s+v^h\sin\tau))\right]\\
&-\Pi_1\left[(\xi_1^u+v^h)^2\sin^2\tau\widetilde{\Pi}(\xi^u)-(\xi_1^s+v^h)^2\sin^2\tau\widetilde{\Pi}(\xi^s)\right]\sin\tau\\
&-\Pi_1\Big[(\xi_1^u+v^h)\sin\tau(\widetilde{\Pi}[\xi^u])^2-(\xi_1^s+v^h)\sin\tau(\widetilde{\Pi}[\xi^s])^2\\
&+\frac{1}{3}\left((\widetilde{\Pi}[\xi^u])^3-(\widetilde{\Pi}[\xi^s])^3\right)\Big]\sin\tau\\
&+\Bigg(-\frac{1}{\varepsilon^3\omega^3}\Pi_1\left[f(\varepsilon\omega(\xi^u+v^h\sin\tau))-f(\varepsilon\omega(\xi^s+v^h\sin\tau))\right]\\
&-\frac{3v^h\left((\xi_1^u)^2-(\xi_1^s)^2\right)}{4}-\frac{(\xi_1^u)^3-(\xi_1^s)^3}{4}\Bigg)\sin\tau.
\end{aligned}$$

The proof follows from calculations based in the power series expansion of g and f and the estimates

$$\left|v^h(y)\right|\leq\frac{M}{|y^2+\pi^2/4|},\qquad \|\xi^{u,s}\|_{\ell_1}(y)\leq\frac{M\varepsilon^2}{|y^2+\pi^2/4|^3}$$

obtained in Theorem 3.1. □

The idea to obtain the exponentially small splitting estimate is that $y^\pm=\pm i(\frac{\pi}{2}-\kappa\varepsilon)$ (see Figure 9) are connected by a vertical segment where the linear operator $\mathcal{L}$ becomes elliptic (in the PDE sense) in the 1-dim variable y except in the direction of the mode $\sin\tau$. This has two implications: a.) the solution is determined by the values at the two boundary points $y^\pm$ and b.) the Green's function principally in the form of exponential functions leads to the desired splitting estimate at $y=0$. The mode $\sin\tau$ seems to be an exception. Recalling $\partial_y\Pi_1[\Delta]|_{y=0}=0$, the splitting in the direction $\Pi_1[\Delta]|_{y=0}$ will be handled by the conservation of energy due to the Hamiltonian structure.

As explained in Section 2.1, to prove that the distance between the stable and unstable manifold is exponentially small is crucial the fact that the model considered is Hamiltonian. Indeed, if the system would not have a first integral, the distance between the invariant manifolds would be "tipically" of order of some power of ε. Therefore, in this section we must rely on the conservation of energy to analyze Δ.

Let us rewrite equation (1.15) as

$$\begin{cases}\partial_y v=w,\\ \partial_y w=\dfrac{1}{\varepsilon^2}\partial_\tau^2 v+\dfrac{1}{\varepsilon^2\omega^2}v-\dfrac{1}{3}v^3-\dfrac{1}{\varepsilon^3\omega^3}f\left(\varepsilon\omega v\right),\end{cases}$$

which is Hamiltonian with respect to

$$\mathcal{H}(v,w)=\frac{1}{\pi}\int_{\mathbb{T}}\left(\frac{w^2}{2}+\frac{(\partial_\tau v)^2}{2\varepsilon^2}-\frac{v^2}{2\varepsilon^2\omega^2}+\frac{v^4}{12}+\frac{F(\varepsilon\omega v)}{\varepsilon^4\omega^4}\right)d\tau,$$

where F is an analytic function such that $F'(z)=f(z)$ and $F(z)=\mathcal{O}(z^6)$.

Notice that the solutions $v^\star(y,\tau)$ of (1.15), $\star=u,s$, obtained in Theorem 3.1 are contained in the energy level $\{\mathcal{H}=0\}$. We use the Hamiltonian $\mathcal{H}$ to obtain the variable $\Pi_1[\Delta]$ in terms of the variables $\widetilde{\Pi}[\Delta]$, $\Pi_1[\Xi]$ and $\widetilde{\Pi}[\Xi]$ where $\Xi=\partial_y\Delta=w^u-w^s=\partial_y v^u-\partial_y v^s$.

Lemma 3.10. *The functions Δ, Ξ satisfy*

$$\Pi_1[\Delta](y)=\frac{\dot v^h(y)}{\ddot v^h(y)}\Pi_1[\Xi](y)+A(\Xi)(y)+B(\widetilde{\Pi}[\Delta])(y), \tag{3.15}$$

where A and B are linear operators such that, for $y\in\mathcal{R}_\kappa$,

(1) $|A(\Xi)(y)|\leq \dfrac{M\varepsilon^2}{|y^2+\pi^2/4|}\|\Xi\|_{\ell_1}(y)$

(2) $|B(\widetilde{\Pi}[\Delta])(y)|\leq M\|\widetilde{\Pi}[\Delta]\|_{\ell_1}(y)$.

Proof. As the projections Π_1 and $\widetilde{\Pi}$ are orthogonal (see (2.2) and (2.6)), $\mathcal{H}$ is given by

$$\mathcal{H}(v,w)=\frac{(\Pi_1[w])^2}{2}-\frac{(\Pi_1[v])^2}{2}+\frac{1}{\pi}\int_{\mathbb{T}}\left(\frac{(\widetilde{\Pi}[w])^2}{2}+\frac{(\partial_\tau\widetilde{\Pi}[v])^2}{2\varepsilon^2}-\frac{(\widetilde{\Pi}[v])^2}{2\varepsilon^2\omega^2}+\frac{v^4}{12}+\frac{F(\varepsilon\omega v)}{\varepsilon^4\omega^4}\right)d\tau.$$

Using that $\mathcal{H}(v^\star,w^\star)=0$, $\star=u,s$, integrating by parts the ∂_τ term and the Mean Value Theorem, we have that

$$\begin{aligned}0=&\mathcal{H}(v^u,w^u)-\mathcal{H}(v^s,w^s)\\ =&\frac{\Pi_1[w^u]+\Pi_1[w^s]}{2}\Pi_1[\Xi]-\frac{\Pi_1[v^u]+\Pi_1[v^s]}{2}\Pi_1[\Delta]\\ &+\frac{1}{\pi}\int_{\mathbb{T}}\left[\frac{\widetilde{\Pi}[w^u]+\widetilde{\Pi}[w^s]}{2}\widetilde{\Pi}[\Xi]-\frac{1}{\varepsilon^2}\frac{\partial_\tau^2\widetilde{\Pi}[v^u]+\partial_\tau^2\widetilde{\Pi}[v^s]}{2}\widetilde{\Pi}[\Delta]-\frac{\widetilde{\Pi}[v^u]+\widetilde{\Pi}[v^s]}{2\varepsilon^2\omega^2}\widetilde{\Pi}[\Delta]\right]d\tau\\ &+\frac{1}{\pi}\int_{\mathbb{T}}\left[\frac{(v^u)^3+(v^u)^2(v^s)+(v^u)(v^s)^2+(v^s)^3}{12}\Delta+\left(\frac{1}{\varepsilon^3\omega^3}\int_0^1 f(\varepsilon\omega(\sigma v^u+(1-\sigma)\sigma v^s))d\sigma\right)\Delta\right]d\tau.\end{aligned}$$

Using

$$v^\star=v^h\sin(\tau)+\xi^\star(y,\tau),\quad \ddot v^h=v^h-(v^h)^3/4=\sqrt{2}(\cosh(2y)-3)\operatorname{sech}^3(y),$$

and observing that $\ddot v^h(y)$ is strictly negative, for every $y=i\widetilde{y}$ with $\widetilde{y}\in(-\pi/2,\pi/2)$, one has

$$0=-\ddot v^h(1+a(y))\Pi_1[\Delta]+\dot v^h\Pi_1[\Xi]+\widetilde{A}(\Xi)+\widetilde{B}(\widetilde{\Pi}[\Delta])$$

By the estimates in Theorem 3.1 and using that $\ddot v^h(y)$ has a third order pole at $y=\pm i\pi/2$, we have

$$|a(y)|\leq\frac{M\varepsilon^2}{|y^2+\pi^2/4|^2}\leq\frac{M}{\kappa^2},\qquad \text{for}\quad y\in\mathcal{R}_\kappa \tag{3.16}$$

and, also for $y\in\mathcal{R}_\kappa$,

$$\left|\widetilde{A}(\Xi)\right|(y)\leq\frac{M\varepsilon^2}{|y^2+\pi^2/4|^4}\|\Xi\|_{\ell_1}(y)\qquad\text{and}\qquad\left|\widetilde{B}(\widetilde{\Pi}[\Delta])\right|(y)\leq\frac{M}{|y^2+\pi^2/4|^3}\|\widetilde{\Pi}[\Delta]\|_{\ell_1}(y).$$

Moreover, using the estimate (3.16) and taking κ big enough, we have

$$|D(y)^{-1}|\leq M|y^2+\pi^2/4|^3,\ y\in\mathcal{R}_\kappa,\ \text{where } D(y)=\ddot v^h(y)\,(1+a(y))\,.$$

Hence, it follows that

$$\Pi_1[\Delta]=\frac{\dot v^h\Pi_1[\Xi]+\widetilde{A}(\Xi)+\widetilde{B}(\widetilde{\Pi}[\Delta])}{\ddot v^h(1+a)}=\frac{\dot v^h}{\ddot v^h}\Pi_1[\Xi]+A(\Xi)+B(\widetilde{\Pi}[\Delta]),$$

where A and B are the linear operators

$$A(\Xi)(y) = \frac{\widetilde{A}(\Xi)(y)}{\ddot{v}^h(y)(1+a(y))} - \frac{\dot{v}^h(y)}{\ddot{v}^h(y)(1+a(y))}a(y)\Pi_1[\Xi](y)$$
$$B(\widetilde{\Pi}[\Delta]) = \frac{\widetilde{B}(\widetilde{\Pi}[\Delta])}{\ddot{v}^h(y)(1+a(y))}.$$

The proof of the proposition follows directly from the estimates of $\widetilde{A}(\Xi)$, $\widetilde{B}(\widetilde{\Pi}[\Delta])$, a and the fact that $\ddot{v}^h$ and $\dot{v}^h$ have a third and second order pole at the points $y = \pm i\pi/2$, respectively. □

Lemma 3.10 allows to study the difference between the invariant manifolds without keeping track of the component Δ_1. In other words, we use coordinates $(\Pi_1 w, \widetilde{\Pi} v, \widetilde{\Pi} w)$ to analyze the level of energy $\mathcal{H} = 0$ and therefore we measure the difference between the functions v^u and v^s through the components $(\Xi_1, \widetilde{\Pi}\Delta, \widetilde{\Pi}\Xi)$. The inconvience of the energy reduction is that the equation loses the second order structure since it also depends on $\Xi = \partial_y \Delta$.

To capture the exponentially small behavior of $(\widetilde{\Pi}\Delta, \widetilde{\Pi}\Xi)$ it is convenient to write the second order equation as a first order system in diagonal form. Thus, we define

$$\begin{aligned}\Gamma &= \sum_{k\geq 1} \Gamma_{2k+1}(y)\sin((2k+1)\tau), \quad \Gamma_{2k+1} = \lambda_{2k+1}\Delta_{2k+1} + i\varepsilon\Xi_{2k+1}\\ \Theta &= \sum_{k\geq 1} \Theta_{2k+1}(y)\sin((2k+1)\tau), \quad \Theta_{2k+1} = \lambda_{2k+1}\Delta_{2k+1} - i\varepsilon\Xi_{2k+1},\end{aligned} \tag{3.17}$$

From now on we measure the difference between the invariant manifolds (within the energy level $\mathcal{H} = 0$) by the difference "vector"

$$\widetilde{\Delta} = (\Xi_1, \Gamma, \Theta) \tag{3.18}$$

Notice that the estimates of Theorem 3.1 imply that Δ satisfies

$$\sum_{k\geq 1} \lambda_{2k+1}^2 |\Delta_{2k+1}(y)| \leq M\|\partial_\tau^2 \xi^u\|_{\ell_1}(y) + M\|\partial_\tau^2 \xi^s\|_{\ell_1}(y) \leq \frac{M\varepsilon^2}{|y^2+\pi^4/4|^3},$$

along with a similar estimate on Ξ, therefore the functions Γ and Θ are well defined for $y \in \mathcal{R}_\kappa$ and satisfy

$$\sum_{k\geq 1} \lambda_{2k+1}|\Gamma_{2k+1}(y)| \leq \frac{M\varepsilon^2}{|y^2+\pi^4/4|^3} \quad \text{and} \quad \sum_{k\geq 1} \lambda_{2k+1}|\Theta_{2k+1}(y)| \leq \frac{M\varepsilon^2}{|y^2+\pi^4/4|^3}.$$

Proposition 3.11. *The function $\widetilde{\Delta} = (\Xi_1, \Gamma, \Theta)$ satisfies the equation*

$$\widetilde{\mathcal{L}}(\widetilde{\Delta}) = \mathcal{M}(\widetilde{\Delta}),$$

where $\widetilde{\mathcal{L}}$ is the differential operator

$$\begin{aligned}\widetilde{\mathcal{L}}(\Xi_1, \Gamma, \Theta) \;=\; &\Bigg(\dot{\Xi}_1 - \frac{\dddot{v}^h}{\ddot{v}^h}\Xi_1, \sum_{k\geq 1}\left(\dot{\Gamma}_{2k+1} + i\frac{\lambda_{2k+1}}{\varepsilon}\Gamma_{2k+1}\right)\sin((2k+1)\tau),\\ &\sum_{k\geq 1}\left(\dot{\Theta}_{2k+1} - i\frac{\lambda_{2k+1}}{\varepsilon}\Theta_{2k+1}\right)\sin((2k+1)\tau)\Bigg)\end{aligned} \tag{3.19}$$

and $\mathcal{M}$ is a linear operator which can be written as

$$\mathcal{M}(\Xi_1, \Gamma, \Theta) = \begin{pmatrix} m_W(y)\Xi_1 + \mathcal{M}_W(\Gamma, \Theta) \\ m_{\rm osc}(y,\tau)\Xi_1 + \mathcal{M}_{\rm osc}(\Gamma, \Theta) \\ -m_{\rm osc}(y,\tau)\Xi_1 - \mathcal{M}_{\rm osc}(\Gamma, \Theta)\end{pmatrix}, \tag{3.20}$$

where $m_W : \mathcal{R}_\kappa \to \mathbb{C}$, $m_{\rm osc} : \mathcal{R}_\kappa \times \mathbb{T} \to \mathbb{C}$ are functions analytic in y satisfying

$$|m_W(y)| \leq \frac{M\varepsilon^2}{|y^2+\pi^2/4|^3} \qquad \text{and} \qquad \|m_{\rm osc}\|_{\ell_1}(y) \leq \frac{M\varepsilon}{|y^2+\pi^2/4|}$$

and $\mathcal{M}_W,\ \mathcal{M}_{\rm osc}$ *are linear operators such that, for* $y\in\mathcal{R}_\kappa$*,*

$$|\mathcal{M}_W(\Gamma,\Theta)(y)|\leq \frac{M}{|y^2+\pi^2/4|^2}\left(\|\Gamma\|_{\ell_1}(y)+\|\Theta\|_{\ell_1}(y)\right)$$
$$\left\|\mathcal{M}_{\rm osc}(\Gamma,\Theta)\right\|_{\ell_1}(y)\leq \frac{M\varepsilon}{|y^2+\pi^2/4|^2}\left(\|\Gamma\|_{\ell_1}(y)+\|\Theta\|_{\ell_1}(y)\right),$$

where $M>0$ *is a constant independent of* ε *and* κ *.*

The proof of this proposition is deferred to Appendix B.

We characterize the function $\widetilde{\Delta}$ as the *unique solution* of a certain integral equation. To this end, we introduce some notation. Given a sequence $a=(a_{2k+1})_{k\geq 1}$, we define the functions

$$\begin{aligned}\mathcal{I}_\Gamma(a)(y,\tau)&=\sum_{k\geq 1}a_{2k+1}e^{-i\frac{\lambda_{2k+1}}{\varepsilon}y}\sin((2k+1)\tau)\\ \mathcal{I}_\Theta(a)(y,\tau)&=\sum_{k\geq 1}a_{2k+1}e^{i\frac{\lambda_{2k+1}}{\varepsilon}y}\sin((2k+1)\tau).\end{aligned}\tag{3.21}$$

We also define the following linear operator, which is a right inverse of the operator $\widetilde{\mathcal{L}}$ in (3.19),

$$\mathcal{P}(f,g,h)=\left(\mathcal{P}^W(f),\mathcal{P}^\Gamma(g),\mathcal{P}^\Theta(h)\right),\tag{3.22}$$

where

$$\begin{aligned}\mathcal{P}^W(f)&=\ddot{v}^h(y)\int_0^y\frac{f(s)}{\ddot{v}^h(s)}ds\\ \mathcal{P}^\Gamma(g)&=\sum_{k\geq 1}\mathcal{P}^\Gamma_{2k+1}(g)\sin((2k+1)\tau),\qquad \mathcal{P}^\Gamma_{2k+1}(g)(y)=\int_{y^+}^y e^{i\frac{\lambda_{2k+1}}{\varepsilon}(s-y)}\Pi_{2k+1}[g](s)ds\\ \mathcal{P}^\Theta(h)&=\sum_{k\geq 1}\mathcal{P}^\Theta_{2k+1}(h)\sin((2k+1)\tau),\qquad \mathcal{P}^\Theta_{2k+1}(h)(y)=\int_{y^-}^y e^{-i\frac{\lambda_{2k+1}}{\varepsilon}(s-y)}\Pi_{2k+1}[h](s)ds.\end{aligned}$$

and

$$y^\pm=\pm i\left(\frac{\pi}{2}-\kappa\varepsilon\right).$$

Using the just introduced funtions and operators and recalling that by Theorem 3.1 $\Xi_1(0)=\partial_y\xi_1^u(0)-\partial_y\xi_1^s(0)=0$, it can be easily checked that the function $\widetilde{\Delta}$ must satisfy the integral equation

$$\widetilde{\Delta}=(0,\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d))+\widetilde{\mathcal{M}}(\widetilde{\Delta}),\qquad\text{with}\qquad \widetilde{\mathcal{M}}(\widetilde{\Delta})=\mathcal{P}\circ\mathcal{M}(\widetilde{\Delta}),\tag{3.23}$$

where $\mathcal{M}$ is given by (3.20) and $\mathcal{I}_\Gamma(c)$, $\mathcal{I}_\Theta(d)$ are given in (3.21) with

$$c_{2k+1}=\Gamma_{2k+1}(y^+)e^{i\frac{\lambda_{2k+1}}{\varepsilon}y^+}\qquad\text{and}\qquad d_{2k+1}=\Theta_{2k+1}(y^-)e^{-i\frac{\lambda_{2k+1}}{\varepsilon}y^-},\tag{3.24}$$

(note that $\Gamma(y^+,\tau)=\mathcal{I}_\Gamma(c)(y^+,\tau)$ and $\Theta(y^-,\tau)=\mathcal{I}_\Theta(d)(y^-,\tau)$).

Now we are ready to define the leading order of the function $\widetilde{\Delta}$. We first give some heuristic explanation. In Section 7, we shall first show that $\widetilde{M}$ is small and thus we expect that the main term of $\widetilde{\Delta}$ for the (Γ,Θ) is given by $(\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d))$. Let us analyze how these functions behave. We do the reasoning for Γ since the one for Θ is analogous.

$$\mathcal{I}_\Gamma(c)(y,\tau)=\sum_{k\geq 1}\Gamma_{2k+1}(y^+)e^{-i\frac{\lambda_{2k+1}}{\varepsilon}(y-y^+)}\sin((2k+1)\tau)$$

Recalling $\lambda_3=\sqrt{8-\varepsilon^2}$, $\mu_3=2\sqrt{2}$, (3.17), that $\Gamma_{2k+1}(y^+)=\lambda_{2k+1}\Delta_{2k+1}(y^+)+i\varepsilon\Xi_{2k+1}(y^+)$ and using Theorem 3.6 to approximate the functions $v^{u,s}$ at the point $y=y^+$ by the corresponding solutions of the inner equation (see Theorem 3.3) and the asymptotic formula for the difference between $\phi^{0,u}$ and $\phi^{0,s}$ at $z^+=(y^+-i\pi/2)/\varepsilon$, also in Theorem 3.3, one has

$$\begin{aligned}\Gamma_3(y^+)&=2\frac{\lambda_3}{\varepsilon}e^{-i\mu_3\frac{y^+-i\frac{\pi}{2}}{\varepsilon}}\left(C_{\rm in}+\mathcal{O}\left(\frac{1}{\kappa}\right)\right)+\text{h.o.t}\\ \Gamma_{2k+1}(y^+)&=\frac{1}{\varepsilon}e^{-i\mu_3\frac{y^+-i\frac{\pi}{2}}{\varepsilon}}\mathcal{O}\left(\frac{1}{\kappa}\right)+\text{h.o.t.}\end{aligned}$$

Therefore,

$$\mathcal{I}_\Gamma(c)(y) = \frac{2\lambda_3}{\varepsilon} e^{-i2\sqrt{2}\frac{y-i\frac{\pi}{2}}{\varepsilon}} \left(C_{\text{in}} \sin 3\tau + \mathcal{O}\left(\frac{1}{\kappa}\right)\right) + \text{h.o.t}$$

To prove Theorem 2.1, it suffices to justify the above leading order expansion of $\widetilde{\Delta}$.

Proposition 3.12. *Take $\kappa = \frac{1}{2\lambda_3}|\log\varepsilon|$. There exists $M > 0$ independent of small ε such that, for any $y \in \mathcal{R}_\kappa$, it holds*

$$|\Xi_1(y)| \le \frac{M}{|y^2 + \pi^2/4|^2} e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2} - |\operatorname{Im}(y)|\right)},$$
$$\left\| \Gamma(y,\tau) - \frac{2\lambda_3}{\varepsilon} C_{\text{in}} e^{-i2\sqrt{2}\frac{y-i\frac{\pi}{2}}{\varepsilon}} \sin 3\tau \right\|_{\ell_1} \le \frac{M}{\varepsilon|\log\varepsilon|} e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2} - |\operatorname{Im}(y)|\right)},$$

for some constant M independent of ε. Moreover $\Theta(\bar{y},\tau) = \overline{\Gamma(y,\tau)}$ satisfies a similar estimate.

The proof of this proposition is deferred to Section 7. Note that for $y = 0$, this proposition gives smaller bounds of the perturbations to the exponentially small leading order term if $C_{\text{in}} \neq 0$. Recall that $\Xi_1(0) = \partial_y v^u(0) - \partial_y v^s(0) = 0$ (see Theorem 3.1). However, we also need to estimate this component for $y \in \mathcal{R}_\kappa$ to obtain the estimate of (Γ, Θ) due to the coupling (see Section 7). The definition of Γ and the above inequality imply inequality (2.4) except for the missing $\sin\tau$ mode, which easily follows from $\Xi_1(0) = 0$ and the estimate on $\Pi_1[\Delta]$ given by Lemma 3.10.

3.4. **Generalized breathers with exponentially small tails.** Finally we consider the intersection of the center-stable manifold $W^{cs}(0)$ and center-unstable manifold $W^{cu}(0)$ of the zero solution which form a tube homoclinic to the center manifold $W^c(0)$ of 0 in the phase space. In the original coordinates, they correspond to an infinite dimensional family of waves of (1.2) which are ω-periodic in t (with ω given in (1.13) with $k = 1$) and of the order

$$u(t,x) = \mathcal{O}(\varepsilon e^{-\varepsilon|x|}) + \mathcal{O}(e^{-\frac{\sqrt{2}\pi}{\varepsilon}}).$$

In particular, the exponentially small oscillating tails do not decay as $|x| \to \infty$. The construction of such generalized breathers is largely based on the approach in [59, 42, 41], so we shall adapt the problem into the framework in [42].

We shall adopt a slightly different coordinate system and phase space in this subsection. Let

$$q = (q_1, q_2)^T := (v_1, \partial_y v_1)^T, \quad Q = \left(\varepsilon^{-1}\widetilde{\Pi}[v], \, (-\partial_\tau^2 - \omega^{-2})^{-\frac{1}{2}}\widetilde{\Pi}[\partial_y v]\right)^T. \tag{3.25}$$

In the above the operator $(-\partial_\tau^2 - \omega^{-2})^{-\frac{1}{2}}$ is well-posed on $\ker \Pi_1$. In the (q, Q) variables, equation (1.2), or equivalently (1.15), takes the form

$$\begin{cases} \partial_y q = Aq + F(q, Q, \varepsilon) \\ \partial_y Q = \frac{J}{\varepsilon} Q + G(q, Q, \varepsilon), \end{cases} \tag{3.26}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = (-\partial_\tau^2 - \omega^{-2})^{\frac{1}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : H^1(\mathbb{S}^1) \cap \ker \Pi_1 \to L^2(\mathbb{S}^1) \cap \ker \Pi_1,$$

and with $v = q_1 \sin\tau + \widetilde{\Pi}[v]$,

$$F(q, Q, \varepsilon) = \left(0, \, -\frac{1}{4}q_1^3 + \left(-\frac{1}{\varepsilon^3\omega^3}\Pi_1\left[g(\varepsilon\omega v)\right] + \frac{q_1^3}{4}\right)\right)^T,$$

$$G(q, Q, \varepsilon) = \left(0, \, -\frac{1}{\varepsilon^3\omega^3}(-\partial_\tau^2 - \omega^{-2})^{-\frac{1}{2}}\widetilde{\Pi}\left[g(\varepsilon\omega v)\right]\right)^T.$$

While $q \in X := \mathbb{R}^2$, we take $Y = L^2(\mathbb{S}^1) \cap \ker \Pi_1$. Apparently

$$J : Y \supset D(J) = Y_1 \to Y, \;\; Y_1 := H^1(\mathbb{S}^1) \cap \ker \Pi_1, \quad J^* = -J,$$

where L^2 and H^1 stand for the standard Sobolev space of square integrable functions and the subspace of L^2 functions with square integrable first order derivatives. It is straight forward to verify that $X_1 = X$, Y, Y_1, A, J, F, and G satisfy all assumptions in Sec. 2–5 in [42]. (In fact G satisfies a stronger estimates

$$\|G\|_{H^1} \leq M, \quad \|D_q^{l_1} D_Q^{l_2} G\|_{L\left((\mathbb{R}^2\times L^2)\otimes(\otimes^{l_1+l_2-1}(\mathbb{R}^2\times H^1)),\mathbb{R}^2\times L^2\right)} \leq M\varepsilon^{3l_2},$$

for some $M > 0$ independent of small $\varepsilon > 0$, on any bounded set in $X \times Y_1$.) Therefore smooth local invariant manifolds of 0, including the 1-dim stable and unstable manifolds analyzed in details in this current paper, exist with sizes and bounds (in (q, Q) variables) uniform in ε.

In the following we consider the homoclinic tube formed by the intersection of the center-stable manifold $W^{cs}(0)$ and the center-unstable manifold $W^{cu}(0)$. The focus will also include the estimate of the minimal value of the Hamiltonian $\mathcal{H}$ on the homoclinic tube which in turn yields an estimate on the minimal amplitude of the oscillating tails of the corresponding generalized breathers.

• *Notation.* In this subsection all differentiation D are only with respect to the variables (q, Q) in the phase space, but never with respect to ε.

• **The local invariant manifolds and the restriction of the Hamiltonian $\mathcal{H}$ there.** Let q_u and q_s be the coordinates of q in the eigenvector expansion

$$q = q_u(1, 1)^T + q_s(1, -1)^T$$

in term of the stable and unstable eigenvectors. According to Theorem 4.2 in [42], locally the center-unstable (or center-stable, center) manifold $W^{cu}(0) \in X \times Y_1$ (or $W^{cs}(0)$, $W^c(0)$) can be represented as the graph of a smooth mapping $h^{cu}(\cdot, \varepsilon) : Y_1 \times \mathbb{R} \to \mathbb{R}$ (or h^{cs}, h^c):

$$\begin{aligned} W^{cu}(0) \cap \{|q_u|, |q_s|, \|Q\|_{Y_1} \leq \delta\} &= \{q_s = h^{cu}(q_u, Q, \varepsilon)\}, \\ W^{cs}(0) \cap \{|q_u|, |q_s|, \|Q\|_{Y_1} \leq \delta\} &= \{q_u = h^{cs}(q_s, Q, \varepsilon)\}, \\ W^{c}(0) \cap \{|q_u|, |q_s|, \|Q\|_{Y_1} \leq \delta\} &= \{(q_u, q_s) = h^{c}(Q, \varepsilon)\}, \end{aligned}$$

for some $\delta > 0$ independent of sufficiently small $\varepsilon > 0$. Moreover $h^{c\star}(q_\star, Q = 0, 0)$, $\star = u, s$, is well-defined and correspond to the 1-dim stable and unstable manifold of (1.21) with $k = 1$. They satisfy for $l \geq 1$ and some $M > 0$ independent of ε, for $\star = u, s$,

$$|h^{c\star}(q_\star, 0, \varepsilon) - h^{c\star}(q_\star, 0, 0)| + |D_{q_\star} h^{c\star}(q_\star, 0, \varepsilon) - D_{q_\star} h^{c\star}(q_\star, 0, 0)| + \|D_Q h^{c\star}(q_\star, 0, \varepsilon)\|_{(H^1)^*} \leq M\varepsilon^4,$$

$$Dh^{c\star}(0, 0, \varepsilon) = 0, \quad Dh^c(0, \varepsilon) = 0, \quad \|D^l h^{c\star}\| + \|D^l h^c\| \leq M.$$

In the (q_u, q_s, w) variables the Hamiltonian $\mathcal{H}$ defined in (1.16) takes the form

$$\mathcal{H}(q_u, q_s, Q, \varepsilon) = -2\pi q_u q_s + \frac{1}{2}\left\|(-\partial_\tau^2 - \omega^{-2})^{\frac{1}{2}} Q\right\|_{L^2}^2 + \int_{\mathbb{T}} \left(\frac{v^4}{12} + \frac{F(\varepsilon\omega v)}{\varepsilon^4\omega^4}\right) d\tau$$

which is smooth in $(q, Q) \in \mathbb{R}^2 \times Y_1$ and ε due to $F(u) = \mathcal{O}(u^6)$ near $u = 0$. Since

$$D_Q^2 \mathcal{H}(0, 0, 0, \varepsilon)(Q, Q) = \|(-\partial_\tau^2 - \omega^{-2})^{\frac{1}{2}} Q\|_{L^2}^2 \geq \frac{1}{2}\|Q\|_{H^1}^2,$$

it is straight forward to obtain the uniform quadratic positivity of $\mathcal{H}$ restricted on the center manifold $W^c(0)$

$$D_Q^2\big(\mathcal{H}\big(h^c(Q, \varepsilon), Q, \varepsilon\big)\big)(\tilde{Q}, \tilde{Q}) \geq \frac{1}{3}\|\tilde{Q}\|_{H^1}^2, \quad \mathcal{H}\big(h^c(Q, \varepsilon), Q, \varepsilon\big) \geq \frac{1}{6}\|Q\|_{H^1}^2, \quad \forall Q \in Y_1, \ \|Q\|_{H_1} \leq \delta. \tag{3.27}$$

The quadratic positivity implies that the center manifold $W^c(0)$ is unique and (1.15) (with $k = 1$) is stable both forward and backward in y on $W^c(0)$. By the conservation of energy and the invariant foliation structure (see [42]), we have that $\mathcal{H} \geq 0$ on $W^{c\star}(0)$ and it achieves 0 exactly at $W^\star(0)$, $\star = u, s$. Therefore, at any $U \in W^\star(0)$, $\|U\|_{H^1} \leq \delta$, $\star = u, s$,

$$T_U W^{c\star}(0) = \ker D\mathcal{H}(U, \varepsilon), \quad \ker\big(D^2\mathcal{H}(U, \varepsilon)|_{T_U W^{c\star}(0)}\big) = T_U W^\star(0).$$

[4] Actually some better estimates have been obtained in this current paper.

Here $\ker D\mathcal{H}(U,\varepsilon)$ is viewed as a linear functional on $\mathbb{R}^2\times Y_1$ and $D^2\mathcal{H}(U,\varepsilon)|_{T_U W^{c\star}(0)}$ a bounded linear operator on $T_U W^{c\star}(0)$ induced by the symmetric quadratic form on $T_U W^{c\star}(0)$. Moreover, for any hyperplane P in the tangent space of $T_U W^{c\star}(0)\subset\mathbb{R}^2\times Y_1$ transversal to $T_U W^{\star}(0)$, there exists $\sigma>0$ such that

$$\|D^2\mathcal{H}(U,\varepsilon)|_P\|_{L(P\otimes P,\mathbb{R})}\geq\sigma. \tag{3.28}$$

• **Analyzing** $W^{cs}(0)\cap W^{cu}(0)$**.** In terms of the (q,Q) coordinates, let

$$\Lambda=\{q_2=0\}\subset\mathbb{R}^2\times Y_1$$

be the hyperplane perpendicular to the unperturbed homoclinic orbit

$$\Gamma^h:=\big\{(v^h(y),\partial_y v^h(y),0)\mid y\in\mathbb{R}\big\}\subset\mathbb{R}^2\times Y_1,\qquad\text{(see (1.22))},$$

at $U_0=(v^h(0),0,0)$.

By Theorem 2.2 and 2.3 in [42], for any fixed time $T>0$ the time–T map of (3.26) is smooth in the phase space $\mathbb{R}^2\times Y_1$ with its derivative bounded uniformly in ε. (Even though only the first differentiation was carefully estimated in [42], the uniform in ε bounds of the higher order derivatives simply follow from a similar argument inductively.) Due the uniform in ε sizes and bounds on $W^{cs}(0)$ and $W^{cu}(0)$, they can be extended to stripes along Γ^h. For $\star=u,s$, consider the following intersections with Λ for the first time after $W^{c\star}(0)$ are extended from a neighborhood of 0 by the flow of (3.26),

$$\widetilde{W}^{c\star}(0)=W^{c\star}(0)\cap\Lambda,\quad U_\star=(q_{1,\star},0,Q_\star)=W^{\star}(0)\cap\Lambda\in\widetilde{W}^{c\star}(0).$$

Clearly, here $U_\star$ corresponds to the values of the stable and unstable solutions $\big(v^\star(0),\partial_y v^\star(0)\big)$ analyzed in Theorem 2.1.

We shall start with the decomposition $\Lambda=\big(\mathbb{R}(1,0)^T\big)\oplus Y_1$ to set up a coordinate system to analyze $\widetilde{W}^{c\star}(0)$, $\star=u,s$. Here in particular we notice

$$\{q=0\}\times Y_1=\ker D\mathcal{H}(U_0,0)\cap\Lambda,\quad\nabla\mathcal{H}(U_0,0)=\frac{10\sqrt{2}}{3}\big(q=(1,0)^T,Q=0\big). \tag{3.29}$$

Clearly $\widetilde{W}^{c\star}(0)$ is a hypersurface in Λ. Due to the conservation of the Hamiltonian $\mathcal{H}$ by the flow map, it holds

$$\mathcal{H}(U_\star,\varepsilon)=0,\quad T_{U_\star}W^{c\star}(0)=\ker D\mathcal{H}(U_\star,\varepsilon),\quad T_{U_\star}\widetilde{W}^{c\star}(0)=\ker D\mathcal{H}(U_\star,\varepsilon)\cap\Lambda,$$

which implies that locally $\{\mathcal{H}(\cdot,\varepsilon)=0\}\cap\Lambda$ and $\widetilde{W}^{c\star}(0)$ can be expressed as the graphs of smooth mapping from $Y_1\to\mathbb{R}$. In fact, due to the smoothness of $\mathcal{H}$ in ε and the uniform in ε bounds of $W^{c\star}(0)$ near 0 and the flow map, there exist $\delta>0$ independent of ε and $\widetilde{h}^0,\widetilde{h}^{c\star}:Y_1\to\mathbb{R}$, $\star=u,s$, such that inside the box $\{|q_1-v^h(0)|,\|Q-Q_\star\|_{H^1}\leq\delta\}$ in Λ,

$$\{\mathcal{H}(\cdot,\varepsilon)=0\}\cap\Lambda=\{q_1=\widetilde{h}^0(Q,\varepsilon)\},\quad\widetilde{h}^0(Q_\star,\varepsilon)=q_{1,\star},\star=u,s$$
$$\widetilde{W}^{c\star}(0)=\{q_1=\widetilde{h}^{c\star}(Q,\varepsilon)\},\quad\widetilde{h}^{c\star}(Q_\star,\varepsilon)=q_{1,\star},\star=u,s,$$

where $\widetilde{h}^0$ and $\widetilde{h}^{c\star}$ along with their derivatives are bounded uniformly in small ε.

Due to (3.29), it is clear

$$(q_1,0,Q)\in\widetilde{W}^{cu}(0)\cap\widetilde{W}^{cs}(0)\Longleftrightarrow q_1=\widetilde{h}^{cu}(Q,\varepsilon)=\widetilde{h}^{cs}(Q,\varepsilon)\Longleftrightarrow\mathcal{H}\big(\widetilde{h}^{cu}(Q,\varepsilon),0,Q,\varepsilon\big)=\mathcal{H}\big(\widetilde{h}^{cs}(Q,\varepsilon),0,Q,\varepsilon\big),$$

By (3.27) and (3.29), there exists $C>0$ such that

$$0\leq\mathcal{H}\big(\widetilde{h}^{c\star}(Q,\varepsilon),0,Q,\varepsilon\big)=\mathcal{H}\big(\widetilde{h}^{c\star}(Q,\varepsilon),0,Q,\varepsilon\big)-\mathcal{H}\big(\widetilde{h}^{0}(Q,\varepsilon),0,Q,\varepsilon\big)\leq C(\widetilde{h}^{c\star}(Q,\varepsilon)-\widetilde{h}^{0}(Q,\varepsilon)), \tag{3.30}$$

which implies

$$\widetilde{h}^{c\star}(Q,\varepsilon)\geq\widetilde{h}^{0}(Q,\varepsilon),\ \text{ and }\ \text{“}=\text{” holds iff } Q=Q_\star,\quad\star=u,s. \tag{3.31}$$

Moreover, from (3.28), the conservation of $\mathcal{H}$, and the uniform in ε bound on the flow map, we have

$$C\|Q-Q_\star\|_{H^1}^2\geq\mathcal{H}\big(\widetilde{h}^{c\star}(Q,\varepsilon),0,Q,\varepsilon\big)\geq\frac{1}{C}\|Q-Q_\star\|_{H^1}^2,\quad\star=u,s, \tag{3.32}$$

Therefore if $Q_u=Q_s$, clearly $U_u=U_s$ and $W^s(0)=W^u(0)$ which gives rises to a homoclinic orbit to 0. In the case of $Q_u\neq Q_s$, (3.31) implies

$$\widetilde{h}^{cu}(Q_s,\varepsilon)>\widetilde{h}^{0}(Q_s,\varepsilon)=\widetilde{h}^{cs}(Q_s,\varepsilon)\ \text{ and }\ \widetilde{h}^{cs}(Q_u,\varepsilon)>\widetilde{h}^{0}(Q_u,e)=\widetilde{h}^{cu}(Q_u,\varepsilon).$$

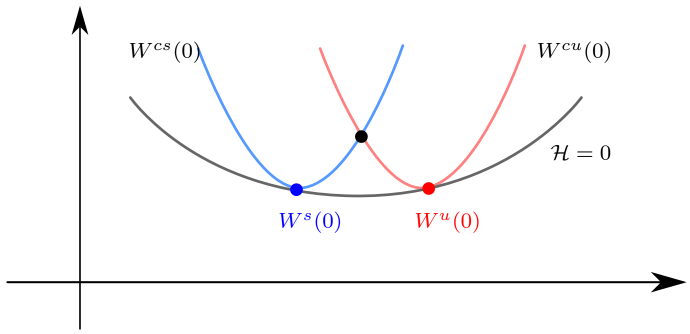

FIGURE 10. Intersection between $W^{cs}(0)$ and $W^{cu}(0)$ giving rise to a generalized breather.

Therefore, there exists $\widetilde{Q}$, e.g. on the segment connecting Q_u and Q_s, such that $\widetilde{h}^{cu}(\widetilde{Q},\varepsilon)=\widetilde{h}^{cs}(\widetilde{Q},\varepsilon)$ and thus

$$\big(\tilde{q}_1=\widetilde{h}^{cu}(\widetilde{Q},\varepsilon),0,\widetilde{Q}\big)\in \widetilde{W}^{cs}(0)\cap\widetilde{W}^{cu}(0)\subset W^{cu}(0)\cap W^{cs}(0).$$

This completes the proof of $W^{cs}(0)\cap W^{cu}(0)\neq\emptyset$, which had been also obtained in [41]. Moreover, (3.30) and (3.32) imply that such that

$$D\widetilde{h}^{c\star}(Q_\star,\varepsilon)=0,\quad D^2\widetilde{h}^{c\star}(Q_\star,\varepsilon)\geq \tfrac{1}{C}>0,\qquad \star=u,s.$$

Since Q_u and Q_s are exponentially close and the derivatives of $h^{c\star}$ are bounded uniformly in ε, we obtain the transversality of the intersection of $W^{cs}(0)\cap W^{cu}(0)$ near the above mentioned $\widetilde{Q}$ on the segment connecting Q_u and Q_s if $Q_u\neq Q_s$. See Figure 10.

Each orbit $\big(q(y),Q(y)\big)$ starting in $\widetilde{W}^{cs}(0)\cap\widetilde{W}^{cu}(0)$ is homoclinic to $W^c(0)$. That is, at $y\to\pm\infty$ it converges to two orbits in $W^c(0)$, which in the original coordinates (see (3.25)) can be written as $\big(v_c^\pm(y),\partial_y v_c^\pm(y)\big)\subset W^c(0)$. Moreover, by (3.27), they satisfy

$$\tfrac{1}{C}\mathcal{H}(q,Q)=\tfrac{1}{C}\mathcal{H}(v_c^\pm,\partial_y v_c^\pm)\leq \varepsilon^{-2}\|v_c^\pm\|_{H^1}^2+\|\partial_y v_c^\pm\|_{L^2}^2\leq C\mathcal{H}(v_c^\pm,\partial_y v_c^\pm)=C\mathcal{H}(q,Q).$$

According to (3.32), $\mathcal{H}\big(q(0),Q(0)\big)$ can be used as an equivalent measure between the $\big(q(0),Q(0)\big)$ and $\big(v^\star(0),\partial_y v^\star(0)\big)$, $\star=u,s$. For y on any finite interval, the distance between $\big(q(y),Q(y)\big)$ and $\big(v^\star(y),\partial_y v^\star(y)\big)$ is proportional to $\mathcal{H}\big(q(0),Q(0)\big)$ simply due to the uniform-in-ε boundedness on the derivatives of the flow maps. When $\big(q(y),Q(y)\big)$ is close to 0 (within a small $\mathcal{O}(1)$ distance), the distance between $\big(q(y),Q(y)\big)$ and $\big(v^\star(y),\partial_y v^\star(y)\big)$, where $\star=u$ for $y\ll 0$ and $\star=s$ for $y\gg 1$, can be estimate using the stable/unstable foliations which along with their derivatives are bounded uniformly in ε (see Sec. 5 of [42]). Combined with

$$\|Q\|_{H^1}^2+|q|^2\sim \big\|\varepsilon^{-1}|-\partial_\tau^2-\omega^{-2}|^{\frac{1}{2}}v\big\|_{L^2}^2+\|\partial_y v\|_{L^2}^2$$

uniformly in ε, this finishes the proof of (2.5).

Finally, we estimate $\inf\mathcal{H}$ on $\widetilde{W}^{cs}(0)\cap\widetilde{W}^{cu}(0)$. Let $Q\in Y_1$ such that $\|Q-Q_\star\|_{H^1}\leq\delta$ and $\big(q_1=\widetilde{h}^{cu}(Q,\varepsilon),0,Q\big)\in\widetilde{W}^{cs}\cap\widetilde{W}^{cu}$, then (3.32) implies

$$\|Q-Q_u\|_{H^1}\leq C\|Q-Q_s\|_{H^1}\ \text{ and }\ \|Q-Q_s\|_{H^1}\leq C\|Q-Q_u\|_{H^1},$$

which further yields

$$C\|Q-Q_\star\|_{H^1}\geq\|Q_u-Q_s\|_{H^1},\quad \star=u,s.$$

Taking into account (3.32) again, one obtains

$$\mathcal{H}\big(\widetilde{h}^{c\star}(Q,\varepsilon),0,Q,\varepsilon\big)\geq\frac{1}{C}\|Q_u-Q_s\|_{H^1}^2.$$

Moreover, if such Q is on the segment connecting Q_u and Q_s, one has

$$\mathcal{H}\big(\widetilde{h}^{c\star}(Q,\varepsilon),0,Q,\varepsilon\big)\leq C\|Q_u-Q_s\|_{H^1}^2$$

Therefore we obtain

$$C\|Q_u-Q_s\|_{H^1}^2\geq\inf_{\widetilde{W}^{cu}(0)\cap\widetilde{W}^{cs}(0)}\mathcal{H}\geq\frac{1}{C}\|Q_u-Q_s\|_{H^1}^2,$$

and item (5) of Theorem 2.1 follows from the above argument.

4. Estimates of the invariant manifolds: Proof of Theorem 3.1

4.1. **Banach Spaces and Linear Operators.** In this section we prove Theorem 3.1 through a fixed point argument in some appropriate Banach spaces. We consider only the unstable case, since the stable one is completely analogous.

Given $\kappa \geq 1$ and a real-analytic function $h : D_\kappa^{\text{out},u} \to \mathbb{C}$ (see (3.5)), we define

$$\|h\|_{m,\alpha} = \sup_{y \in D_\kappa^{\text{out},u} \cap \{\text{Re}(y) \leq -1\}} |\cosh(y)^m h(y)| + \sup_{y \in D_\kappa^{\text{out},u} \cap \{\text{Re}(y) \geq -1\}} |(y^2 + \pi^2/4)^\alpha h(y)|, \tag{4.1}$$

and given a function $\xi : D_\kappa^{\text{out},u} \times \mathbb{T} \to \mathbb{C}$ which is real analytic in $y \in D_\kappa^{\text{out},u}$, we define

$$\|\xi\|_{\ell_1,m,\alpha} = \sum_{n \geq 1} \|\Pi_n[\xi]\|_{m,\alpha}.$$

and the Banach spaces

$$\mathcal{E}_{m,\alpha} = \{\xi : D_\kappa^{\text{out},u} \to \mathbb{C};\ \xi \text{ is real-analytic in } y, \text{ and } \|\xi\|_{m,\alpha} < \infty\}$$
$$\mathcal{E}_{\ell_1,m,\alpha} = \{\xi : D_\kappa^{\text{out},u} \times \mathbb{T} \to \mathbb{C};\ \xi(y,\tau) \text{ is real-analytic in } y \text{ and } \|\xi\|_{\ell_1,m,\alpha} < \infty\}.$$

Lemma 4.1. *There exists $M > 0$ depending only on β such that, for any $g, h : D_\kappa^{\text{out},u} \times \mathbb{T} \to \mathbb{C}$, it holds*

(1) *If $\alpha_2 \geq \alpha_1 \geq 0$, then*

$$\|h\|_{\ell_1,m,\alpha_2} \leq M\|h\|_{\ell_1,m,\alpha_1} \quad \textit{and} \quad \|h\|_{\ell_1,m,\alpha_1} \leq \frac{M}{(\kappa\varepsilon)^{\alpha_2-\alpha_1}}\|h\|_{\ell_1,m,\alpha_2}.$$

(2) *If $\alpha_1, \alpha_2 \geq 0$, and $\|g\|_{\ell_1,m_1,\alpha_1}, \|h\|_{\ell_1,m_2,\alpha_2} < \infty$, then*

$$\|gh\|_{\ell_1,m_1+m_2,\alpha_1+\alpha_2} \leq \|g\|_{\ell_1,m_1,\alpha_1}\|h\|_{\ell_1,m_2,\alpha_2}.$$

This lemma actually applies to general functions 2π-periodic in τ, not just to odd functions. The proof of this lemma is straight forward and we omit it.

Firstly to solve the linear equation $\mathcal{L}\xi = h$, we introduce the operator $\mathcal{G}(h)$ acting on the Fourier coefficients of h as

$$\mathcal{G}(h) = \sum_{n \geq 1} \mathcal{G}_n(h_n)\sin(n\tau), \quad \widetilde{\mathcal{G}}(h) = \sum_{n \geq 2} \mathcal{G}_n(h_n)\sin(n\tau) = \widetilde{\Pi}[\mathcal{G}(h)],$$

with

$$\mathcal{G}_1(h_1) = -\zeta_1(y)\int_0^y \zeta_2(s)h_1(s)ds + \zeta_2(y)\int_{-\infty}^y \zeta_1(s)h_1(s)ds \tag{4.2}$$

$$\mathcal{G}_n(h_n) = -\frac{i\varepsilon}{2\lambda_n}e^{i\frac{\lambda_n}{\varepsilon}y}\int_{-\infty}^y e^{-i\frac{\lambda_n}{\varepsilon}s}h_n(s)ds + \frac{i\varepsilon}{2\lambda_n}e^{-i\frac{\lambda_n}{\varepsilon}y}\int_{-\infty}^y e^{i\frac{\lambda_n}{\varepsilon}s}h_n(s)ds,\ n \geq 2. \tag{4.3}$$

where

$$\zeta_1(y) = -2\sqrt{2}\frac{\sinh(y)}{\cosh^2(y)} \quad \textit{and} \quad \zeta_2(y) = -\frac{\sqrt{2}}{16}\frac{\sinh(y)}{\cosh^2(y)}(6y - 4\coth(y) + \sinh(2y)), \tag{4.4}$$

are linearly independent solutions of

$$\ddot{\zeta} - \zeta + \frac{3(v^h)^2}{4}\zeta = 0 \qquad \text{(see (3.1))}.$$

Remark 4.2. *When $-\infty$ is involved in the above integrals, it should be understood that the integral is along horizontal lines. As the integrands are analytic functions, integral paths may be modified to yield better estimates in certain cases.*

Proposition 4.3. *The following statements hold.*

(1) $\partial_y\Pi_1(\mathcal{G}(\xi))(0) = 0$.
(2) $\mathcal{G} \circ \mathcal{L}(\xi) = \mathcal{L} \circ \mathcal{G}(\xi) = \xi$.
(3) *For any $m > 1$ and $\alpha \geq 5$, there exists a constant $M > 0$ independent of ε and κ such that, for every $h \in \mathcal{E}_{m,\alpha}$,*

$$\|\mathcal{G}_1(h)\|_{1,\alpha-2} + \|\partial_y\mathcal{G}_1(h)\|_{1,\alpha-1} \leq M\|h\|_{m,\alpha}.$$

(4) *For any $m \geq 1$, $\alpha \geq 0$, there exists $M > 0$ such that for every $n \geq 2$ and $h \in \mathcal{E}_{m,\alpha}$,*

$$\|\mathcal{G}_n(h)\|_{m,\alpha} \leq M\frac{\varepsilon^2}{\lambda_n^2}\|h\|_{m,\alpha}, \quad \|\partial_y \mathcal{G}_n(h)\|_{m,\alpha} \leq M\frac{\varepsilon}{\lambda_n}\|h\|_{m,\alpha}, \quad \|\partial_y \mathcal{G}_n(h)\|_{m,\alpha} \leq M\frac{\varepsilon^2}{\lambda_n^2}\|\partial_y h\|_{m,\alpha}.$$

The proof of this proposition is deferred to Appendix A. In particular, the last item indicates a gain of an extra order regularity in τ for $\mathcal{G}(\xi)$ compared to general solution to wave equations and an improvement in the estimate of $\partial_y \mathcal{G}_n(h)$ when $\partial_y h \in \mathcal{E}_{m,\alpha}$, which is a typical trading between the smoothness and the smallness in problems involving rapid oscillations.

4.2. **Fixed Point Argument.** Now, we use Proposition 4.3 to rewrite (3.3) as $\xi = \mathcal{G} \circ \mathcal{F}(\xi)$, where $\mathcal{F}$ is given in (3.2). We analyze the operator

$$\mathcal{F}^\sharp = \mathcal{G} \circ \mathcal{F}$$

defined on the closed ball

$$\mathcal{B}_0(R\varepsilon^2) = \left\{\xi \in \mathcal{E}_{\ell_1,1,3} \mid \|\xi\|_{\ell_1,1,3} + \|\partial_y \xi\|_{\ell_1,1,4} \leq R\varepsilon^2\right\}$$

for some $R > 0$.

Proposition 4.4. *There exists $M, \kappa_0, \varepsilon_0 > 0$, such that, if $\varepsilon \in (0, \varepsilon_0)$, $R > 0$, and $\kappa > \kappa_0 R^{\frac{1}{2}}$, then the operator*

$$\mathcal{F}^\sharp : \mathcal{E}_{\ell^1,1,3} \supset \mathcal{B}_0(R\varepsilon^2) \to \mathcal{E}_{\ell^1,1,3}$$

is well defined and satisfies

$$\|\partial_\tau^2 \mathcal{F}^\sharp(0)\|_{\ell_1,1,3} + \|\partial_\tau^2 \partial_y \mathcal{F}^\sharp(0)\|_{\ell_1,1,4} \leq M\varepsilon^2,$$

$$\|\partial_\tau^2 \widetilde{\Pi}[\mathcal{F}^\sharp(\xi) - \mathcal{F}^\sharp(\xi')]\|_{\ell_1,1,3} + \|\partial_\tau^2 \partial_y \widetilde{\Pi}[\mathcal{F}^\sharp(\xi) - \mathcal{F}^\sharp(\xi')]\|_{\ell_1,1,4} \leq \frac{M}{\kappa^2}\left(\|\xi - \xi'\|_{\ell_1,1,3} + \|\partial_y \xi - \partial_y \xi'\|_{\ell_1,1,4}\right),$$

$$\begin{aligned}&\|\Pi_1[\mathcal{F}^\sharp(\xi) - \mathcal{F}^\sharp(\xi')]\|_{1,3} + \|\partial_y \Pi_1\left[\mathcal{F}^\sharp(\xi) - \widetilde{\mathcal{F}}(\xi')\right]\|_{1,4} \\ &\quad \leq M\frac{1+R}{\kappa^2}\left(\|\xi - \xi'\|_{\ell_1,1,3} + \|\partial_y \xi - \partial_y \xi'\|_{\ell_1,1,4}\right) + M\left(\|\widetilde{\Pi}[\xi - \xi']\|_{\ell_1,1,3} + \|\partial_y \widetilde{\Pi}[\xi - \xi']\|_{\ell_1,1,4}\right).\end{aligned}$$

Notice that the above bounds on $\partial_\tau^2 \mathcal{F}^\sharp(\xi)$ immediately implies those on $\mathcal{F}^\sharp(\xi)$ as the zeroth mode is not included.

Proof. First, we rewrite the operator $\mathcal{F}$ given in (3.2), in order to make explicit some cancellations. Recall that $g(u) = u^3/3 + f(u)$ is given by (1.11). Then,

$$\begin{aligned}\mathcal{F}(\xi) &= -\frac{1}{\varepsilon^3\omega^3} g(\varepsilon\omega(\xi + v^h \sin\tau)) + \left(\frac{(\xi_1 + v^h)^3}{4} - \frac{3v^h \xi_1^2}{4} - \frac{\xi_1^3}{4}\right)\sin\tau \\ &= -\frac{1}{\varepsilon^3\omega^3}\widetilde{\Pi}\left[g(\varepsilon\omega(\xi + v^h \sin\tau))\right] + \left\{-\frac{1}{3}\Pi_1\left[\left((\xi_1 + v^h)\sin(\tau) + \widetilde{\Pi}(\xi)\right)^3\right]\right. \\ &\quad \left. -\frac{1}{\varepsilon^3\omega^3}\Pi_1\left[f(\varepsilon\omega(\xi + v^h \sin\tau))\right] + \frac{(\xi_1 + v^h)^3}{4} - \frac{3v^h \xi_1^2}{4} - \frac{\xi_1^3}{4}\right\}\sin\tau \\ &= -\frac{1}{\varepsilon^3\omega^3}\widetilde{\Pi}\left[g(\varepsilon\omega(\xi + v^h \sin\tau))\right] + \left\{-\frac{1}{3}\Pi_1\left[(\xi_1 + v^h)^3 \sin^3\tau + 3(\xi_1 + v^h)^2 \sin^2\tau \widetilde{\Pi}[\xi]\right.\right. \\ &\quad \left. + 3(\xi_1 + v^h)\sin\tau(\widetilde{\Pi}[\xi])^2 + (\widetilde{\Pi}[\xi])^3\right] - \frac{1}{\varepsilon^3\omega^3}\Pi_1\left[f(\varepsilon\omega(\xi + v^h \sin\tau))\right] \\ &\quad \left. + \frac{(\xi_1 + v^h)^3}{4} - \frac{3v^h \xi_1^2}{4} - \frac{\xi_1^3}{4}\right\}\sin\tau.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{F}(\xi) = &-\frac{1}{\varepsilon^3\omega^3}\widetilde{\Pi}\left[g(\varepsilon\omega(\xi + v^h \sin\tau))\right] + \left\{\Pi_1\left[-(\xi_1 + v^h)^2(\sin^2\tau)\widetilde{\Pi}[\xi] - (\xi_1 + v^h)(\sin\tau)(\widetilde{\Pi}[\xi])^2\right.\right. \\ &\left.\left. -\frac{1}{3}(\widetilde{\Pi}[\xi])^3\right] - \frac{1}{\varepsilon^3\omega^3}\Pi_1\left[f(\varepsilon\omega(\xi + v^h \sin\tau))\right] - \frac{3v^h \xi_1^2}{4} - \frac{\xi_1^3}{4}\right\}\sin\tau.\end{aligned} \tag{4.5}$$

which implies

$$\mathcal{F}(0) = -\frac{1}{\varepsilon^3\omega^3}\widetilde{\Pi}\left[g(\varepsilon\omega(v^h\sin\tau))\right] - \frac{1}{\varepsilon^3\omega^3}\Pi_1\left[f(\varepsilon\omega(v^h\sin\tau))\right]\sin\tau.$$

Let g and f have the power series expansion

$$g(u) = \sum_{d=1}^{\infty} g_{2d+1}u^{2d+1}, \quad f(u) = \sum_{d=2}^{\infty} g_{2d+1}u^{2d+1}, \quad g_3 = \frac{1}{3},$$

with a positive radius of convergence. Using Lemma 4.1 and Proposition 4.3, one may estimate

$$\begin{aligned}\|\partial_\tau^2\widetilde{\mathcal{G}}\mathcal{F}(0)\|_{\ell_1,1,3} &\lesssim \varepsilon^2\|\widetilde{\Pi}\mathcal{F}(0)\|_{\ell_1,1,3} \lesssim \varepsilon^2\sum_{d=1}^{\infty}(\varepsilon\omega)^{2d-2}|g_{2d+1}|\|(v^h\sin\tau)^{2d+1}\|_{\ell_1,1,3} \\ &\lesssim \varepsilon^2\sum_{d=1}^{\infty}\left(\frac{\omega}{\kappa}\right)^{2d-2}|g_{2d+1}|\|(v^h\sin\tau)^{2d+1}\|_{\ell_1,2d+1,2d+1} \lesssim \varepsilon^2\sum_{d=1}^{\infty}\left(\frac{\omega}{\kappa}\right)^{2d-2}|g_{2d+1}|\|v^h\|_{1,1}^{2d+1} \lesssim \varepsilon^2,\end{aligned}$$

for reasonably large κ. In particular, in the above the operator ∂_τ^2 creates a Fourier multiplier of n^2 to the mode of $\sin n\tau$, which is cancelled by the λ_n^{-2} in the estimate of $\mathcal{G}_n$ in Proposition 4.3. In order to obtain the desired estimate on $\|\partial_y\widetilde{\mathcal{G}}\mathcal{F}(0)\|_{\ell_1,1,4}$, we also need

$$\partial_y\mathcal{F}(0) = -\frac{1}{\varepsilon^2\omega^2}g'(\varepsilon\omega(v^h\sin\tau))(v^h)'\sin\tau = -\sum_{d=1}^{\infty}(2d+1)(\varepsilon\omega)^{2d-2}g_{2d+1}(v^h)^{2d}\partial_y v^h\sin^{2d+1}\tau$$

which implies

$$\begin{aligned}\|\partial_y\mathcal{F}(0)\|_{\ell_1,1,4} &\lesssim \sum_{d=1}^{\infty}(2d+1)(\varepsilon\omega)^{2d-2}|g_{2d+1}|\|(v^h)^{2d}\partial_y v^h\sin^{2d+1}\tau\|_{\ell_1,1,4} \\ &\lesssim \sum_{d=1}^{\infty}(2d+1)\left(\frac{\omega}{\kappa}\right)^{2d-2}|g_{2d+1}|\|(v^h)^{2d}\partial_y v^h\sin^{2d+1}\tau\|_{\ell_1,2d+1,2d+2} \\ &\lesssim \sum_{d=1}^{\infty}(2d+1)\left(\frac{\omega}{\kappa}\right)^{2d-2}|g_{2d+1}|\|v^h\|_{1,1}^{2d}\|\partial_y v^h\|_{1,2} \lesssim 1,\end{aligned}$$

for reasonably large κ. Hence the estimates related to $\|\cdot\|_{\ell_1,1,4}$ estimate related to $\partial_y\widetilde{\mathcal{G}}\mathcal{F}(0)$ follows from Proposition 4.3. Again, using Lemma 4.1 and Proposition 4.3, one may also estimate

$$\begin{aligned}\|\mathcal{G}_1\Pi_1\mathcal{F}(0)\|_{1,3} &\lesssim \|\Pi_1\mathcal{F}(0)\|_{3,5} \lesssim \sum_{d=2}^{\infty}(\varepsilon\omega)^{2d-2}|g_{2d+1}|\|(v^h)^{2d+1}\|_{3,5} \\ &\lesssim (\varepsilon\omega)^2\sum_{d=2}^{\infty}\left(\frac{\omega}{\kappa}\right)^{2d-4}|g_{2d+1}|\|(v^h)^{2d+1}\|_{2d+1,2d+1} \lesssim (\varepsilon\omega)^2\sum_{d=2}^{\infty}\left(\frac{\omega}{\kappa}\right)^{2d-4}|g_{2d+1}|\|v^h\|_{1,1}^{2d+1} \lesssim \varepsilon^2,\end{aligned}$$

for reasonably large κ. The estimate on $\partial_y\mathcal{G}\Pi_1[\mathcal{F}(0)]$ is obtained in a similar fashion. The sum of these inequalities imply the estimate on $\mathcal{F}^\sharp(0)$.

To estimate the Lipschitz constant of $\mathcal{F}^\sharp$, let $\xi, \xi' \in \mathcal{B}_0(R\varepsilon^2)$, we have

$$\begin{aligned}\mathcal{F}(\xi) - \mathcal{F}(\xi') = &-\frac{1}{(\varepsilon\omega)^3}\widetilde{\Pi}\left[g(\varepsilon\omega(\xi + v^h\sin\tau)) - g(\varepsilon\omega(\xi' + v^h\sin\tau))\right] \\ &+ \Bigg\{ -\Pi_1\left[(\xi_1 + v^h)^2(\sin^2\tau)(\widetilde{\Pi}[\xi] - \widetilde{\Pi}[\xi']) - \left((\xi_1 + v^h)^2 - (\xi_1' + v^h)^2\right)(\sin^2\tau)\widetilde{\Pi}[\xi']\right] \\ &- \Pi_1\left[(\xi_1 + v^h)(\sin\tau)(\widetilde{\Pi}[\xi]^2 - \widetilde{\Pi}[\xi']^2) - (\xi_1 - \xi_1')(\sin\tau)\widetilde{\Pi}[\xi']^2\right] - \frac{1}{3}\Pi_1\left[\widetilde{\Pi}[\xi]^3 - \widetilde{\Pi}[\xi']^3\right] \\ &- \frac{1}{(\varepsilon\omega)^3}\Pi_1\left[f(\varepsilon\omega(\xi + v^h\sin\tau)) - f(\varepsilon\omega(\xi' + v^h\sin\tau))\right] - \frac{3v^h(\xi_1^2 - (\xi_1')^2)}{4} - \frac{\xi_1^3 - (\xi_1')^3}{4}\Bigg\}\sin\tau.\end{aligned}$$

For any $d \geq 2$, $m \geq 0$, $\alpha \geq 0$, and $\zeta, \zeta' \in \mathcal{E}_{\ell_1,m,\alpha}$, it is straight forward to estimate

$$\|\zeta^d - (\zeta')^d\|_{\ell_1,dm,d\alpha} \lesssim d(\|\zeta\|_{\ell_1,m,\alpha}^{d-1} + \|\zeta'\|_{\ell_1,m,\alpha}^{d-1})\|\zeta - \zeta'\|_{\ell_1,m,\alpha}$$

where the constant is independent of d. Another useful inequality is

$$\|\xi\|_{\ell_1,1,1} + \|\partial_y \xi\|_{\ell_1,1,2} \lesssim (\kappa\varepsilon)^{-2}(\|\xi\|_{\ell_1,1,3} + \|\partial_y \xi\|_{\ell_1,1,4}) \lesssim \frac{R}{\kappa^2} \lesssim 1, \quad \xi \in \mathcal{B}_0(R\varepsilon^2).$$

Hence one may use Lemma 4.1 and Proposition 4.3 to estimate

$$\begin{aligned}
\|\partial_\tau^2 \widetilde{\mathcal{G}}[\mathcal{F}(\xi) - \mathcal{F}(\xi')]\|_{\ell_1,1,3} &\lesssim \varepsilon^2 \|\widetilde{\Pi}[\mathcal{F}(\xi) - \mathcal{F}(\xi')]\|_{\ell_1,1,3} \\
&\lesssim \varepsilon^2 \sum_{d=1}^{\infty} (\varepsilon\omega)^{2d-2} |g_{2d+1}| \|(\xi + v^h \sin\tau)^{2d+1} - (\xi' + v^h \sin\tau)^{2d+1}\|_{\ell_1,1,3} \\
&\lesssim \varepsilon^2 \sum_{d=1}^{\infty} \left(\frac{\omega}{\kappa}\right)^{2d-2} |g_{2d+1}| \|(\xi + v^h \sin\tau)^{2d+1} - (\xi' + v^h \sin\tau)^{2d+1}\|_{\ell_1,2d+1,2d+1} \\
&\lesssim \varepsilon^2 \sum_{d=1}^{\infty} d \left(\frac{\omega}{\kappa}\right)^{2d-2} |g_{2d+1}| (1 + \|\xi\|_{\ell_1,1,1}^{2d} + \|\xi'\|_{\ell_1,1,1}^{2d}) \|\xi - \xi'\|_{\ell_1,1,1} \\
&\lesssim \kappa^{-2} \sum_{d=1}^{\infty} d \left(\frac{\omega}{\kappa}\right)^{2d-2} |g_{2d+1}| \|\xi - \xi'\|_{\ell_1,1,3} \lesssim \kappa^{-2} \|\xi - \xi'\|_{\ell_1,1,3}
\end{aligned}$$

AQUI for $\kappa \geq R$ reasonably large. To estimates $\partial_y \widetilde{\mathcal{G}}[\mathcal{F}(\xi) - \mathcal{F}(\xi')]$, in a similar fashion one needs to compute

$$\begin{aligned}
&\|\partial_y\big(\mathcal{F}(\xi) - \mathcal{F}(\xi')\big)\|_{\ell_1,1,4} \\
&\quad \lesssim \sum_{d=1}^{\infty} d (\varepsilon\omega)^{2d-2} |g_{2d+1}| \Big\|(\xi + v^h \sin\tau)^{2d}(\partial_y \xi + \partial_y \sin\tau) - (\xi' + v^h \sin\tau)^{2d}(\partial_y \xi' + \partial_y \sin\tau)\Big\|_{\ell_1,1,4} \\
&\quad \lesssim \sum_{d=1}^{\infty} d \left(\frac{\omega}{\kappa}\right)^{2d-2} |g_{2d+1}| \Big\|(\xi + v^h \sin\tau)^{2d}(\partial_y \xi + \partial_y v^h \sin\tau) \\
&\qquad\qquad - (\xi' + v^h \sin\tau)^{2d}(\partial_y \xi' + \partial_y v^h \sin\tau)\Big\|_{\ell_1,2d+1,2d+2} \\
&\quad \lesssim \|\xi - \xi'\|_{\ell_1,1,1} + \|\partial_y \xi - \partial_y \xi'\|_{\ell_1,1,2} \lesssim (\kappa\varepsilon)^{-2}(\|\xi - \xi'\|_{\ell_1,1,3} + \|\partial_y \xi - \partial_y \xi'\|_{\ell_1,1,4}),
\end{aligned}$$

where in the derivation of the third $\lesssim$ we applied $\|\cdot\|_{\ell_1,1,1}$ norm to all ξ, ξ', and v^h and $\|\cdot\|_{\ell_1,1,2}$ norm to all $\partial_y \xi$, $\partial_y \xi'$, and $\partial_y v^h$. Along with Proposition 4.3 this inequality yields the desired estimate on $\partial_y \widetilde{\mathcal{G}}[\mathcal{F}(\xi) - \mathcal{F}(\xi')]$. The $\mathcal{G}_1$ component can be estimated much as in the above. In fact,

$$\begin{aligned}
\|\mathcal{G}_1 \Pi_1[\mathcal{F}(\xi) - \mathcal{F}(\xi')]\|_{1,3} + \|\partial_y \mathcal{G}_1 \Pi_1[\mathcal{F}(\xi) - \mathcal{F}(\xi')]\|_{1,4} &\lesssim \|\Pi_1[\mathcal{F}(\xi) - \mathcal{F}(\xi')]\|_{3,5} \\
&\lesssim \frac{1}{(\varepsilon\omega)^3} \|f(\varepsilon\omega(\xi + v^h \sin\tau)) - f(\varepsilon\omega(\xi' + v^h \sin\tau))\|_{\ell_1,3,5} + \|\widetilde{\Pi}[\xi] - \widetilde{\Pi}[\xi']\|_{\ell_1,1,3} \\
&\quad + (\|\xi\|_{\ell_1,1,1} + \|\xi\|_{\ell_1,1,1}^2 + \|\xi'\|_{\ell_1,1,1} + \|\xi'\|_{\ell_1,1,1}^2) \|\xi - \xi'\|_{\ell_1,1,3}
\end{aligned}$$

where all the ξ, ξ', and $v^h \sin\tau$ in front of $\xi - \xi'$ were taken the $\|\cdot\|_{\ell_1,1,1}$ norm. The f terms can be estimated much as in the above

$$\begin{aligned}
&\frac{1}{(\varepsilon\omega)^3} \|f(\varepsilon\omega(\xi + v^h \sin\tau)) - f(\varepsilon\omega(\xi' + v^h \sin\tau))\|_{\ell_1,3,5} \\
&\quad \lesssim \sum_{d=2}^{\infty} (\varepsilon\omega)^{2d-2} |g_{2d+1}| (\kappa\varepsilon)^{-(2d-4)} \|(\xi + v^h \sin\tau)^{2d+1} - (\xi' + v^h \sin\tau)^{2d+1}\|_{\ell_1,2d+1,2d+1} \\
&\quad \lesssim \kappa^{-2} \sum_{d=1}^{\infty} d (\frac{\omega}{\kappa})^{2d-4} |g_{2d+1}| \|\xi - \xi'\|_{\ell_1,1,3} \lesssim \kappa^{-2} \|\xi - \xi'\|_{\ell_1,1,3}
\end{aligned}$$

for $\kappa \geq R$ reasonably large. Summarizing the above estimates, the proposition follows. □

With the above preparations, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We claim that, if κ is sufficiently large, then $\mathcal{F}^\sharp$ is a contraction on the set

$$S=\{\xi\in\mathcal{E}_{\ell_1,1,3}\ |\|\Pi_1[\xi]\|_{1,3}+\|\partial_y\Pi_1[\xi]\|_{1,4}\leq(1+M)^2\varepsilon^2,$$
$$\|\widetilde{\Pi}[\xi]\|_{\ell_1,1,3}+\|\partial_y\widetilde{\Pi}[\xi]\|_{\ell_1,1,4}\leq(1+M)\varepsilon^2\}\subset\mathcal{B}_0(R\varepsilon^2),\quad R=(1+M)(2+M),$$

equipped with the metric

$$|\xi|_M:=\|\Pi_1[\xi]\|_{1,3}+\|\partial_y\Pi_1[\xi]\|_{1,4}+(1+M)\big(\|\widetilde{\Pi}[\xi]\|_{\ell_1,1,3}+\|\partial_y\widetilde{\Pi}[\xi]\|_{\ell_1,1,4}\big),$$

where M is the constant from Proposition 4.4. In fact, using Proposition 4.4 it is straight forward to estimate that, for any $\xi\in S$,

$$\begin{aligned}\|\Pi_1[\mathcal{F}^\sharp(\xi)]\|_{1,3}+\|\partial_y\Pi_1[\mathcal{F}^\sharp(\xi)]\|_{1,4}\leq&\|\mathcal{F}^\sharp(0)\|_{\ell_1,1,3}+\|\partial_y\mathcal{F}^\sharp(0)\|_{\ell_1,1,4}+M\frac{1+R}{\kappa^2}R\varepsilon^2+M(1+M)\varepsilon^2\\ \leq&\left(M+M\frac{1+R}{\kappa^2}R+M(1+M)\right)\varepsilon^2\leq(1+M)^2\varepsilon^2,\\ \|\widetilde{\Pi}[\mathcal{F}^\sharp(\xi)]\|_{\ell_1,1,3}+\|\partial_y\widetilde{\Pi}[\mathcal{F}^\sharp(\xi)]\|_{\ell_1,1,4}\leq&\|\mathcal{F}^\sharp(0)\|_{\ell_1,1,3}+\|\partial_y\mathcal{F}^\sharp(0)\|_{\ell_1,1,4}+\frac{M}{\kappa^2}R\varepsilon^2\\ \leq&\left(M+\frac{M}{\kappa^2}R\right)\varepsilon^2\leq(1+M)\varepsilon^2,\end{aligned}$$

and for any $\xi,\xi'\in S$,

$$\begin{aligned}|\mathcal{F}^\sharp(\xi)-\mathcal{F}^\sharp(\xi')|_M\leq&\left(M\frac{1+R}{\kappa^2}+(1+M)\frac{M}{\kappa^2}\right)\big(\|\Pi_1[\xi-\xi']\|_{1,3}+\|\widetilde{\Pi}[\xi-\xi']\|_{\ell_1,1,3}+\|\partial_y\Pi_1[\xi-\xi']\|_{1,4}\\ &+\|\partial_y\widetilde{\Pi}[\xi-\xi']\|_{\ell_1,1,4}\big)+M(\|\widetilde{\Pi}[\xi-\xi']\|_{\ell_1,1,3}+\|\partial_y\widetilde{\Pi}[\xi-\xi']\|_{\ell_1,1,4})\\ \leq&\left(M\frac{1+R}{\kappa^2}+(1+M)\frac{M}{\kappa^2}+\frac{M}{1+M}\right)|\xi-\xi'|_M.\end{aligned}$$

Therefore our above claim holds if κ is large and $\mathcal{F}^\sharp$ has a unique fixed point $\xi^u\in S\subset\mathcal{B}_0(R\varepsilon^2)$. It clearly satisfies all desired properties in Theorem 3.1. Using that g given in (1.11) is an odd function, a straightforward computation shows that the operator $\mathcal{F}$ in (3.2) leaves invariant the subspace of functions $\xi:D^{\mathrm{out},u}_\kappa\times\mathbb{T}\to\mathbb{C}$ satisfying $\Pi_{2l}[\xi]=0$, $\forall l\geq0$. Consequently, ξ^u satisfies that $\Pi_{2l}[\xi^u]=0$, $\forall l\geq0$ which completes the proof of Theorem (3.1).

5. The Inner equation: Proof of Theorem 3.3

We look for solutions odd in τ of the inner equation (3.9) as

$$\phi^0=\sum_{n\geq1}\phi^0_n\sin(n\tau).\tag{5.1}$$

Replacing (5.1) in (3.9), we obtain that

$$(\partial_z^2+(n^2-1))\phi^0_n+\Pi_n\left[\frac{1}{3}(\phi^0)^3+f(\phi^0)\right]=0,\qquad n\geq1.\tag{5.2}$$

As explained in Section 3, we look for solutions of the form

$$\phi^0(z,\tau)=\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi(z,\tau)\qquad\text{with}\qquad\psi=\mathcal{O}\left(\frac{1}{z^3}\right).\tag{5.3}$$

Then, by (5.2), $\psi(z,\tau)=\sum_{n\geq1}\psi_n(z)\sin(n\tau)$ must satisfy

$$\left\{\begin{aligned}&\partial_z^2\psi_1-\frac{6}{z^2}\psi_1=-\Pi_1\left[-\frac{8}{z^2}\sin^2(\tau)\widetilde{\Pi}\left[\psi\right]-\frac{2\sqrt{2}i}{z}\sin(\tau)\psi^2+\frac{1}{3}\psi^3+f\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\right)\right],\\ &\partial_z^2\psi_n+\mu_n^2\psi_n=-\Pi_n\left[\frac{1}{3}\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\right)^3+f\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\right)\right],\ n\geq2,\end{aligned}\right.\tag{5.4}$$

where $'=d/dz$, and $\mu_n=\sqrt{n^2-1}$.

We define the operators

$$\mathcal{I}(\psi) = \left(\partial_z^2\psi_1 - \frac{6}{z^2}\psi_1\right)\sin(\tau) + \sum_{n\geq 2}\left(\partial_z^2\psi_n + \mu_n^2\psi_n\right)\sin(n\tau) \tag{5.5}$$

$$\begin{aligned}\mathcal{W}(\psi) = &-\Pi_1\left[-\frac{8}{z^2}\sin^2(\tau)\widetilde{\Pi}\left[\psi\right] - \frac{2\sqrt{2}i}{z}\sin(\tau)\psi^2 + \frac{1}{3}\psi^3\right]\sin(\tau)\\ &-\widetilde{\Pi}\left[\frac{1}{3}\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\right)^3\right] - f\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)+\psi\right)\end{aligned} \tag{5.6}$$

and notice that, for $\star = u, s$, to find a solution $\phi^{0,\star}$ of (3.9) satisfying (5.3) is equivalent to find a solution $\psi^\star$ of the functional equation

$$\mathcal{I}(\psi) = \mathcal{W}(\psi), \tag{5.7}$$

which satisfies $\psi^* \sim \mathcal{O}(z^{-3})$ for $z \in D^{*,\text{in}}_{\theta,\kappa}$, $* = u, s$. In the remainder of this section, we look for solutions of (5.7) with such asymptotics through a fixed point argument. As before, we consider only the unstable case, since the stable one is completely analogous.

5.1. **Banach Spaces and Linear Operators.** Given $\alpha \geq 0$ and an analytic function $f : D^{u,\text{in}}_{\theta,\kappa} \to \mathbb{C}$, where $D^{u,\text{in}}_{\theta,\kappa}$ is given in (3.10), consider the norm

$$\|f\|_\alpha = \sup_{z\in D^{u,\text{in}}_{\theta,\kappa}} |z^\alpha f(z)|,$$

and the Banach space

$$\mathcal{X}_\alpha = \{f : D^{u,\text{in}}_{\theta,\kappa} \to \mathbb{C};\ f \text{ is an analytic function and } \|f\|_\alpha < \infty\}.$$

Moreover, for $f : D^{u,\text{in}}_{\theta,\kappa} \times \mathbb{T} \to \mathbb{C}$, analytic in the variable z, we define

$$\|f\|_{\ell_1,\alpha} = \sum_{n\geq 1} \|f_n\|_\alpha,$$

and the Banach space

$$\mathcal{X}_{\ell_1,\alpha} = \left\{f : D^{u,\text{in}}_{\theta,\kappa} \times \mathbb{T} \to \mathbb{C};\ f \text{ is an analytic function in the variable } z \text{ and } \|f\|_{\ell_1,\alpha} < \infty\right\}.$$

Lemma 5.1. *Given an analytic function $f : B(R_0) \to \mathbb{C}$, and $g, h : D^{u,\text{in}}_{\theta,\kappa} \times \mathbb{T} \to \mathbb{C}$, where $B(R_0) \subset \mathbb{C}$ is a ball with center at the origin and radius R_0, the following statements hold for some M depending only on θ and f,*

(1) *If $\alpha \geq \beta \geq 0$, then*
$$\|h\|_{\ell_1,\alpha-\beta} \leq \frac{M}{\kappa^\beta}\|h\|_{\ell_1,\alpha}.$$
(2) *If $\alpha, \beta \geq 0$, and $\|g\|_{\ell_1,\alpha}, \|h\|_{\ell_1,\beta} < \infty$, then*
$$\|gh\|_{\ell_1,\alpha+\beta} \leq \|g\|_{\ell_1,\alpha}\|h\|_{\ell_1,\beta}.$$
(3) *If $\alpha \geq 0$, $f, g \in \mathcal{X}_{\ell_1,\alpha}$ and $\|g\|_{\ell_1,0}$, $\|h\|_{\ell_1,0} \leq R_0/4$, then*
$$\|f(g) - f(h)\|_{\ell_1,\alpha} \leq M\|g - h\|_{\ell_1,\alpha}.$$
(4) *Given $n \geq 0$, if $f^{(k)}(0) = 0$, for every $0 \leq k \leq n-1$, and $\|g\|_{\ell_1,\alpha} \leq R_0/4$, then*
$$\|f(g)\|_{\ell_1,n\alpha} \leq M(\|g\|_{\ell_1,\alpha})^n.$$
(5) *If $h \in \mathcal{X}_{\ell_1,\alpha}$ (with respect to the inner domain $D^{u,\text{in}}_{\theta,\kappa}$), then $\partial_z h \in \mathcal{X}_{\ell_1,\alpha+1}$ (with respect to the inner domain $D^{u,\text{in}}_{2\theta,4\kappa}$), and*
$$\|\partial_z h\|_{\ell_1,\alpha+1} \leq M\|h\|_{\ell_1,\alpha}.$$

This lemma is proved in [3].

Now, define the linear operator acting on the Fourier coefficients of ψ

$$\mathcal{J}(\psi) = \sum_{n\geq 1} \mathcal{J}_n(\psi_n)\sin(n\tau),$$

where

$$\begin{aligned} \mathcal{J}_1(\psi_1)(z) &= \frac{z^3}{5}\int_{-\infty}^{z}\frac{\psi_1(s)}{s^2}ds - \frac{1}{5z^2}\int_{-\infty}^{z} s^3\psi_1(s)ds \\ \mathcal{J}_n(\psi_n)(z) &= \frac{1}{2i\mu_n}\int_{-\infty}^{z} e^{-i\mu_n(s-z)}\psi_n(s)ds - \frac{1}{2i\mu_n}\int_{-\infty}^{z} e^{i\mu_n(s-z)}\psi_n(s)ds,\ n\geq 2. \end{aligned} \tag{5.8}$$

See Remark 4.2 regarding the integral paths.

Proposition 5.2. *Consider $\kappa \geq 1$ big enough. Given $\alpha > 2$, the operator $(\partial_\tau^2)\circ\mathcal{J} : \mathcal{X}_{\ell_1,\alpha+2} \to \mathcal{X}_{\ell_1,\alpha}$ is well defined and the following statements hold.*

1. *$\mathcal{J}\circ\mathcal{I}(\psi) = \mathcal{I}\circ\mathcal{J}(\psi) = \psi$.*
2. *For any $\alpha > 2$, there exists a constant $M > 0$ independent of κ such that, for every $h \in \mathcal{X}_{\alpha+2}$,*
$$\|\mathcal{J}_1(h)\|_\alpha \leq M\|h\|_{\alpha+2}.$$
3. *For any $\alpha > 1$, there exists a constant $M > 0$ independent of κ and n such that, for every $h \in \mathcal{X}_\alpha$,*
$$\|\mathcal{J}_n(h)\|_\alpha \leq \frac{M}{\mu_n^2}\|h\|_\alpha.$$

Again the above estimates represent the gain of one more order of derivative in τ. The assumption $\alpha > 1$ in the above last inequality ensures the convergence of the integral in the definition of $\mathcal{J}_n$ and also allows one to adjust the path of the integral in certain ways.

Proof. The proof of Item (1) is straightforward. For $\mathcal{J}_n$, $n \geq 2$ and $\alpha > 1$, one can use the results in [3] to obtain that for $h \in \mathcal{X}_\alpha$ and $z \in D^{u,\mathrm{in}}_{\theta,\kappa}$,

$$|z^\alpha \mathcal{J}_n(h)(z)| \leq \frac{M}{\mu_n^2}\|h\|_\alpha.$$

For $\mathcal{J}_1$, taking $h \in \mathcal{X}_{\alpha+2}$, $\alpha > 2$ and $z \in D^{u,\mathrm{in}}_{\theta,\kappa}$,

$$\begin{aligned} |z^\alpha \mathcal{J}_1(h)(z)| =& \left|\frac{z^{\alpha+3}}{5}\int_{-\infty}^{z}\frac{h(s)}{s^2}ds - \frac{z^{\alpha-2}}{5}\int_{-\infty}^{z} s^3 h_1(s)ds\right| \\ \leq& M\|h\|_{\alpha+2}\left(\int_{-\infty}^{z}\frac{|z|^{\alpha+3}}{|s|^{\alpha+4}}ds + \int_{-\infty}^{z}\frac{|z|^{\alpha-2}}{|s|^{\alpha-1}}ds\right) \leq M\|h\|_{\alpha+2}. \end{aligned}$$

The proof of the proposition is complete. □

5.2. **The fixed point argument.** By Proposition 5.2, we rewrite (5.7) as

$$\psi = \mathcal{W}^\sharp(\psi), \qquad \mathcal{W}^\sharp = \mathcal{J}\circ\mathcal{W}.$$

where $\mathcal{W}$ is given by (5.6). In the following proposition we study some properties of the operator $\widetilde{\mathcal{W}}$.

Proposition 5.3. *Given $R > 0$, for big enough $\kappa \geq 1$, the operator $\mathcal{W}^\sharp : \mathcal{B}_0(R) \subset \mathcal{X}_{\ell_1,3} \to \mathcal{X}_{\ell_1,3}$ is well defined and the following statements hold.*

1. *There exists a constant $M_1 > 0$ independent of κ such that $\|\partial_\tau^2\mathcal{W}^\sharp(0)\|_{\ell_1,3} \leq M_1$.*
2. *There exists a constant $M_2 > 1$ independent of κ such that, for every $\psi, \psi' \in \mathcal{B}_0(R) \subset \mathcal{X}_{\ell_1,3}$,*
$$\left\|\mathcal{W}^\sharp(\psi) - \mathcal{W}^\sharp(\psi')\right\|_{\ell_1,3} \leq M_2\left(\frac{1}{\kappa^2}\|\psi - \psi'\|_{\ell_1,3} + \|\widetilde{\Pi}[\psi] - \widetilde{\Pi}[\psi]\|_{\ell_1,3}\right).$$
$\pi_1(\xi) = \pi_1(\xi') = 0$, then Furthermore,
$$\left\|\partial_\tau^2\left(\widetilde{\Pi}[\mathcal{W}^\sharp(\psi)] - \widetilde{\Pi}[\mathcal{W}^\sharp(\psi')]\right)\right\|_{\ell_1,3} \leq \frac{M_2}{\kappa^2}\|\psi - \psi'\|_{\ell_1,3}.$$

Proof. $\mathcal{W}(0)$ is given by

$$\mathcal{W}(0) = -\widetilde{\Pi}\left[\frac{1}{3}\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)\right)^3\right] - f\left(\frac{-2\sqrt{2}i}{z}\sin(\tau)\right).$$

Thus, since $f(z) = \mathcal{O}(z^5)$, it follows from Proposition 5.1 that

$$\|\Pi_1[\mathcal{W}(0)]\|_5 \leq M\left\|\frac{-2\sqrt{2}i}{z}\sin(\tau)\right\|_{\ell_1,1}^5 \leq M,$$

$$\left\|\widetilde{\Pi}[\mathcal{W}(0)]\right\|_{\ell_1,3} \leq M\left(\left\|\frac{-2\sqrt{2}i}{z}\sin(\tau)\right\|_{\ell_1,1}^3 + \frac{1}{\kappa^2}\left\|\frac{-2\sqrt{2}i}{z}\sin(\tau)\right\|_{\ell_1,1}^5\right) \leq M.$$

Hence, from Proposition 5.2, there exists $M_1 > 0$ such that

$$\begin{aligned}\left\|\partial_\tau^2 W^\sharp(0)\right\|_{\ell_1,3} &\leq \left\|\partial_\tau^2 \mathcal{J}\left(\Pi_1[\mathcal{W}(0)]\sin(\tau)\right)\right\|_{\ell_1,3} + \left\|\partial_\tau^2\mathcal{J}\left(\widetilde{\Pi}[\mathcal{W}(0)]\right)\right\|_{\ell_1,3}\\ &\leq M\left(\|\Pi_1[\mathcal{W}(0)]\|_5 + \left\|\widetilde{\Pi}[\mathcal{W}(0)]\right\|_{\ell_1,3}\right) \leq M_1.\end{aligned}$$

To prove item (2) on the Lipschitz property, assume that $\|\psi\|_{\ell_1,3}, \|\psi'\|_{\ell_1,3} \leq R$, and notice that

$$\begin{aligned}\mathcal{W}(\psi) - \mathcal{W}(\psi') \;=\; & -\Pi_1\left[-\frac{8}{z^2}\sin^2(\tau)\left(\widetilde{\Pi}[\psi] - \widetilde{\Pi}[\psi']\right) - \frac{2\sqrt{2}i}{z}\sin(\tau)\left(\psi^2 - (\psi')^2\right)\right.\\ & \left.+\frac{1}{3}\left(\psi^3 - (\psi')^3\right)\right]\sin(\tau) - \tfrac{1}{3}\widetilde{\Pi}\left[\left(\frac{-2\sqrt{2}i}{z}\sin(\tau) + \psi\right)^3 - \left(\frac{-2\sqrt{2}i}{z}\sin(\tau) + \psi'\right)^3\right]\\ & - f\left(\frac{-2\sqrt{2}i}{z}\sin(\tau) + \psi\right) + f\left(\frac{-2\sqrt{2}i}{z}\sin(\tau) + \psi'\right).\end{aligned}$$

Thus,

$$\begin{aligned}\|\Pi_1[\mathcal{W}(\psi) - \mathcal{W}(\psi')]\|_5 \;\leq\; & \left\|\frac{8}{z^2}\sin^2(\tau)\right\|_{\ell_1,2}\left\|\widetilde{\Pi}[\psi] - \widetilde{\Pi}[\psi']\right\|_{\ell_1,3}\\ & + \left(\left\|\frac{2\sqrt{2}i}{z}\sin(\tau)\right\|_{\ell_1,1}\|\psi + \psi'\|_{\ell_1,1} + \left\|\psi^2 + \psi\psi' + (\psi')^2\right\|_{\ell_1,2}\right)\|\psi - \psi'\|_{\ell_1,3}\\ & + \left\|\int_0^1 f'\left(\frac{-2\sqrt{2}i}{z}\sin(\tau) + s\psi + (1-s)\psi'\right)ds\right\|_{\ell_1,2}\|\psi - \psi'\|_{\ell_1,3}\\ \leq\; & M\left(\left\|\widetilde{\Pi}[\psi] - \widetilde{\Pi}[\psi']\right\|_{\ell_1,3} + \frac{1}{\kappa^2}\|\psi - \psi'\|_{\ell_1,3}\right),\end{aligned}$$

and, recalling that $g(z) = \mathcal{O}(z^3)$, we have that

$$\begin{aligned}\left\|\widetilde{\Pi}[\mathcal{W}(\psi) - \mathcal{W}(\psi')]\right\|_{\ell_1,3} \;\leq\; & \left\|\int_0^1 g'\left(\frac{-2\sqrt{2}i}{z}\sin(\tau) + s\psi + (1-s)\psi'\right)ds\right\|_{\ell_1,0}\|\psi - \psi'\|_{\ell_1,3}\\ \leq\; & \frac{M}{\kappa^2}\|\psi - \psi'\|_{\ell_1,3}.\end{aligned}$$

Item (2) follows from the estimates above and Proposition 5.2. □

Proof of Item 1 Theorem 3.3. Much as in the proof of Theorem 3.1, we use an equivalent norm on $\mathcal{X}_{\ell_1,3}$

$$\|\psi\|_* := \|\psi_1\|_3 + 2M_2\|\widetilde{\Pi}[\psi]\|_{\ell_1,3}.$$

Using Proposition 5.3, it is straight forward to verify that, with in the above norm for sufficiently large $\kappa > 0$, $\mathcal{W}^\sharp$ is a contraction on the closed ball of $\mathcal{X}_{\ell_1,3}$ with radius $3M_1(1+2M_2)$ with Lipschitz constant $\kappa^{-2}M_2(1+2M_2)+\frac{1}{2} < \frac{2}{3}$. The unique fixed point ψ^u gives the unstable solution $\phi^{0,u}$ in the form of (5.3) which satisfies the desired estimates. Using the same arguments in the proof of Theorem 3.1, one can conclude $\Pi_{2l}[\psi^u] \equiv 0$, $\forall l \geq 0$. □

5.3. **The difference between the solutions of the Inner Equation.** This section is devoted to prove the second statement of Theorem 3.3. We consider the two solutions $\phi_0^{u,s}$ of the inner equation (3.9) which are given by (3.11) and we study the difference

$$\Delta\psi(z,\tau) = \phi^{0,u}(z,\tau) - \phi^{0,s}(z,\tau) = \psi^u(z,\tau) - \psi^s(z,\tau),$$

for $z \in \mathcal{R}^{\text{in},+}_{\theta,\kappa} = D^{u,\text{in}}_{\theta,\kappa} \cap D^{s,\text{in}}_{\theta,\kappa} \cap \{z;\ z \in i\mathbb{R} \text{ and } \operatorname{Im}(z) < 0\}$ and $\tau \in \mathbb{T}$. For this purpose, we actually work on (3.9) as an ill-posed dynamical system of real independent variable along $\mathcal{R}^{\text{in},+}_{\theta,\kappa}$.

Remark 5.4. *We are interested in the behavior of the difference in the connected component $\mathcal{R}^{\text{in},+}_{\theta,\kappa}$ of $D^{u,\text{in}}_{\theta,\kappa} \cap D^{s,\text{in}}_{\theta,\kappa} \cap i\mathbb{R}$ because the change $z = \varepsilon^{-1}(y - i\pi/2)$ brings the origin $y = 0$ into $z = -i\varepsilon^{-1}\pi/2 \in \mathcal{R}^{\text{in},+}_{\theta,\kappa}$.*

Let $r \gg 1$. We define

$$z = -ir, \quad \Psi_1(r) = \phi^0_1(-ir), \quad \Psi_{n\pm}(r) = \partial_r\big(\phi^0_n(-ir)\big) \pm \sqrt{n^2-1}\phi^0_n(-ir), \ \ n \geq 3. \tag{5.9}$$

That is

$$\phi^0(-ir,\tau) = \Psi_1(r)\sin\tau + \sum_{n\geq 3} \frac{1}{2\sqrt{n^2-1}}(\Psi_{n+}(r) - \Psi_{n-}(r))\sin n\tau.$$

Then, equation (3.9) takes the form

$$\begin{cases} \partial_r^2\Psi_1 - \frac{1}{4}\Psi_1^3 = F_1(\Psi) \\ \partial_r\Psi_{n\pm} = \pm\sqrt{n^2-1}\Psi_{n\pm} + F_n(\Psi), \end{cases} \tag{5.10}$$

where

$$F_1(\Psi) = \Pi_1\Big[\frac{1}{3}(\phi^0)^3 + f(\phi^0)\Big] - \frac{1}{4}\Psi_1^3, \qquad F_n(\Psi) = \Pi_n\Big[\frac{1}{3}(\phi^0)^3 + f(\phi^0)\Big], \quad n \geq 3.$$

Since $\frac{1}{4}\Psi_1^3$ in the nonlinearity is isolated into the left side the (5.10), the cubic terms in F_1 do not include Ψ_1^3. Note that by Theorem 3.3 we can restrict to the space of odd n's.

Let $\Psi^*(r)$, $* = u,s$, be the functions $\phi^{0,*}$, $* = u,s$, expressed in the coordinates introduced in (5.9). We are interested in $\Psi^s - \Psi^u$ as $r \to +\infty$ where, since we shall consider certain local invariant manifolds/foliation which are not necessarily analytic submanifolds, we work in the space ℓ_2 with the smooth norm

$$\|\Psi\|^2_{\ell_2} := |\Psi_1|^2 + |\partial_r\Psi_1|^2 + \sum_{n=3,\text{odd}}^{\infty} n^2(|\Psi_{n+}|^2 + |\Psi_{n-}|^2)$$

and treat $\Psi_1, \partial_r\Psi_1, \Psi_{n\pm}$ as 2-dim real vectors. We also define

$$\Psi_c = (\Psi_1, \partial_r\Psi_1), \qquad \Psi_\pm = (\Psi_{n\pm})_{n=3}^\infty.$$

Part (1) of Theorem 3.3 implies that $\Psi^{u,s}$ do belong to ℓ_2 space.

It is easy to see that (F_1, F_n) defines a smooth mapping on the ℓ_2 space. Due to both positively and negatively unbounded exponential growth rates caused by the linear parts, (5.10) is ill-posed both forward and backward in r. However, after multiplying a smooth cut-off function based on $\|\cdot\|_{\ell_2}$ to the nonlinearities (F_1, F_n), the standard Lyapunov-Perron approach (see e.g. [13, 14]) still yields smooth local invariant manifolds and foliations near $\Psi = 0$, including an infinite dimensional center-stable manifold W^{cs} where (5.10) is well-posed for $r > 0$, the 4-dim center manifold $W^c \subset W^{cs}$, and stable fibers inside W^{cs} transversal to W^c.

Center-stable manifold W^{cs} can be represented as a graph of an odd mapping h^{cs} as

$$W^{cs} = \{\Psi_+ = h^{cs}(\Psi_c, \Psi_-) \mid |\Psi_c|, \|\Psi_-\|_{\ell_2} < \delta\}, \quad h^{cs} \in C^\infty,\ D^j h^{cs}(0) = 0,\ j = 0,1,2,$$

for some $\delta > 0$.

Normally $Dh^{cs}(0) = 0$ is due to the tangency of W^{cs} to the center-stable subspace. Here the extra $D^2h^{cs}(0) = 0$ comes from the odd symmetry of (5.10), whose W^{cs} can be constructed to be odd. Usually the center-stable manifold is not unique, however, each such local invariant manifold contains $\Psi^{u,s}(r)$ as they converge to 0 as $r \to +\infty$.

Inside W^{cs}: 4-dim center manifold W^c and the stable foliation. Inside the center-stable manifold W^{cs}, the gaps in the exponential decay rates imply that there exists a smooth (real) 4-dim local invariant center submanifold $W^c \subset W^{cs}$ also represented as a graph of an odd mapping h^c

$$W^c = \{\Psi \in W^{cs} \mid \Psi_- = h^c(\Psi_c)\}, \quad h^c \in C^\infty, \quad D^j h^c(0) = 0, \quad j = 0, 1, 2.$$

Moreover, the invariant foliation theorem implies that each point Ψ on W^{cs} belongs to a unique stable fiber which intersects W^c at a unique point Ψ_b (the base point of the fiber). Orbits starting on the same fiber have the same asymptotic behavior as $r \to +\infty$, up to an error of order $\mathcal{O}(e^{-2r})$ as $2 \in (0, \sqrt{3^2-1})$. This invariant stable foliation induces a smooth coordinate system $\Psi = \Gamma(\tilde{\Psi}_c, \tilde{\Psi}_-)$ on W^{cs}, where $\tilde{\Psi}_c$ is the Ψ_c coordinate of the base point $\Psi_b \in W^c$ of Ψ and $\tilde{\Psi}_-$ is the Ψ_- coordinate of $\Psi - \Psi_b$. More precisely, there exists an odd stable foliation mapping $h^s(\tilde{\Psi}_c, \tilde{\Psi}_-) \in \mathbb{R}^4$ such that

$$\Psi = (\Psi_c, \Psi_-, \Psi_+) = \Gamma(\tilde{\Psi}_c, \tilde{\Psi}_-), \text{ where } \Psi_c = \tilde{\Psi}_c + h^s(\tilde{\Psi}_c, \tilde{\Psi}_-), \;\; \Psi_- = h^c(\tilde{\Psi}_c) + \tilde{\Psi}_-, \;\; \Psi_+ = h^{cs}(\Psi_c, \Psi_-),$$

and

$$h^s \in C^\infty, \quad h^s(\tilde{\Psi}_c, 0) = 0, \quad Dh^s(0,0) = 0,$$

where the smoothness of h^s is due to the lack of exponential growth or decay of the linear flow in the center directions.

By Item 1 of Theorem 3.3 and (5.9), the stable/unstable solutions $\Psi^{u,s}$ satisfy $\lim_{r\to+\infty} \Psi^{u,s}(r,\tau) = 0$ and therefore they belong to the center-stable manifold. Thus, we can express them in the stable fiber coordinates,

$$\Psi^{u,s}(r) = \Gamma\big(\tilde{\Psi}_c^{u,s}, \tilde{\Psi}_-^{u,s}(r)\big), \quad \tilde{\Psi}_c^{u,s}(r) = \big(\tilde{\Psi}_1^{u,s}(r), \partial_r \tilde{\Psi}_1^{u,s}(r)\big), \tag{5.11}$$

and let $\Psi_b^{u,s}$ be their base points

$$\Psi_b^{u,s}(r) = \Gamma(\tilde{\Psi}_c^{u,s}, 0) = \Big(\tilde{\Psi}_c^{u,s}(r), h^c\big(\tilde{\Psi}_c^{u,s}(r)\big), h^{cs}\big(\tilde{\Psi}_c^{u,s}(r), h^c\big(\tilde{\Psi}_c^{u,s}(r)\big)\big)\Big) \in W^c,$$

which satisfy

$$\|\Psi^{u,s}(r) - \Psi_b^{u,s}(r)\|_{\ell_2} \le \mathcal{O}(e^{-2r}), \quad \text{as } r \to +\infty.$$

Lemma 5.5. $\tilde{\Psi}_c^u(r) = \tilde{\Psi}_c^s(r)$.

Proof. From item (1) of Theorem 3.3 and Lemma 5.1, $\partial_\tau \Psi^{u,s}(r)$ have exactly the same leading order term proportional to $r^{-1}\sin\tau$ with remainders of $\mathcal{O}(r^{-3})$ in ℓ_2 metric and $\partial_r \Psi^{u,s}(r)$ with remainder of $\mathcal{O}(r^{-4})$. Since $Dh^c(0) = 0$ and $Dh^{cs}(0) = 0$, we have

$$\mathcal{O}(r^{-3}) \ge \|\Psi^u(r) - \Psi^s(r)\|_{\ell_2} \ge \|\tilde{\Psi}_b^u(r) - \tilde{\Psi}_b^s(r)\|_{\ell_2} - \mathcal{O}(e^{-2r}) \ge \frac{1}{2}|\tilde{\Psi}_c^u(r) - \tilde{\Psi}_c^s(r)| - \mathcal{O}(e^{-2r}),$$

and thus

$$|\tilde{\Psi}_c^u(r) - \tilde{\Psi}_c^s(r)| \le \mathcal{O}(r^{-3}). \tag{5.12}$$

According to the invariant foliation theory, $\Psi_b^{u,s}(r)$ are solutions to (5.10) contained in the center manifold W^c, governed by the dynamics of their center coordinates $\tilde{\Psi}^{u,s}(r)$. Let

$$\big(\tilde{\beta}_1(r), \partial_r \tilde{\beta}_1(r)\big) = \tilde{\Psi}_c^u(r) - \tilde{\Psi}_c^s(r), \quad B(r) = \sup_{r' \ge r} (r')^3 |\tilde{\Psi}_c^u(r') - \tilde{\Psi}_c^s(r')| < \infty,$$

where (5.12) was also used. Substituting h^c and h^{cs} into (5.10), using $Dh^c(0) = 0$ and $Dh^{cs}(0) = 0$ along with the leading order expansion of $\Psi^{u,s}(r)$, and observing that the cubic nonlinearity $F_1(\Psi)$ does not contain the term Ψ_1^3 in its Taylor expansion, we have

$$\partial_r^2 \tilde{\beta}_1 - \frac{6}{r^2}\tilde{\beta}_1 = \tilde{G}(r) = \mathcal{O}\left(\frac{1}{r^3}(|\tilde{\beta}_1| + |\partial_r \tilde{\beta}_1|)\right) \le \mathcal{O}\left(\frac{B(r)}{r^6}\right).$$

As in the definition of $\mathcal{J}_1$ in (5.8), a fundamental set of solutions of $\partial_r^2\tilde\beta_1 - \frac{6}{r^2}\tilde\beta_1 = 0$ is given by r^{-2} and r^3. Therefore the general solutions of the above equation is

$$\tilde\beta_1(r) = c_1 r^{-2} + c_2 r^3 + \frac{r^3}{5}\int_{+\infty}^r \frac{\tilde G(s)}{s^2}ds - \frac{1}{5r^2}\int_{+\infty}^r s^3\tilde G(s)ds,$$

which implies

$$|\tilde\beta_1(r) - c_1r^{-2} - c_2r^3| \le \mathcal{O}\big(r^{-4}B(r)\big).$$

In the view of (5.12), we conclude $c_1 = c_2 = 0$ and thus $|\tilde\beta_1(r)| \le \mathcal{O}\big(r^{-4}B(r)\big)$ which again leads to a contradiction to the definition of $B(r)$ for $r \gg 1$, unless $B \equiv 0$. The lemma is proved. □

Finally we are ready to prove the estimate on the difference between $\Psi^{u,s}(r)$.

Proof of Item 2 Theorem 3.3. To complete the proof of the theorem, we need to estimate the difference

$$\tilde\beta_- = (\tilde\beta_{n-})_{n=3}^{+\infty} := \tilde\Psi_-^u(r) - \tilde\Psi_-^s(r) = \big(\tilde\Psi_{n-}^u(r) - \tilde\Psi_{n-}^s(r)\big)_{n=3}^{+\infty}$$

between the stable components of the fiber coordinates of $\Psi^{u,s}(r)$ given in (5.11). As $\Psi^{u,s}(r)$ belong to the same stable fiber, through standard calculations using the stable fiber coordinates system $\Gamma(\tilde\Psi_c, \tilde\Psi_-)$, the odd symmetry of the system (5.10), one may compute the equation satisfied by $\tilde\Psi_-^{u,s}(r)$

$$\partial_r\tilde\Psi_-^{u,s} = A\tilde\Psi_-^{u,s} + \tilde F_-(r, \tilde\Psi_-^{u,s}), \quad A\tilde\Psi_- = \big(-\sqrt{n^2-1}\tilde\Psi_{n-}\big)_{n=3}^{+\infty},$$

and

$$\tilde F_-(r,\tilde\Psi_-) = \widetilde\Pi F\big(\Gamma(\tilde\Psi_c^{u,s}(r), \tilde\Psi_-)\big) - \widetilde\Pi F\big(\Gamma(\tilde\Psi_c^{u,s}(r), 0)\big) = \mathcal{O}\big(\|\tilde\Psi_-\|_{\ell_2}(r^{-2} + \|\tilde\Psi_-\|_{\ell_2}^2)\big).$$

smooth on ℓ_2 with r dependence coming from the presence of $\tilde\Psi_c^{u,s}(r)$ (recall that). manifold W^c. Since $\tilde\Psi_-^{u,s}(r)$ decay exponentially as $r \to +\infty$, $e^{\sqrt8 r}\tilde\beta_-(r)$ satisfies linear equation of the form

$$\partial_r\big(e^{\sqrt8 r}\tilde\beta_-\big) = (A+\sqrt8)e^{\sqrt8 r}\tilde\beta_- + A_-(r)e^{\sqrt8 r}\tilde\beta_-, \quad \text{with} \quad \|A_-(r)\|_{L(\ell_2)} = \mathcal{O}(r^{-2}).$$

As $A + \sqrt8 \le 0$, the Gronwall inequality implies

$$\sup\Big\{e^{\sqrt8 r}\|\tilde\beta_-(r)\|_{\ell_2}\Big\} < +\infty.$$

Moreover, using the variation of constants formula, one gets:

$$e^{\sqrt8 r}\tilde\beta_-(r) = e^{(r-r_0)(A+\sqrt8)}e^{\sqrt8 r_0}\tilde\beta_-(r_0) + \int_{r_0}^r e^{(r-r')(A+\sqrt8)}A_-(r')e^{\sqrt8 r'}\tilde\beta_-(r')]dr'$$

Now, using that $\Pi_3(A+\sqrt8) = 0$ and $(I-\Pi_3)(A+\sqrt8) \le \sqrt{24} - \sqrt8 < 0$, one easily obtains

$$\lim_{r\to\infty} e^{\sqrt8 r}\tilde\beta_-(r) = \left[e^{\sqrt8 r_0}\tilde\beta_{3-}(r_0) + \int_{r_0}^{+\infty}\Pi_3(A_-(r')e^{\sqrt8 r'}\tilde\beta_-(r'))dr'\right]\sin 3\tau$$

Defining

$$\tilde C_{\rm in} = e^{\sqrt8 r_0}\tilde\beta_{3-}(r_0) + \int_{r_0}^{+\infty}\Pi_3\left[A_-(r')e^{\sqrt8 r'}\tilde\beta_-(r')\right]dr'$$

we have seen that

$$\lim_{r\to\infty} e^{\sqrt8 r}\tilde\beta_-(r) - \tilde C_{\rm in}\sin 3\tau = 0$$

In fact, for $r \ge 2r_0 \gg 1$,

$$\begin{aligned}&\left\|e^{\sqrt8 r}\tilde\beta_-(r) - \tilde C_{\rm in}\sin 3\tau\right\|_{\ell_2}\\ &\quad= \Bigg\|(I-\Pi_3)\left[e^{(r-r_0)(A+\sqrt8)}e^{\sqrt8 r_0}\tilde\beta_-(r_0) + \int_{r_0}^r e^{(r-r')(A+\sqrt8)}(I-\Pi_3)[A_-(r')e^{\sqrt8 r'}\tilde\beta_-(r')]dr'\right]\\ &\qquad - \int_r^{+\infty}\Pi_3[A_-(r')e^{\sqrt8 r'}\tilde\beta_-(r')]\sin 3\tau dr'\Bigg\|_{\ell_2} \le \mathcal{O}\left(\frac1r\right)\end{aligned}$$

To complete the proof of Theorem 3.3, we need to estimate $e^{\sqrt{8}r}(\phi^{0,u}-\phi^{0,s})$ and $\partial_r\big(e^{\sqrt{8}r}(\phi^{0,u}-\phi^{0,s})\big)$. From the definition of the stable fiber coordinates $\Gamma(\tilde\Psi_c,\tilde\Psi_-)$, recalling that $\tilde\Psi_c^u=\tilde\Psi_c^s$, and the cubic leading order of $h^{c,s,cs}$, we have

$$\Psi_-^u(r)-\Psi_-^s(r)=\tilde\beta_-(r),\quad \|\Psi^u(r)-\Psi^s(r)-\tilde\beta_-(r)\|_{\ell_2}\le\mathcal{O}\big(r^{-2}\|\tilde\beta_-(r)\|_{\ell_2}\big)\le\mathcal{O}(r^{-2}e^{-\sqrt{8}r}).$$

Going back from Ψ to the ϕ^0, the above estimate first implies, for $C_{\rm in}=-\frac{\sqrt{2}}{8}\tilde C_{\rm in}$,

$$\|\partial_\tau\big(e^{\sqrt{8}r}\Delta\phi^0(-ir)-C_{\rm in}\sin 3\tau\big)\|_{\ell_1}\le\|e^{\sqrt{8}r}\big(\Psi_-^u(r)-\Psi_-^s(r)\big)-\tilde C_{\rm in}\sin 3\tau\|_{\ell_2}+\mathcal{O}(r^{-2})\le\mathcal{O}(r^{-1}).$$

Moreover,

$$\|\partial_r\big(e^{\sqrt{8}r}\Delta\phi^0(-ir)\big)\|_{\ell_1}\le\left\|e^{\sqrt{8}r}\left(\sqrt{8}\frac{-1}{2\sqrt{8}}\big(\Psi_-^u(r)-\Psi_-^s(r)\big)+\frac{1}{2}\big(\Psi_-^u(r)-\Psi_-^s(r)\big)\right\|_{\ell_2}+\mathcal{O}\left(\frac{1}{r^2}\right)\le\mathcal{O}\left(\frac{1}{r^2}\right).$$

Since $\Delta\phi^0(z)$ is analytic, the estimate on $\partial_r\big(e^{\sqrt{8}r}\Delta\phi^0(-ir)\big)$ implies the same estimate on $\partial_z\big(e^{i\sqrt{8}z}\Delta\phi^0(z)\big)$ and this completes the proof of Theorem 3.3. ☐

6. Complex matching estimates: Proof of Theorem 3.6

As usual, we consider only the unstable case, and in order to simplify the notation, we omit the superscript "u" of the solutions. Moreover, in this section, we use the domain $D^{\mathrm{mch},u}_{+,\kappa}$ instead of $D^{u,\mathrm{in}}_{\theta,\kappa}$ (see (3.10) and (3.13)) but we work on the same notation for the norms and Banach spaces introduced in Section 5.1.

Proposition 6.1. *Let $\phi(z,\tau)$ and $\phi^0(z,\tau)$ be solutions to* (3.8) *and* (3.9)*, respectively. The function $\varphi : D^{\mathrm{mch},u}_{+,\kappa}\times\mathbb{T}\to\mathbb{C}$ defined as*

$$\varphi(z,\tau)=\phi(z,\tau)-\phi^0(z,\tau). \tag{6.1}$$

satisfies the following differential equation

$$\mathcal{I}(\varphi)(z,\tau)=\Big(L(\varphi)(z)+\widehat L(\widetilde\Pi[\varphi])(z)\Big)\sin(\tau)+K(\varphi)(z,\tau)+\mathcal{C}_{\rm mch}(z,\tau), \tag{6.2}$$

where $\mathcal{I}$ is the operator given by (5.5)*, $L:\mathcal{X}_{\ell_1,\alpha}\to\mathcal{X}_{\alpha+4}$, $\widehat L:\mathcal{X}_{\ell_1,\alpha}\to\mathcal{X}_{\alpha+2}$, and $K:\mathcal{X}_{\ell_1,\alpha}\to\mathcal{X}_{\ell_1,\alpha+2}$ are linear operators and $\mathcal{C}_{\rm mch}:D^{\mathrm{mch},u}_{+,\kappa}\times\mathbb{T}\to\mathbb{C}$ is an analytic function in the variable z. Moreover, $\Pi_1\circ K\equiv 0$ and there exists a constant $M>0$ independent of ε and κ such that, for $0<\gamma<1$, ε sufficiently small and κ big enough*

1. $\|\Pi_1[\mathcal{C}_{\rm mch}]\|_4\le M\varepsilon^{3\gamma-1}$ *and* $\left\|\partial_\tau^2\widetilde\Pi[\mathcal{C}_{\rm mch}]\right\|_{\ell_1,3}\le M\varepsilon^2$;
2. $\|L(\varphi)\|_{\alpha+4}\le M\|\varphi\|_{\ell_1,\alpha}$;
3. $\|\widehat L(\varphi)\|_{\alpha+2}\le M\|\varphi\|_{\ell_1,\alpha}$;
4. $\|K(\varphi)\|_{\ell_1,\alpha+2}\le M\|\varphi\|_{\ell_1,\alpha}$, $j=0,1,2$.

Proof. Since ϕ and ϕ^0 satisfy (3.8) and (3.9), respectively, we have that $\varphi(z,\tau)$ satisfies

$$\partial_z^2\varphi-\partial_\tau^2\varphi-\varphi=\varepsilon^2\phi-\frac{1}{3}(\phi^3-(\phi^0)^3)-\frac{1}{\omega^3}f(\omega\phi)+f(\phi^0),\quad \omega=(1+\varepsilon^2)^{-\frac12}. \tag{6.3}$$

Now, recall that $\phi(z,\tau)=\varepsilon v(i\pi/2+\varepsilon z,\tau)$, where $v(y,\tau)=v^h(y)\sin(\tau)+\xi(y,\tau)$, v^h is given by (1.22) and ξ is given by Theorem 3.1. An easy computation shows that

$$\varepsilon v^h(i\pi/2+\varepsilon z)=-\frac{2\sqrt{2}i}{z}+l_1(z),$$

where l_1 is an analytic function such that $|l_1(z)|\le M\varepsilon^2|z|$, for each $z\in D^{\mathrm{mch},u}_{+,\kappa}$. Thus,

$$\phi(z,\tau)=-\frac{2\sqrt{2}i}{z}\sin(\tau)+l_1(z)\sin(\tau)+\varepsilon\xi(i\pi/2+\varepsilon z,\tau). \tag{6.4}$$

Using Theorem 3.1 and $y=i\pi/2+\varepsilon z$, we have

$$\|\varepsilon\partial_\tau^2\xi(i\pi/2+\varepsilon z,\tau)\|_{\ell_1,3}\le\frac{1}{\varepsilon^2}\|\partial_\tau^2\xi(y,\tau)\|_{\ell_1,1,3}\le M,$$

where $\|\cdot\|_{\ell_1,1,3}$ is the norm introduced in Section 4.1.

Since $M\kappa \le |z| \le M\varepsilon^{\gamma-1}$ for every $z \in D^{\mathrm{mch},u}_{+,\kappa}$, it holds

$$\left\| \partial_\tau^2 \left(\phi^0(z,\tau) + \frac{2\sqrt{2}i}{z}\sin(\tau)\right)\right\|_{\ell_1,3} \le M, \tag{6.5}$$

and using that $\|\phi\|_{\ell_1,1}, \|\phi^0\|_{\ell_1,1} \le M$, $f(z) = \mathcal{O}(z^5)$, we obtain from the Mean Value Theorem that

$$\begin{aligned} -\frac{1}{3}\left(\phi^3 - (\phi^0)^3\right) - f(\phi) + f(\phi^0) &= -\frac{1}{3}\left(\phi^2 + \phi\phi^0 + (\phi^0)^2\right)\varphi - \varphi\int_0^1 f'(s\phi + (1-s)\phi^0)ds \\ &= \frac{6}{z^2}\Pi_1[\varphi]\sin(\tau) - \frac{2}{z^2}\Pi_1[\varphi]\sin(3\tau) + l_2(\varphi) + l_3(\widetilde{\Pi}[\varphi]), \end{aligned} \tag{6.6}$$

where $l_2 : \mathcal{X}_{\ell_1,\alpha} \to \mathcal{X}_{\ell_1,\alpha+4}$ and $l_3 : \mathcal{X}_{\ell_1,\alpha} \to \mathcal{X}_{\ell_1,\alpha+2}$ are linear operators such that,

$$\|l_2(\varphi)\|_{\ell_1,\alpha+4} \le M\|\varphi\|_{\ell_1,\alpha} \quad \textit{and} \quad \|l_3(\varphi)\|_{\ell_1,\alpha+2} \le M\|\varphi\|_{\ell_1,\alpha}.$$

The proof of the proposition follows from (6.3), (6.4), and (6.6) and by taking,

- $\mathcal{C}_{\mathrm{mch}} = \varepsilon^2\phi + f(\phi) - \omega^{-3}f(\omega\phi)$,
- $L(\varphi) = \Pi_1\left[l_2(\varphi)\right]$,
- $\widehat{L}(\varphi) = \Pi_1\left[l_3(\widetilde{\Pi}[\varphi])\right]$,
- $K(\varphi) = \widetilde{\Pi}\left[-\frac{2}{z^2}\Pi_1[\varphi]\sin(3\tau) + l_2(\varphi) + l_3(\widetilde{\Pi}[\varphi])\right]$.

□

Let $z_j = \varepsilon^{-1}(y_j - i\pi/2)$, $j = 1,2$, where y_1 and y_2 are the vertices of the matching domain $D^{\mathrm{mch},u}_{+,\kappa}$ given by (3.13). Consider the following linear operator acting on the Fourier coefficients of $h(z,\tau) = \sum_{k\ge0} h_{2k+1}(z)\sin((2k+1)\tau)$.

$$\mathcal{T}(h) = \sum_{k\ge0}\mathcal{T}_{2k+1}(h_{2k+1})\sin((2k+1)\tau), \tag{6.7}$$

where

$$\begin{aligned} \mathcal{T}_1(h_1) &= \frac{z^3}{5}\int_{z_1}^{z}\frac{h_1(s)}{s^2}ds - \frac{1}{5z^2}\int_{z_2}^{z}h_1(s)s^3ds \\ &\quad - \frac{1}{5(z_2^5 - z_1^5)}\left[\left(z^3 - \frac{z_2^5}{z^2}\right)\int_{z_2}^{z_1}h_1(s)s^3ds + \left(z^3z_2^5 - \frac{(z_1z_2)^5}{z^2}\right)\int_{z_1}^{z_2}\frac{h_1(s)}{s^2}ds\right] \\ \mathcal{T}_{2k+1}(h_{2k+1}) &= \int_{z_2}^{z}\frac{h_{2k+1}(s)e^{-i\mu_{2k+1}(s-z)}}{2i\mu_{2k+1}}ds - \int_{z_1}^{z}\frac{h_{2k+1}(s)e^{i\mu_{2k+1}(s-z)}}{2i\mu_{2k+1}}ds, \\ &\quad + \frac{\sin(\mu_{2k+1}(z_2 - z))}{\sin(\mu_{2k+1}(z_1 - z_2))}\int_{z_2}^{z_1}\frac{h_{2k+1}(s)e^{-i\mu_{2k+1}(s-z_1)}}{2i\mu_{2k+1}}ds \\ &\quad + \frac{\sin(\mu_{2k+1}(z_1 - z))}{\sin(\mu_{2k+1}(z_1 - z_2))}\int_{z_1}^{z_2}\frac{h_{2k+1}(s)e^{-i\mu_{2k+1}(s-z_2)}}{2i\mu_{2k+1}}ds, \qquad \text{for } k \ge 1. \end{aligned}$$

Observe that $\mathcal{T}$ is chosen such that $\mathcal{I} \circ \mathcal{T} = \mathrm{Id}$ and $\mathcal{T}(h)(z_j,\tau) = 0$, $j = 1,2$.

Moreover, consider the analytic in z function $\mathcal{Q} : D^{\mathrm{mch},u}_{+,\kappa} \times \mathbb{T} \to \mathbb{C}$ given by

$$\mathcal{Q}(z,\tau) = \sum_{k\ge0}\mathcal{Q}_{2k+1}(z)\sin((2k+1)\tau), \tag{6.8}$$

which is defined using φ in (6.1) as follows, where $k \ge 1$,

$$\begin{aligned} \mathcal{Q}_1(z) &= \frac{1}{z_2^5 - z_1^5}\left(z^3(z_2^2\varphi_1(z_2) - z_1^2\varphi_1(z_1)) - \frac{1}{z^2}\left(z_1^5z_2^2\varphi_1(z_2) - z_1^2z_2^5\varphi_1(z_1)\right)\right), \\ \mathcal{Q}_{2k+1}(z) &= \frac{\sin(\mu_{2k+1}(z - z_2))}{\sin(\mu_{2k+1}(z_1 - z_2))}\varphi_{2k+1}(z_1) - \frac{\sin(\mu_{2k+1}(z - z_1))}{\sin(\mu_{2k+1}(z_1 - z_2))}\varphi_{2k+1}(z_2). \end{aligned}$$

Observe that $\mathcal{Q}$ satisfies $\mathcal{I}\mathcal{Q} = 0$ and $\mathcal{Q}_{2k+1}(z_j) = \varphi_{2k+1}(z_j)$, $j = 1,2$.

In conclusion, observe that if $h, \widehat{\varphi} : D^{\mathrm{mch},u}_{+,\kappa} \times \mathbb{C} \to \mathbb{C}$ are analytic in z functions such that

$$\mathcal{I}(\widehat{\varphi}) = h, \qquad \widehat{\varphi}(z_j) = \varphi(z_j), \quad j = 1, 2,$$

where φ is given in (6.1), then, we have that

$$\widehat{\varphi}(z,\tau) = \mathcal{Q}(z,\tau) + \mathcal{T}(h)(z,\tau),$$

where $\mathcal{T}$ and $\mathcal{Q}$ are given by (6.7) and (6.8). In particular, as the function φ satisfies (6.2) by Proposition 6.1, it can be written as

$$\varphi(z,\tau) = \mathcal{Q}(z_1,z_2)(z,\tau) + \mathcal{T}\left(\mathcal{C}_{\mathrm{mch}}(z,\tau) + \left(L(\varphi)(z) + \widehat{L}(\widetilde{\Pi}[\varphi])(z)\right)\sin(\tau) + K(\varphi)(z,\tau)\right). \tag{6.9}$$

We use this expression for φ to obtain estimates of this function for $z \in D^{\mathrm{mch},u}_{+,\kappa}$.

Next lemma gives estimates for the operators $\mathcal{T}$ and $\mathcal{Q}$ given in (6.7) and (6.8).

Lemma 6.2. *There exists $\delta > 0$ depending only on $\beta_{1,2}$ (see (3.13)), such that, for $\kappa\varepsilon^{1-\gamma} \le \delta$, the following statements hold.*

(1) *The linear operator $\mathcal{T}_1 : \mathcal{X}_\alpha \to \mathcal{X}_{\alpha-2}$ is well defined and*

$$\|\mathcal{T}_1(h)\|_{\alpha-2} \le M\,\|h\|_\alpha\,, \ \ \alpha > 4; \quad \|\mathcal{T}_1(h)\|_2 \le M|\log\varepsilon|\,\|h\|_4\,, \ \ \alpha = 4.$$

(2) *For $k \ge 1$ and $h \in \mathcal{X}_\alpha$, with $\alpha \ge 0$,*

$$\|\mathcal{T}_{2k+1}(h)\|_\alpha \le \frac{M}{k^2}\,\|h\|_\alpha\,.$$

(3) *$\mathcal{Q}$ satisfies*

$$\|\mathcal{Q}_1\|_\alpha \le M\left(\varepsilon^{(\alpha-3)(\gamma-1)} + \varepsilon^{2+(\alpha+1)(\gamma-1)}\right), \ \ \alpha \ge 2; \qquad \left\|\partial_\tau^2\widetilde{\Pi}[\mathcal{Q}]\right\|_\alpha \le M\varepsilon^{(\alpha-3)(\gamma-1)}, \ \ \alpha \ge 0.$$

Proof. Due to the assumption $e^{5(\pi-\beta_1)} - e^{-5\beta_2} \neq 0$, when δ is small, it holds

$$\frac{1}{M}\varepsilon^{\gamma-1} \le |z_1|,\, |z_2|,\, |z_1^5 - z_2^5|^{\frac{1}{5}} \le M\varepsilon^{\gamma-1}; \quad \kappa \le |z| \le M\varepsilon^{\gamma-1}, \ \ \forall z \in D^{\mathrm{mch,u}}_{+,\kappa}. \tag{6.10}$$

Therefore,

$$\left|\frac{1}{5z^2}\int_{z_2}^{z} h(s)s^3 ds\right| \le \frac{M\|h\|_\alpha}{|z|^2}\int_{z_2}^{z}|s|^{3-\alpha}ds \le \begin{cases} M\|h\|_\alpha |z|^{2-\alpha}, & \alpha > 4, \\ M|\log\varepsilon|\|h\|_\alpha |z|^{2-\alpha}, & \alpha = 4, \end{cases}$$

and, for $\alpha \ge 4$,

$$\begin{aligned}
&\left|\frac{z^3}{5}\int_{z_1}^{z}\frac{h(s)}{s^2}ds\right| \le M\|h\|_\alpha |z|^3\int_{z_1}^{z}\frac{1}{|s|^{2+\alpha}}ds \le M\|h\|_\alpha|z|^{2-\alpha} \\
&\left|\frac{1}{5(z_2^5 - z_1^5)}\left(z^3 - \frac{z_2^5}{z^2}\right)\int_{z_2}^{z_1} h(s)s^3 ds\right| \le \frac{M\|h\|_\alpha}{|z|^2}\int_{z_2}^{z_1}|s|^{3-\alpha}ds \le \frac{M\|h\|_\alpha\varepsilon^{(\gamma-1)(4-\alpha)}}{|z|^2} \le M\|h\|_\alpha|z|^{2-\alpha} \\
&\left|\frac{1}{5(z_2^5 - z_1^5)}\left(z^3 z_2^5 - \frac{(z_1z_2)^5}{z^2}\right)\int_{z_1}^{z_2}\frac{h(s)}{s^2}ds\right| \le M\|h\|_\alpha\left(|z|^3 + \frac{|z_2|^5}{|z|^2}\right)\int_{z_1}^{z_2}\frac{1}{|s|^{2+\alpha}}ds \\
&\qquad\qquad \le M\|h\|_\alpha\left(|z|^3 + \frac{|z_2|^5}{|z|^2}\right)\varepsilon^{(\gamma-1)(-1-\alpha)} \le M\|h\|_\alpha|z|^{2-\alpha},
\end{aligned}$$

where the integral $\int_{z_1}^{z_2}$ was simply taken along the arc of the circle centered at $-i\kappa\varepsilon$. Hence, we finish the proof of item (1) of the theorem.

To deal with the higher modes, we will see that

$$\left|\frac{\sin(\mu_{2k+1}(z_j - z))}{\sin(\mu_{2k+1}(z_1 - z_2))}\right| \le M, \ \ j = 1,2, \ \ \forall\, z \in D^{\mathrm{mch},u}_{+,\kappa}, \ \ \forall\, k \ge 1. \tag{6.11}$$

In fact, recalling that $|\sin^2(z)| = \frac{1}{2}(\cosh(2\,\mathrm{Im}(z)) - \cos(2\,\mathrm{Re}(z)))$, we have

$$\left|\frac{\sin(\mu_{2k+1}(z_j - z))}{\sin(\mu_{2k+1}(z_1 - z_2))}\right|^2 \le \frac{\cosh(2\mu_{2k+1}\,\mathrm{Im}(z_j - z)) + 1}{\cosh(2\mu_{2k+1}\,\mathrm{Im}(z_1 - z_2)) - 1}.$$

Since $\mathrm{Im}(z_1 - z_2) = K\varepsilon^{\gamma-1}$ and $|\,\mathrm{Im}(z_j - z)| \le |\,\mathrm{Im}(z_1 - z_2)|$, we obtain (6.11).

Assume that $\alpha \geq 0$. For each $z \in D^{\mathrm{mch},u}_{+,\kappa}$, there exist β_1^*, β_2^* (depending on z) between β_1 and β_2 and $t_2^*, t_1^* > 0$ (depending on z) such that $z_2 = z + e^{-i\beta_2^*} t_2^*$ and $z_1 = z + e^{i(\pi-\beta_1^*)} t_1^*$. Thus, we have that

$$\begin{aligned}\left|\int_{z_2}^{z} h_{2k+1}(s) e^{-i\mu_{2k+1}(s-z)} ds\right| &\leq \int_0^{t_2^*} \left| h_{2k+1}\left(z + e^{-i\beta_2^*} t\right)\right| e^{-\mu_{2k+1}\sin(\beta_2^* t)} dt \\ &\leq \|h_{2k+1}\|_\alpha \int_0^{t_2^*} \frac{e^{-\mu_{2k+1}\sin(\beta_2^* t)}}{|z + e^{-i\beta_2^*} t|^\alpha} dt \leq \frac{\|h_{2k+1}\|_\alpha}{|z|^\alpha} \int_0^\infty e^{-\mu_{2k+1}\sin(\beta_2^* t)} dt \\ &\leq \frac{M\|h_{2k+1}\|_\alpha}{\mu_{2k+1}|z|^\alpha}.\end{aligned}$$

Analogously, we prove that

$$\left|\int_{z_1}^{z} h_{2k+1}(s) e^{i\mu_{2k+1}(s-z)} ds\right| \leq \frac{M\|h_{2k+1}\|_\alpha}{\mu_{2k+1}|z|^\alpha},$$

and in particular, using that $|z_j| \geq M|z|$, $j = 1, 2$,

$$\begin{aligned}\left|\int_{z_2}^{z_1} h_{2k+1}(s) e^{-i\mu_{2k+1}(s-z_1)} ds\right| &\leq \frac{M\|h_{2k+1}\|_\alpha}{\mu_{2k+1}|z_1|^\alpha} \leq \frac{M\|h_{2k+1}\|_\alpha}{\mu_{2k+1}|z|^\alpha} \\ \left|\int_{z_1}^{z_2} h_{2k+1}(s) e^{i\mu_{2k+1}(s-z_2)} ds\right| &\leq \frac{M\|h_{2k+1}\|_\alpha}{\mu_{2k+1}|z_2|^\alpha} \leq \frac{M\|h_{2k+1}\|_\alpha}{\mu_{2k+1}|z|^\alpha}.\end{aligned}$$

Hence,

$$\|\mathcal{T}_{2k+1}(h_{2k+1})\|_\alpha \leq \frac{M}{\mu_{2k+1}^2} \|h_{2k+1}\|_\alpha,\ k \geq 1,\ \alpha \geq 0. \tag{6.12}$$

Items (2) follows (6.12).

To estimate $\mathcal{Q}$, observe that using (6.4) and (6.5), one has

$$\varphi(z,\tau) = l_1(z)\sin\tau + b(z,\tau), \qquad \text{with } b(z,\tau) = \varepsilon\xi(i\pi/2 + \varepsilon z, \tau) - \left(\phi_0(z,\tau) + \frac{2\sqrt{2}i}{z}\sin\tau\right),$$

where l_1 is given in (6.4). Then, $\|\partial_\tau^2 b\|_{\ell_1,3} \leq M$ and $|l_1(z)| \leq M\varepsilon^2|z|$, for each $z \in D^{\mathrm{mch},u}_{+,\kappa}$. Thus, from (6.10), we can see that

$$\begin{aligned}|\mathcal{Q}_1(z_1,z_2)(z)| &= \left|\frac{1}{z_2^5 - z_1^5}\left(z^3(z_2^2\varphi_1(z_2) - z_1^2\varphi_1(z_1)) - \frac{1}{z^2}\left(z_1^5 z_2^2\varphi_1(z_2) - z_1^2 z_2^5\varphi_1(z_1)\right)\right)\right| \\ &\leq M\left(|\varphi_1(z_1)| + |\varphi_1(z_2)| + \frac{|z_1^2|}{|z|^2}|\varphi_1(z_1)| + \frac{|z_2^2|}{|z|^2}|\varphi_1(z_2)|\right) \\ &\leq M\left(\frac{1}{|z_2||z|^2} + \varepsilon^2|z_2| + \frac{\varepsilon^2|z_2|^3}{|z|^2}\right).\end{aligned}$$

Therefore for $\alpha \geq 2$,

$$\|\mathcal{Q}_1(z_1,z_2)\|_\alpha \leq M\left(\varepsilon^{(\alpha-3)(\gamma-1)} + \varepsilon^{2+(\alpha+1)(\gamma-1)}\right).$$

Finally, from (6.11) and (6.8), we can see that, for $\alpha \geq 0$ and $k \geq 1$,

$$\begin{aligned}|z^\alpha\partial_\tau^2\mathcal{Q}_{2k+1}(z_1,z_2)(z)| &= \left|\frac{\sin(\mu_{2k+1}(z-z_2))}{\sin(\mu_{2k+1}(z_1-z_2))} z^\alpha\partial_\tau^2\varphi_{2k+1}(z_1) - \frac{\sin(\mu_{2k+1}(z-z_1))}{\sin(\mu_{2k+1}(z_1-z_2))} z^\alpha\partial_\tau^2\varphi_{2k+1}(z_2)\right| \\ &\leq Mk^2\|\Pi_{2k+1}[b]\|_3\frac{|z|^\alpha}{|z_2|^3} \leq Mk^2\|\Pi_{2k+1}[b]\|_3\varepsilon^{(\alpha-3)(\gamma-1)},\end{aligned}$$

and thus

$$\|\partial_\tau^2\mathcal{Q}_{2k+1}(z_1,z_2)\|_\alpha \leq M\varepsilon^{(\alpha-3)(\gamma-1)},\ \alpha \geq 0,\ k \geq 1,$$

which completes the proof of item (3). □

End of the proof of Theorem 3.6. To obtain the estimates for φ stated in the theorem, we just need to estimate $\|\varphi\|_{\ell^1,2}$. From (6.9), and Propositions 6.1 and 6.2, we have that

$$\begin{aligned}\|\varphi_1\|_2 &= \left\|\mathcal{Q}_1(z_1,z_2)+\mathcal{T}_1\left(\Pi_1\left[\mathcal{C}_{\mathrm{mch}}\right]+L(\varphi)+\widehat{L}(\widetilde{\Pi}[\varphi])\right)\right\|_2\\ &\leq \|\mathcal{Q}_1(z_1,z_2)\|_2+M|\log\varepsilon|\left(\|\Pi_1\left[\mathcal{C}_{\mathrm{mch}}\right]\|_4+\|L(\varphi)\|_4+\left\|\widehat{L}(\widetilde{\Pi}[\varphi])\right\|_4\right)\\ &\leq M(\varepsilon^{1-\gamma}+\varepsilon^{2+3(\gamma-1)})+M|\log\varepsilon|\left(\varepsilon^{3\gamma-1}+\|\varphi\|_{\ell_1,0}+\left\|\widetilde{\Pi}[\varphi]\right\|_{\ell_1,2}\right)\\ &\leq M\left(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1}|\log\varepsilon|+\frac{|\log\varepsilon|}{\kappa^2}\|\varphi\|_{\ell_1,2}+|\log\varepsilon|\left\|\widetilde{\Pi}[\varphi]\right\|_{\ell_1,2}\right).\end{aligned}$$

Moreover, since $\Pi_1\circ K\equiv 0$, we have that

$$\begin{aligned}\left\|\partial_\tau^2\widetilde{\Pi}[\varphi]\right\|_{\ell_1,2} &= \left\|\partial_\tau^2\widetilde{\Pi}\circ\mathcal{Q}(z_1,z_2,\varphi)+\partial_\tau^2\mathcal{T}\left(\widetilde{\Pi}\left[\mathcal{C}_{\mathrm{mch}}\right]+K(\varphi)\right)\right\|_{\ell_1,2}\\ &\leq \left\|\partial_\tau^2\widetilde{\Pi}\circ\mathcal{Q}(z_1,z_2,\varphi)\right\|_{\ell_1,2}+M\left(\left\|\widetilde{\Pi}\left[\mathcal{C}_{\mathrm{mch}}\right]\right\|_{\ell_1,2}+\|K(\varphi)\|_{\ell_1,2}\right)\\ &\leq M(\varepsilon^{1-\gamma}+\varepsilon^{2+3(\gamma-1)})+M\left(\frac{\varepsilon^2}{\kappa}+\|\varphi\|_{\ell_1,0}\right)\\ &\leq M\left(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1}+\frac{1}{\kappa^2}\|\varphi\|_{\ell_1,2}\right).\end{aligned}$$

Since $\kappa^{-2}|\log\varepsilon|$ is assumed to be small, it follows from multiplying the second inequality by $2M|\log\varepsilon|$ and adding it to the first one that

$$\|\varphi_1\|_2+M|\log\varepsilon|\|\partial_\tau^2\widetilde{\Pi}[\varphi]\|_{\ell_1,2}\leq 2M|\log\varepsilon|\left(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1}\right).$$

Finally, the estimate on $\partial_z\varphi$ could be derived by differentiating the formula of φ with respect to z. Alternatively, from Lemma 8.1 of [7], reducing the domain $D^{\mathrm{mch},u}_{+,\kappa}$ (see (3.13)), with vertices y_1 and y_2 such that $|y_j-i(\pi/2-\kappa\varepsilon)|=\varepsilon^\gamma$, $j=1,2$, to $D^{\mathrm{mch},u}_{+,2\kappa}\subset D^{\mathrm{mch},u}_{+,\kappa}$ having vertices $\widetilde{y}_1$ and $\widetilde{y}_2$ such that $|\widetilde{y}_j-i(\pi/2-2\kappa\varepsilon)|=\widetilde{c}\varepsilon^\gamma$, $j=1,2$, and $0<\widetilde{c}<1$, we obtain that

$$\|\partial_\tau^2\partial_z\varphi\|_{\ell_1,2}\leq\frac{M}{\kappa}|\log\varepsilon|(\varepsilon^{1-\gamma}+\varepsilon^{3\gamma-1}).$$

It completes the proof of this theorem. In order to simplify the notation, we make no distinction between $D^{\mathrm{mch},u}_{+,\kappa}$ and $D^{\mathrm{mch},u}_{+,2\kappa}$. □

7. The distance between the manifolds: Proof of Proposition 3.12

7.1. **Banach Space and Operators.** We devote this section to prove Proposition 3.12. We start by defining the functional setting. Given an analytic function $f:\mathcal{R}_\kappa\to\mathbb{C}$ (see Figure 9), we define the norm

$$\|f\|_{\alpha,\exp}=\sup_{y\in\mathcal{R}_\kappa}\left|(y^2+\pi^2/4)^\alpha e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}f(y)\right|,$$

and the Banach space

$$\mathcal{X}_{\alpha,\exp}=\{f:\mathcal{R}_\kappa\to\mathbb{C};\ f\ \text{ analytic},\|f\|_{\alpha,\exp}<\infty\}.$$

Moreover, given an analytic function $f:\mathcal{R}_\kappa\times\mathbb{T}\to\mathbb{C}$ odd in $\tau\in\mathbb{T}$, we define the corresponding norm and the associated Banach space

$$\begin{aligned}\|f\|_{\ell_1,\alpha,\exp} &= \sum_{k\geq 1}\|\Pi_{2k+1}[f]\|_{\alpha,\exp}\\ \mathcal{X}_{\ell_1,\alpha,\exp} &= \{f:\mathcal{R}_\kappa\times\mathbb{T}\to\mathbb{C};\ f\text{ is an analytic function in the variable }y\text{ such that}\\ &\qquad \Pi_1[f]=\Pi_{2l}[f]=0,\forall l\geq 0\text{ and }\|f\|_{\ell_1,\alpha,\exp}<\infty\}.\end{aligned}$$

Finally, we consider the product Banach space

$$\mathcal{Y}_{\ell_1,2,\exp}=\mathcal{X}_{2,\exp}\times\mathcal{X}_{\ell_1,0,\exp}\times\mathcal{X}_{\ell_1,0,\exp},$$

endowed with the weighted norm

$$[\![(f,g,h)]\!]_{\ell_1,2,\exp} = \frac{1}{\varepsilon}\|f\|_{2,\exp} + \kappa\|g\|_{\ell_1,0,\exp} + \kappa\|h\|_{\ell_1,0,\exp}.$$

The next lemmas give estimates for the operators and functions given in Section 3.3.

Lemma 7.1. *The components of the operator $\mathcal{P}$ in (3.22) have the following properties.*

(1) *For $\alpha = 2,5$, the operator $\mathcal{P}^W : \mathcal{X}_{\alpha,\exp} \to \mathcal{X}_{2,\exp}$ is well defined. Moreover, there exists a constant $M > 0$ independent of ε and κ such that,*
   - *For $h \in \mathcal{X}_{2,\exp}$, $\|\mathcal{P}^W(h)\|_{2,\exp} \le M\varepsilon\|h\|_{2,\exp}$.*
   - *For $h \in \mathcal{X}_{5,\exp}$, $\|\mathcal{P}^W(h)\|_{2,\exp} \le \dfrac{M}{\varepsilon^2\kappa^3}\|h\|_{5,\exp}$.*

(2) *For $\alpha > 1$, the operators $\mathcal{P}^\Gamma, \mathcal{P}^\Theta : \mathcal{X}_{\ell_1,\alpha,\exp} \to \mathcal{X}_{\ell_1,0,\exp}$ are well-defined. Moreover, there exists a constant $M > 0$ independent of ε and κ such that, for every $h \in \mathcal{X}_{\ell_1,\alpha,\exp}$,*

$$\|\mathcal{P}^\Gamma(h)\|_{\ell_1,0,\exp}, \|\mathcal{P}^\Theta(h)\|_{\ell_1,0,\exp} \le \frac{M}{(\kappa\varepsilon)^{\alpha-1}}\|h\|_{\ell_1,\alpha,\exp}.$$

Proof. We first prove Item 1. We take $h \in \mathcal{X}_{\ell_1,2,\exp}$ and, recalling that $\ddot{v}^h$ has a a pole of order 3, we obtain the following estimate for $\mathrm{Im}(y) > 0$,

$$\begin{aligned}
\left|e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}|y^2+\pi^2/4|^2\ddot{v}^h(y)\int_0^y \frac{h(s)}{\ddot{v}^h(s)}ds\right| &\le \frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y \left|\frac{h(s)}{\ddot{v}^h(s)}\right| ds\\
&\le M\|h\|_{2,\exp}\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(s)|\right)}|s^2+\pi^2/4|ds\\
&\le M\|h\|_{2,\exp}\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\mathrm{Im}(y)\right)}}{|y-i\pi/2|}\int_0^{\mathrm{Im}(y)} |\sigma-\pi/2|e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\sigma\right)}d\sigma\\
&\le M\|h\|_{2,\exp}\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-\mathrm{Im}(y)\right)}}{|y-i\pi/2|}\int_{\frac{\frac{\pi}{2}-\mathrm{Im}(y)}{\varepsilon}}^{\frac{\pi}{2\varepsilon}} \varepsilon r e^{-\lambda_3 r}\varepsilon dr\\
&\le \frac{M\varepsilon\|h\|_{2,\exp}}{|y-i\pi/2|}\left(\frac{\varepsilon}{\lambda_3}+\frac{\pi}{2}-\mathrm{Im}(y)-e^{-\frac{\lambda_3}{\varepsilon}\mathrm{Im}(y)}\left(\frac{\varepsilon}{\lambda_3}+\frac{\pi}{2}\right)\right)\\
&\le M\varepsilon\|h\|_{2,\exp}\left(\frac{1}{\kappa}+1\right) \le M\varepsilon\|h\|_{2,\exp}.
\end{aligned}$$

Analogously, one can obtain the same estimate for $\mathrm{Im}(y) < 0$.

For $h \in \mathcal{X}_{\ell_1,5,\exp}$, one obtains,

$$\begin{aligned}
\left|e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}|y^2+\pi^2/4|^2\ddot{v}^h(y)\int_0^y \frac{h(s)}{\ddot{v}^h(s)}ds\right| &\le \frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y \left|\frac{h(s)}{\ddot{v}^h(s)}\right| ds\\
&\le M\|h\|_{5,\exp}\frac{e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}}{|y^2+\pi^2/4|}\int_0^y \frac{e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(s)|\right)}}{|s^2+\pi^2/4|^2}ds\\
&\le \frac{M\|h\|_{5,\exp}}{\kappa^3\varepsilon^3}e^{-\frac{\lambda_3}{\varepsilon}|\mathrm{Im}(y)|}\int_0^y e^{\frac{\lambda_3}{\varepsilon}|\mathrm{Im}(s)|}ds\\
&\le \frac{M\|h\|_{5,\exp}}{\kappa^3\varepsilon^2}.
\end{aligned}$$

We prove Item 2 only for the operator $\mathcal{P}^\Gamma$, since the result for $\mathcal{P}^\Theta$ follows analogously. Let $h(y,\tau) = \sum_{k\ge1} h_{2k+1}(y)\sin((2k+1)\tau)$. We bound each component of the operator $\mathcal{P}^\Gamma$ as

$$\begin{aligned}
\left|\mathcal{P}^\Gamma_{2k+1}(h_{2k+1})e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}\right| &\le \|h_{2k+1}\|_{\alpha,\exp}\int_y^{y^+}\left|e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(y)|\right)}\frac{e^{-\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\mathrm{Im}(s)|\right)}}{|s^2+\pi^2/4|^\alpha}e^{i\frac{\lambda_{2k+1}}{\varepsilon}(s-y)}\right|ds\\
&\le \|h_{2k+1}\|_{\alpha,\exp}\int_{\mathrm{Im}(y)}^{\frac{\pi}{2}-\kappa\varepsilon}\frac{e^{\frac{1}{\varepsilon}(\lambda_3|\sigma|-\lambda_{2k+1}\sigma-(\lambda_3|\mathrm{Im}(y)|-\lambda_{2k+1}\mathrm{Im}(y)))}}{|\sigma^2-\pi^2/4|^\alpha}d\sigma.
\end{aligned}$$

Now, since the functions $f_k(t) = \lambda_3|t| - \lambda_{2k+1}t$ are decreasing for $t \in \mathbb{R}$ and $k \geq 1$, $\sigma > \operatorname{Im}(y)$, and recalling that $\alpha > 1$, we obtain

$$\left|\mathcal{P}^{\Gamma}_{2k+1}(h_{2k+1})e^{\frac{\lambda_3}{\varepsilon}\left(\frac{\pi}{2}-|\operatorname{Im}(y)|\right)}\right| \leq \|h_{2k+1}\|_{\alpha,\exp}\int_{\operatorname{Im}(y)}^{\frac{\pi}{2}-\kappa\varepsilon}\frac{1}{|\sigma^2-\pi^2/4|^{\alpha}}d\sigma \leq \frac{M}{(\kappa\varepsilon)^{\alpha-1}}\|h_{2k+1}\|_{\alpha,\exp}.$$

□

In next proposition, we obtain estimates for the right hand side of equation (3.23).

Proposition 7.2. *There exists a constant M independent of ε and κ such that the following statements hold.*

(1) *The operator $\widetilde{\mathcal{M}} : \mathcal{Y}_{\ell_1,2,\exp} \to \mathcal{Y}_{\ell_1,2,\exp}$ introduced in (3.23) is well-defined and*

$$\left[\!\left[\widetilde{\mathcal{M}}(\Xi_1,\Gamma,\Theta)\right]\!\right]_{\ell_1,2,\exp} \leq \frac{M}{\kappa}\left[\!\left[(\Xi_1,\Gamma,\Theta)\right]\!\right]_{\ell_1,2,\exp}.$$

Moreover, denoting $\widetilde{\mathcal{M}} = (\widetilde{\mathcal{M}}_1, \widetilde{\mathcal{M}}_2, \widetilde{\mathcal{M}}_3)$, we have that

$$\left\|\widetilde{\mathcal{M}}_1(\Xi_1,\Gamma,\Theta)\right\|_{2,\exp} \leq \frac{M}{\kappa^3}\|\Xi_1\|_{2,\exp} + M\varepsilon\left(\|\Gamma\|_{\ell_1,0,\exp} + \|\Theta\|_{\ell_1,0,\exp}\right),$$

$$\left\|\widetilde{\mathcal{M}}_j(\Xi_1,\Gamma,\Theta)\right\|_{\ell_1,0,\exp} \leq \frac{M}{\kappa^2\varepsilon}\|\Xi_1\|_{2,\exp} + \frac{M}{\kappa}\left(\|\Gamma\|_{\ell_1,0,\exp} + \|\Theta\|_{\ell_1,0,\exp}\right),\ j=2,3.$$

(2) *The function $\widetilde{\Delta}$ defined in (3.18) satisfies*

$$\widetilde{\Delta} = (I-\widetilde{M})^{-1}\big(0,\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d)\big) \quad \text{and} \quad [\![\widetilde{\Delta} - \big(0,\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d)\big)]\!]_{\ell_1,2,\exp} \leq \frac{M}{\kappa}[\![\big(0,\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d)\big)]\!]_{\ell_1,2,\exp},$$

where $\mathcal{I}_\Gamma(c)$, $\mathcal{I}_\Theta(d)$ are the functions defined in (3.21) and (3.24).

Proof. Assume that $(\Xi_1,\Gamma,\Theta) \in \mathcal{Y}_{\ell_1,2,\exp}$. To estimate the first component of $\mathcal{M}$, using the estimates for m_W and $\mathcal{M}_W$ in Proposition 3.11 and Lemma 7.1 for the estimates on $\mathcal{P}^W$,

$$\begin{aligned}\left\|\widetilde{\mathcal{M}}_1(\Xi_1,\Gamma,\Theta)\right\|_{2,\exp} &\leq \left\|\mathcal{P}^W(m_W\Xi_1)\right\|_{2,\exp} + \left\|\mathcal{P}^W(M_W(\Gamma,\Theta)\right\|_{2,\exp}\\ &\leq \frac{M}{\varepsilon^2\kappa^3}\|m_W\Xi_1\|_{5,\exp} + M\varepsilon\|M_W(\Gamma,\Theta)\|_{2,\exp}\\ &\leq \frac{M}{\kappa^3}\|\Xi_1\|_{2,\exp} + M\varepsilon\left(\|\Gamma\|_{\ell_1,0,\exp} + \|\Theta\|_{\ell_1,0,\exp}\right).\end{aligned}$$

Now we estimate $\widetilde{\mathcal{M}}_2$. The estimates for $\widetilde{\mathcal{M}}_3$ can be done analogously. Using as before Proposition 3.11 and Lemma 7.1,

$$\begin{aligned}\left\|\widetilde{\mathcal{M}}_2(\Xi_1,\Gamma,\Theta)\right\|_{\ell_1,0,\exp} &\leq \left\|\mathcal{P}^\Gamma(m_{\mathrm{osc}}\Xi_1)\right\|_{\ell_1,0,\exp} + \left\|\mathcal{P}^\Gamma(M_{\mathrm{osc}}(\Gamma,\Theta)\right\|_{\ell_1,0,\exp}\\ &\leq \frac{M}{\kappa^2\varepsilon^2}\|m_{\mathrm{osc}}\Xi_1\|_{\ell_1,3,\exp} + \frac{M}{\kappa\varepsilon}\|M_{\mathrm{osc}}(\Gamma,\Theta)\|_{\ell_1,2,\exp}\\ &\leq \frac{M}{\kappa^2\varepsilon}\|\Xi_1\|_{\ell_1,2,\exp} + \frac{M}{\kappa}\left(\|\Gamma\|_{\ell_1,0,\exp} + \|\Theta\|_{\ell_1,0,\exp}\right).\end{aligned}$$

Item (2) of the proposition is simply a direct consequence of item (1) and (3.23). □

The rest of this section is devoted to estimating $\big(0,\mathcal{I}_\Gamma(c),\mathcal{I}_\Theta(d)\big)$.

Lemma 7.3. *Take $\kappa = \frac{1}{2\mu_3}|\log\varepsilon|$. There exist $\varepsilon_0 > 0$ and a constant $M > 0$ independent of ε such that, for each $\varepsilon \in (0,\varepsilon_0)$,*

$$\left\|\mathcal{I}_\Gamma(c) - \frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1,0,\exp},\ \left\|\mathcal{I}_\Theta(d) - \frac{2\mu_3}{\varepsilon}\overline{C_{\mathrm{in}}}e^{i\frac{\lambda_3}{\varepsilon}(y+i\pi/2)}\sin(3\tau)\right\|_{\ell_1,0,\exp} \leq \frac{M}{\varepsilon|\log\varepsilon|}.$$

Proof. From Theorems 3.3 and 3.6 (see also (3.7)), the function Δ given in (3.14) can be written as

$$\begin{aligned}\Delta(y,\tau) &= \frac{1}{\varepsilon}\phi^u\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)-\frac{1}{\varepsilon}\phi^s\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)\\ &= \frac{1}{\varepsilon}\Delta\phi^0\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)+\frac{1}{\varepsilon}\varphi^u\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)-\frac{1}{\varepsilon}\varphi^s\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)\\ &= \frac{1}{\varepsilon}e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}\left(C_{\text{in}}\sin(3\tau)+\chi\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)\right)\\ &\quad+\frac{1}{\varepsilon}\varphi^u\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)-\frac{1}{\varepsilon}\varphi^s\left(\frac{y-i\pi/2}{\varepsilon},\tau\right)\\ &= \frac{1}{\varepsilon}C_{\text{in}}e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}\sin(3\tau)+E_1^+(y,\tau)+E_2^+(y,\tau),\end{aligned}$$

for every $y\in\mathcal{R}^+_{\text{mch},\kappa}=D^{\text{mch},u}_{+,\kappa}\cap D^{\text{mch},s}_{+,\kappa}\cap i\mathbb{R}$ and κ satisfying assumptions in Theorems 3.3 and 3.6, where $E_1^+,E_2^+:\mathcal{R}_{\text{mch},\kappa}\times\mathbb{T}\to\mathbb{C}$ are analytic functions in the variable y. It follows from Theorem 3.3 that

$$\|\partial_\tau E_1^+\|_{\ell_1}(y)\leq\frac{M|e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|}\quad\textit{and}\quad\|\partial_y E_1^+\|_{\ell_1}(y)\leq\frac{M|e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|^2},\tag{7.1}$$

and from Theorem 3.6, choosing $\gamma=1/2$, we obtain

$$\|\partial^2_\tau E_2^+\|_{\ell_1}(y)\leq\frac{M\varepsilon^{3/2}|\log\varepsilon|}{|y-i\pi/2|^2}\quad\textit{and}\quad\|\partial^2_\tau\partial_y E_2^+\|_{\ell_1}(y)\leq\frac{M\varepsilon^{1/2}|\log\varepsilon|}{\kappa|y-i\pi/2|^2}.\tag{7.2}$$

Analogously, since Δ is real-analytic one can deduce that for $y\in\mathcal{R}^-_{\text{mch},\kappa}=\{z:\bar z\in\mathcal{R}^+_{\text{mch},\kappa}\}$,

$$\Delta(y,\tau)=\frac{1}{\varepsilon}\overline{C_{\text{in}}}e^{i\mu_3\frac{y+i\pi/2}{\varepsilon}}\sin(3\tau)+E_1^-(y,\tau)+E_2^-(y,\tau),$$

where $E_j^-(y,\tau)=\overline{E_j^+(\bar y,\tau)}$, which satisfy

$$\begin{aligned}&\|\partial_\tau E_1^-\|_{\ell_1}(y)\leq\frac{M|e^{i\mu_3\frac{y+i\pi/2}{\varepsilon}}|}{|y+i\pi/2|}\quad\text{and}\quad\|\partial_y E_1^-\|_{\ell_1}(y)\leq\frac{M|e^{i\mu_3\frac{y+i\pi/2}{\varepsilon}}|}{|y+i\pi/2|^2},\\ &\|\partial^2_\tau E_2^-\|_{\ell_1}(y)\leq\frac{M\varepsilon^{3/2}|\log\varepsilon|}{|y+i\pi/2|^2}\quad\text{and}\quad\|\partial_y E_2^-\|_{\ell_1}(y)\leq\frac{M\varepsilon^{1/2}|\log\varepsilon|}{\kappa|y+i\pi/2|^2}.\end{aligned}$$

Using (3.17) and recalling that $\lambda_3=\mu_3+\mathcal{O}(\varepsilon^2)$, we obtain that for $(y,\tau)\in\mathcal{R}^+_{\text{mch},\kappa}\times\mathbb{T}$,

$$\begin{aligned}\Gamma(y,\tau)&=\sum_{k\geq1}(\lambda_{2k+1}\Delta_{2k+1}(y)+i\varepsilon\partial_y\Delta_{2k+1}(y))\sin((2k+1)\tau)\\ &=\frac{2\mu_3}{\varepsilon}C_{\text{in}}e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}(1+\mathcal{O}(\varepsilon^2))\sin(3\tau)\\ &\quad+\sum_{k\geq1}\lambda_{2k+1}\Pi_{2k+1}\left[E_1^++E_2^+\right]\sin((2k+1)\tau)+i\varepsilon\widetilde\Pi\left[\partial_yE_1^++\partial_yE_2^+\right](y,\tau).\end{aligned}$$

Moreover, using (7.1) and (7.2), we have that

$$\begin{aligned}\left\|\sum_{k\geq0}\lambda_{2k+1}\Pi_{2k+1}\left[E_1^++E_2^+\right]\sin((2k+1)\tau)\right\|_{\ell_1}(y)&\leq M\left(\|\partial_\tau E_1^+\|_{\ell_1}(y)+\|\partial_\tau E_2^+\|_{\ell_1}(y)\right)\\ &\leq M\left(\frac{|e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|}+\frac{\varepsilon^{3/2}|\log\varepsilon|}{|y-i\pi/2|^2}\right)\\ \left\|i\varepsilon\widetilde\Pi\left[\partial_yE_1^++\partial_yE_2^+\right]\right\|_{\ell_1}(y)&\leq M\left(\frac{\varepsilon|e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|^2}+\frac{\varepsilon^{3/2}|\log\varepsilon|}{\kappa|y-i\pi/2|^2}\right).\end{aligned}$$

Then, for $(y,\tau)\in\mathcal{R}^+_{\mathrm{mch},\kappa}\times\mathbb{T}$, Γ satisfies

$$\Gamma(y,\tau)=\frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}\sin(3\tau)+E^+_\Gamma(y,\tau),$$

where $E^+_\Gamma:\mathcal{R}^+_{\mathrm{mch},\kappa}\times\mathbb{T}\to\mathbb{C}$ is an analytic function in the variable τ such that

$$\left\|E^+_\Gamma\right\|_{\ell_1}(y)\leq M\left(\frac{|e^{-i\mu_3\frac{y-i\pi/2}{\varepsilon}}|}{|y-i\pi/2|}+\frac{\varepsilon^{3/2}|\log\varepsilon|}{|y-i\pi/2|^2}\right).$$

Proceeding in the same way for the function

$$\Theta(y,\tau)=\sum_{k\geq 0}(\lambda_{2k+1}\Delta_{2k+1}(y)-i\varepsilon\partial_y\Delta_{2k+1}(y))\sin((2k+1)\tau),$$

we conclude that there exists a function $E^-_\Theta:\mathcal{R}^-_{\mathrm{mch},\kappa}\times\mathbb{T}\to\mathbb{C}$ analytic in the variable y such that Θ can be written as

$$\Theta(y,\tau)=\frac{2\mu_3}{\varepsilon}\overline{C_{\mathrm{in}}}e^{i\mu_3\frac{y+i\pi/2}{\varepsilon}}\sin(3\tau)+E^-_\Theta(y,\tau),\qquad\text{for }\ (y,\tau)\in\mathcal{R}^-_{\mathrm{mch},\kappa}\times\mathbb{T}$$

and

$$\left\|E^-_\Theta\right\|_{\ell_1}(y)\leq M\left(\frac{|e^{i\mu_3\frac{y+i\pi/2}{\varepsilon}}|}{|y+i\pi/2|}+\frac{\varepsilon^{3/2}|\log\varepsilon|}{|y+i\pi/2|^2}\right),\qquad\text{for } y\in\mathcal{R}^-_{\mathrm{mch},\kappa}.$$

Now that we have good estimates for the functions Γ and Θ in the domains $\mathcal{R}^\pm_{\mathrm{mch},\kappa}$, we analyze the functions $\mathcal{I}_\Gamma(c)$, $\mathcal{I}_\Theta(d)$. Recall that $\mathcal{I}_\Gamma(c)(y^+)=\Gamma(y^+)$. Therefore

$$\begin{aligned}\left\|\mathcal{I}_\Gamma(c)-\frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1}(y^+)&=\left\|\Gamma-\frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1}(y^+)\\&=\left\|E^+_\Gamma\right\|_{\ell_1}(y^+)\\&\leq M\left(\frac{|e^{-i\mu_3\frac{y^+-i\pi/2}{\varepsilon}}|}{|y^+-i\pi/2|}+\frac{\varepsilon^{3/2}|\log\varepsilon|}{|y^+-i\pi/2|^2}\right)\\&\leq M\left(\frac{e^{-\mu_3\kappa}}{\kappa\varepsilon}+\frac{\varepsilon^{3/2}|\log\varepsilon|}{\kappa^2\varepsilon^2}\right),\end{aligned}$$

and notice that, from (3.21), we have that

$$\left\|\mathcal{I}_\Gamma(c)-\frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1,0,\exp}=e^{\lambda_3\kappa}\left\|\mathcal{I}_\Gamma(c)-\frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1}(y^+),$$

and thus, taking $\kappa=\frac{1}{2\lambda_3}\log(\varepsilon^{-1})$, we have that

$$\left\|\mathcal{I}_\Gamma(c)-\frac{2\mu_3}{\varepsilon}C_{\mathrm{in}}e^{-i\frac{\lambda_3}{\varepsilon}(y-i\pi/2)}\sin(3\tau)\right\|_{\ell_1,0,\exp}\leq M\left(\frac{e^{(\lambda_3-\mu_3)\kappa}}{\kappa\varepsilon}+\frac{\varepsilon^{3/2}|\log\varepsilon|e^{\lambda_3\kappa}}{\kappa^2\varepsilon^2}\right)\leq\frac{M}{\varepsilon|\log\varepsilon|}.$$

The estimate on $\mathcal{I}_\Theta(d)$ follows analogously and it completes the proof of the lemma. □

Proposition 3.12 follows directly from Proposition 7.2 and Lemma 7.3.

8. Non-existence of small breathers: strongly hyperbolic case $\omega\in J_k(\varepsilon_0)$

This section is devoted to proving statement (1) of Theorem 1.3, that is the results for the case $\omega\in J_k(\varepsilon_0)$, $k\geq 0$. The other case $\omega\in I_k(\varepsilon_0)$ will be proved in Section 9. The oddness of u in t is not assumed in these two sections.

For any $\omega\in J_k(\varepsilon_0)$, $k>0$, we adopt the rescaling $\tau=\omega t$ and the nonlinear Klein-Gordon equation (1.2) turns into the form of (1.10). Treating x as the dynamic variable and recallint that u is 2π-periodic in τ, the unknown $u(x,\tau)$ can be expanded in Fourier series

$$u(x,\tau)=\sum_{n=-\infty}^{\infty}u_n(x)e^{in\tau},\quad u_{-n}=\overline{u_n}.$$

(Note this Fourier series is different from the rest of the paper by a ratio of $-\frac{i}{2}$. The latter was adapted so that u_n is the real coefficient of the Fourier sine series when u is odd in t.) The eigenvalues of the linearization of (1.10) at 0

$$\partial_x^2 u - \omega^2 \partial_\tau^2 u - u = 0$$

are $\pm\nu_n$, where

$$\nu_n = \sqrt{1 - n^2\omega^2}, \tag{8.1}$$

and their eigenfunctions can be calculated using the Fourier series. The hyperbolic eigenvalues correspond to $0 \le n \le k$ and

$$\nu_0 \ge \ldots \ge \nu_k \in \left(\frac{\varepsilon_0}{\sqrt{k+\varepsilon_0^2}}, \frac{\sqrt{2k+1}}{k+1} \right], \tag{8.2}$$

while the center eigenvalues correspond to $n \ge k+1$ and

$$\nu_n = i\vartheta_n, \quad 0 \le \vartheta_{k+1} \le \vartheta_{k+2} \le \ldots. \tag{8.3}$$

Let $W_\omega^\star(0)$, $\star = c, s, u$, denote locally invariant center, stable, and unstable manifolds of 0 for the equation (1.10) in the energy space $H_\tau^1 \times L_\tau^2$. Their existence and smoothness follow from standard arguments (see [14] for example) since the nonlinearity $g(u) : H_\tau^1 \to H_\tau^2 \hookrightarrow L_\tau^2$ is analytic. Due to the uniqueness, $W_\omega^\star(0)$, $\star = s, u$, are also obviously the local stable and unstable manifolds of 0 in the ℓ_1 based phase space $(u, \partial_x u) \in \mathbf{X}$ defined in (2.1). Clearly $\dim W_\omega^\star(0) = 2k+1$, $\star = s, u$, while $W_\omega^c(0)$ is of codim-$(4k+2)$. Statement (1) of Theorem 1.3 for the case of $\omega \in J_k(\varepsilon_0)$, $k \ge 0$, will be proved by showing a.) some uniform-in-k-and-ω estimates on the size of $W_\omega^\star(0)$ in ℓ_1, $\star = s, u$, where the norm is dominated by the energy norm, and b.) no solutions converging to 0 along $W_\omega^c(0) \subset H_\tau^1 \times L_\tau^2$.

• **Estimates on the local stable/unstable manifolds for $\omega \in J_k(\varepsilon_0)$.** Usually the sizes of the local stable/unstable manifolds in phase spaces are determined by the power nonlinearity and the minimal absolute value of the real parts of the stable/unstable eigenvalues, which is $\nu_k > \frac{\varepsilon_0}{\sqrt{k+\varepsilon_0^2}}$ according to (8.2). We prove the following proposition on a lower bound of the sizes of $W_\omega^\star(0)$ in $\mathbf{X}$.

Proposition 8.1. *There exists $\rho, M > 0$ such that, for any $\varepsilon_0 \in (0, 1/2)$, $\omega \in J_k(\varepsilon_0)$, $k \ge 0$, there exist $\Omega^u, \Omega^s : B_{\mathbb{R}^{2k+1}}(0, \rho\nu_k) \to \mathbf{X}$, where $B_{\mathbb{R}^{2k+1}}(0, \rho\nu_k)$ is the ball in $\mathbb{R}^{2k+1}$ centered at 0 and with radius $\rho\nu_k$, such that, the image $\Omega^\star\big(B_{\mathbb{R}^{2k+1}}(0, \rho\nu_k)\big)$ is an open subset of $W_\omega^\star(0)$, $\star = s, u$, and*

$$\Omega^\star(0, \tau) = 0, \quad \|\Omega_1^\star(a, \cdot) - \Omega_1^\star(\tilde a, \cdot)\|_{\ell_1} + \nu_k^{-1} \|\Omega_2^\star(a, \cdot) - \Omega_2^\star(\tilde a, \cdot)\|_{\ell_1} \le M\nu_k^{-2}(|a|_1^2 + |\tilde a|_1^2)|a - \tilde a|_1$$

where

$$\Omega^\star(a, \tau) = \left(\sum_{n=-k}^{k} a_n e^{in\tau} + \Omega_1^\star(a, \tau), \ \sum_{n=-k}^{k} (\mp\nu_n) a_n e^{in\tau} + \Omega_2^\star(a, \tau) \right),$$

and a and $\tilde a$ are parameters of $(2k+1)$-dim (real) satisfying

$$a = (a_{-k}, \ldots, a_k), \ \ a_n \in \mathbb{C}, \ \ a_{-n} = \overline{a_n}, \ \ -k \le n \le k, \quad |a|_1 := \sum_{n=-k}^{k} |a_n| < \rho\nu_k. \tag{8.4}$$

Here we identified complex numbers a_n with 2-dim real vectors. These $\Omega^\star$ can be viewed as coordinate mappings of $W_\omega^\star(0)$. They can actually be proved to be analytic in a, but our main focus here is the sizes of their domains and the error estimates.

We use the classical Perron method and will only outline the argument to prove the proposition for the stable manifold. Consider the following Banach space

$$\mathcal{E}_{\mathcal{S}} = \{h : [0, +\infty) \times \mathbb{T} \to \mathbb{R};\ h \text{ is analytic in } x, \text{ and } \|h\|_{\nu_k, \ell_1} < \infty\},$$

where

$$\|h\|_{\nu_k, \ell_1} = \sum_{n \ge 1} \|h_n\|_{\nu_k} \quad \textit{and} \quad \|h_n\|_{\nu_k} = \sup_{x \ge 0} |e^{\nu_k x} h_n(x)|,$$

and define the linear operator $\mathcal{S}$ acting on the Fourier modes of a function $h(x,\tau)$

$$\mathcal{S}\,(h) = \sum_{n\geq 1} \mathcal{S}_n(h_n)\sin(n\tau), \tag{8.5}$$

with

$$\mathcal{S}_n(h) = \frac{1}{2\nu_n}\left(\int_{+\infty}^{x} e^{\nu_n(x-s)}h(s)ds - \int_0^x e^{-\nu_n(x-s)}h(s)ds\right) \qquad \text{for } \ 1\leq n\leq k,$$
$$\mathcal{S}_n(h) = \int_{+\infty}^{x} \frac{\sin(\vartheta_n(x-s))}{\vartheta_n}h(s)ds \qquad \text{for } \ n>k.$$

where we recall $\nu_n = i\vartheta_n$.

Note that we are including in $J_k(\varepsilon_0)$ the case $\omega = 1/(k+1)$. For this value of ω, one has that $\vartheta_{k+1} = 0$. In this case, one can take the limit $\vartheta_{k+1}\to 0$ in $\mathcal{S}_{k+1}(h)$ to obtain

$$\mathcal{S}_{k+1}(h) = \int_{+\infty}^{x} (x-s)h(s)ds.$$

We also define the function

$$\Xi(a,x,\tau) = \sum_{n=-k}^{k} a_n e^{-\nu_n x + in\tau},$$

where $a = (a_{-k},\dots,a_k)$ are parameters satisfying (8.4). One can check that a solution $u(x,\tau)$ of (1.10) belongs to the stable manifold of $u=0$ if, and only if, it is a fixed point of the operator

$$\widetilde{\mathcal{S}}(a,u) = \Xi(a) + \mathcal{S}(g(u)),$$

for some a as in (8.4), where g is the nonlinearity introduced in (1.11).

The following lemma is a direct consequence of the particular form of the operator $\mathcal{S}$ in (8.5) and the fact that the function g is of order 3 near $u=0$.

Lemma 8.2. *There exists $M, r_1 > 0$ independent of $k\geq 0$ and $\omega\in J_k(\varepsilon_0)$ such that, for any $0<r\leq r_1$ and $a\in\mathbb{R}^{2k+1}$, the operator $\widetilde{\mathcal{S}} : B(0,r)\subset\mathcal{E}_{\mathcal{S}}\to\mathcal{E}_{\mathcal{S}}$ is a well-defined Lipschitz operator which satisfies*

$$\left\|\widetilde{\mathcal{S}}(a,u)\right\|_{\nu_k,\ell_1} \leq |a|_1 + \frac{Mr^3}{\nu_k^2}, \quad \left\|\partial_x\widetilde{\mathcal{S}}(a,u)\right\|_{\nu_k,\ell_1} \leq |a|_1 + \frac{Mr^3}{\nu_k}, \qquad \forall\, u\in B(0,r)\subset\mathcal{E}_{\mathcal{S}}$$

and its Lipschitz constant on $B(0,r)$ satisfies

$$\mathrm{Lip}_u(\widetilde{\mathcal{S}}) \leq \frac{Mr^2}{\nu_k^2}, \quad \mathrm{Lip}_u(\partial_x\widetilde{\mathcal{S}}) \leq \frac{Mr^2}{\nu_k}.$$

Consequently, there exists $\rho>0$ independent of $k\geq 0$ and $\omega\in J_k(\varepsilon_0)$ such that, for any $a\in B_{\mathbb{R}^{2k+1}}(0,\rho\nu_k)$, by taking $r = 2|a|_1$, there exists a unique fixed point $h_*(a)\in B(0,r)\subset\mathcal{E}_{\mathcal{S}}$ of $\widetilde{\mathcal{S}}(a,\cdot)$ which also satisfies $h_*(a=0)=0$ and

$$\left\|\big(h_*(a)-\Xi(a)\big) - \big(h_*(\tilde a)-\Xi(\tilde a)\big)\right\|_{\nu_k,\ell_1} \leq M\nu_k^{-2}(|a|_1^2+|\tilde a|_1^2)|a-\tilde a|_1$$
$$\left\|\partial_x\big(h_*(a)-\Xi(a)\big) - \partial_x\big(h_*(\tilde a)-\Xi(\tilde a)\big)\right\|_{\nu_k,\ell_1} \leq M\nu_k^{-1}(|a|_1^2+|\tilde a|_1^2)|a-\tilde a|_1.$$

Let

$$\Omega^s(a,\tau) = \big(h_*(a,0,\tau), \partial_x h_*(a,0,\tau)\big).$$

The conclusions of Proposition 8.1 follow from standard and straight forward arguments.

• **Nonexistence of decaying solutions on the center manifold.** Recall that, when x is viewed as the dynamic variable, the nonlinear Klein-Gordon equation (1.10) conserves the Hamiltonian $\mathcal{H}(u,\partial_x u)$ where

$$\mathcal{H}(u_1,u_2) = \int_{-\pi}^{\pi} \frac{1}{2}u_2^2 + \frac{\omega^2}{2}(\partial_\tau u_1)^2 - \frac{1}{2}u_1^2 + \frac{1}{12}u_1^4 + F(u_1)d\tau$$

is smoothly defined on the energy space $H^1_\tau\times L^2_\tau$. Let $W^c_\omega(0)$ be a center manifold of $(0,0)$ for (1.10). The following lemma holds for all $\omega>0$, not just those in $I_k(\varepsilon_0)$ or $J_k(\varepsilon_0)$.

Lemma 8.3. *For any $\omega>0$, $(0,0)$ is a strict local minimum of $\mathcal{H}$ restricted on its local center manifold .*

Proof. There exists unique $k \geq 0$ such that $\omega \in [1/(k+1), 1/k)$. Let $\mathbf{Y}^c, \mathbf{Y}^h \subset H^1_\tau \times L^2_\tau$ denote the center and hyperbolic subspaces of the linearization of (1.10) at $(0,0)$

$$\mathbf{Y}^c = \{(u_1, u_2) \in H^1_\tau \times L^2_\tau \mid u_j(\tau) = \sum_{|n| \geq k+1} u_{j,n} e^{in\tau},\ u_{j,-n} = \overline{u_{j,n}},\ j = 1,2\},$$

$$\mathbf{Y}^h = \{(u_1, u_2) \mid u_j(\tau) = \sum_{|n| \leq k} u_{j,n} e^{in\tau},\ u_{j,-n} = \overline{u_{j,n}}.\ j = 1,2\}.$$

Locally $W^c_\omega(0)$ can be represented as the graph of a smooth mapping $\gamma^c(u_1, u_2)$ from a small neighborhood of $(0,0)$ in $\mathbf{Y}^c$ to $\mathbf{Y}^h$. Due to the lack of quadratic nonlinear terms in (1.10), γ^c satisfies

$$\gamma^c(u_1, u_2) = \mathcal{O}\big(\|u_1\|^3_{H^1_\tau} + \|u_2\|^3_{L^2_\tau}\big).$$

Due to $F(u) = \mathcal{O}(|u|^6)$ for small u and the orthogonality between $\mathbf{Y}^c$ and $\mathbf{Y}^h$, for small $(u_1, u_2) \in \mathbf{Y}^c$,

$$\begin{aligned}\mathcal{H}\big((u_1, u_2) + \gamma^c(u_1, u_2)\big) &= \int_{-\pi}^{\pi} \frac{1}{2}u_2^2 + \frac{\omega^2}{2}(\partial_\tau u_1)^2 - \frac{1}{2}u_1^2 + \frac{1}{12}u_1^4 d\tau + \mathcal{O}\big(\|u_1\|^6_{H^1_\tau} + \|u_2\|^6_{L^2_\tau}\big) \\ &\geq \frac{1}{2}\|u_2\|^2_{L^2_\tau} + \pi \sum_{|n| \geq k+1}^{\infty} \vartheta_n^2 u_{1,n}^2 + \frac{1}{24\pi}\|u_1\|^4_{L^2_\tau} - \mathcal{O}\big(\|u_1\|^6_{H^1_\tau} + \|u_2\|^6_{L^2_\tau}\big).\end{aligned}$$

If $\omega \neq \frac{1}{k+1}$, then there exists $\delta > 0$ such that

$$\frac{\vartheta_n^2}{1+n^2} \geq \delta,\ \forall |n| \geq k+1 \implies \sum_{n=k+1}^{\infty} \vartheta_n^2 u_{1,n}^2 \geq \frac{\delta}{2\pi}\|u_1\|^2_{H^1_\tau}.$$

Therefore in this case $(0,0)$ is clearly a non-degenerate local minimum of $\mathcal{H}$ on $W^c_\omega(0)$. If $\omega = \frac{1}{k+1}$, then $\vartheta_{\pm(k+1)} = 0$ and there exists $\delta > 0$ such that

$$\frac{\vartheta_n^2}{1+n^2} \geq \delta,\ \forall |n| \geq k+2.$$

Let

$$\tilde{u}_1 = \sum_{|n| \geq k+2} u_n(x) e^{in\tau}$$

and then we have

$$\mathcal{H}\big((u_1, u_2) + \gamma^c(u_1, u_2)\big) \geq \frac{1}{2}\|u_2\|^2_{L^2_\tau} + \frac{\delta}{2}\|\tilde{u}_1\|^2_{H^1_\tau} + \frac{1}{24\pi}\|u_1\|^4_{L^2_\tau} - \mathcal{O}\big(\|\tilde{u}_1\|^6_{H^1_\tau} + |u_{1,\pm(k+1)}|^6 + \|u_2\|^6_{L^2_\tau}\big).$$

Again $(0,0)$ is clearly a strict local minimum of $\mathcal{H}$ on $W^c_\omega(0)$. □

Due the conservation of $\mathcal{H}$, we immediately obtain

Corollary 8.4. *For any $\omega > 0$, $(0,0)$ has a locally unique center manifold $W^c_\omega(0)$ and is stable on $W^c_\omega(0)$ both forward and backward in x. Moreover, except $(0,0)$ no solution on $W^c_\omega(0)$ converges to $(0,0)$ as $x \to +\infty$ or $-\infty$.*

Finally we are ready to complete the proof of statement (1) of Theorem 1.3.

Proof of statement (1) of Theorem 1.3. Let $\varepsilon_0 \in (0, 1/2)$, $k \geq 0$, and $\omega \in J_k(\varepsilon_0)$. Since the $\|\cdot\|_{\ell_1}$ norm is invariant under a rescaling in τ, we can work on (1.10) equivalently. Without loss of generality, assume $u(x,\tau)$ is a solution such that $(u, \partial_x u)$ converging to $(0,0)$ in $H^1_\tau \times L^2_\tau$ as $x \to +\infty$. Such a solution must belong to the local center-stable manifold of $(0,0)$ for $x \gg 1$. It is well-known that the center-stable manifold is foliated into stable fibers based on the local center manifold $W^s_\omega(0)$ (for example, see [13]). The dynamics of all initial data on each fiber is shadowed by that of the based point on $W^c_\omega(0)$. According to Corollary 8.4, no non-trivial solutions on $W^c_\omega(0)$ converges to $(0,0)$ as $x \to +\infty$, the based point of the decaying solution $u(x,\tau)$ must be $(0,0) \in W^s_\omega(0)$ and thus it belongs to the stable manifold $W^s_\omega(0)$. From Proposition 8.1, locally the stable manifold $W^s_\omega(0) = \Omega^s\big(B_{\mathbb{R}^{2k+1}}(0, \rho\nu_k)\big)$ and Ω^s is a small perturbation of the isomorphism $\big(\Xi(a), \mathrm{diag}(\nu_{-k}, \ldots, \nu_k)\Xi(a)\big)$. In the coordinates $(a_{-k}, \ldots, a_k)$, the dynamics on $W^s_\omega(0)$ is governed by

$$\frac{da}{dx} = \big(D\Xi + D_a\Omega^s_1(a,\cdot)\big)^{-1}\big(-\mathrm{diag}(\nu_{-k}, \ldots, \nu_k)\Xi(a) + \Omega^s_2(a,\cdot)\big) = -\mathrm{diag}(\nu_{-k}, \ldots, \nu_k)a + \mathcal{O}(\nu_k\rho^2|a|_1),$$

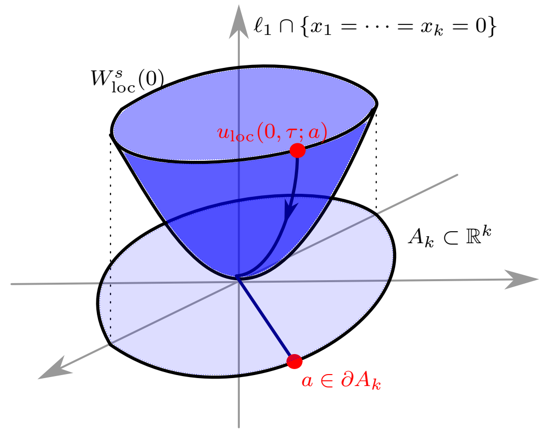

FIGURE 11. Parameterization of the local stable manifold $W^s_{\mathrm{loc}}(0)$ by u_{loc}.

where the estimates on Ω_1^s and Ω_2^s given in Proposition 8.1 were used. It is straightforward to prove that, as x evolves backwards, every solution on $W^s_\omega(0)$ must exit through its boundary where the u_1 component (corresponding to $u(x, \cdot)$ itself) satisfies $\|u_1\|_{\ell_1} \geq \frac{1}{2}\rho\nu_k$. Finally statement (1) of Theorem 1.3 follows from

$$\frac{\nu_k}{\omega^{\frac{1}{2}}} = \left(\frac{1}{\omega} - k^2\omega\right)^{\frac{1}{2}} \geq \left(\sqrt{k(k+\varepsilon_0^2)} - \frac{k^{\frac{3}{2}}}{\sqrt{k+\varepsilon_0^2}}\right)^{\frac{1}{4}} \geq \varepsilon_0 \left(\frac{k}{k+\varepsilon_0^2}\right)^{\frac{1}{2}} \geq \frac{\varepsilon_0}{2}, \quad \text{if } k \geq 1,$$

and $\nu_0 = 1$ if $k = 0$. □

9. BIFURCATION ANALYSIS FOR $\omega \in I_k(\varepsilon_0)$

We devote this section to the proof of Statements 2 and 3 of Theorem 1.3, that is the statements concerning $\omega \in I_k(\varepsilon_0)$. For such ω, there are two pair of (weakly) hyperbolic eigenvalues along with $2k-1$ pairs of stronger ones (see (8.1)). Our strategy is to reduce the problem to $\omega \in I_1(\varepsilon_0)$ and $u(x,t)$ odd in t.

We analyze the birth of small homoclinic loops taking

$$\omega = \sqrt{\frac{1}{k(k+\varepsilon^2)}} \qquad \text{with} \qquad k \geq 1, \;\; 0 < \varepsilon \leq \varepsilon_0 \leq \frac{1}{2}.$$

We expand the (real) solution $u(x,\tau)$ to the nonlinear Klein-Gordon equation (1.10) in Fourier series in τ as

$$u(x,\tau) = \sum_{n=-\infty}^{+\infty} \left(-\frac{i}{2}\right) u_n(y) e^{in\tau}, \quad u_{-n} = -\overline{u_n},$$

where the $-i/2$ factor is simply for the technical convenience that, if $u(x,\tau)$ is odd in τ, then $u_n(y)$, $n > 0$, coincides with the coefficient in its Fourier sine series expansion. Subsequently (1.10) is equivalent to a coupled system of equations in the form of

$$\partial_x^2 u_n = \nu_n^2 u_n - \Pi_n\left[g(u)\right], \quad n \in \mathbb{Z}, \tag{9.1}$$

where Π_n is the projection from $u(\tau)$ to the n-th mode u_n as in the above expansion and

$$\nu_n = \sqrt{1 - n^2\omega^2}, \quad 1 = \nu_0 > \ldots > \nu_k = \varepsilon(k+\varepsilon^2)^{-\frac{1}{2}}, \;\; \nu_n = i\vartheta_n, \;\; \vartheta_{k+1} < \vartheta_{k+2} < \ldots, n \geq k+1, \tag{9.2}$$

are same as those in (8.1) and (8.3). In particular,

$$\nu_{k-1} = \sqrt{\frac{(2+\varepsilon^2)k - 1}{k(k+\varepsilon^2)}} \geq \sqrt{\frac{1}{k}}, \quad \vartheta_{k+1} = \sqrt{\frac{(2-\varepsilon^2)k+1}{k(k+\varepsilon^2)}} \geq \sqrt{\frac{1}{k}}. \tag{9.3}$$

Linearizing at $u \equiv 0$, clearly $|n| \leq k$ corresponds to $2k+1$ pairs of hyperbolic directions, and $|n| \geq k+1$ to codim-$(4k+2)$ center directions. From the same argument as in the proof of statement (1) of Theorem 1.3 (see Section 8) based on Lemma 8.3, a solution $u(x,\tau)$ satisfies $\|(u, \partial_x u)\|_{\mathbf{X}} \to 0$ as $x \to \pm\infty$ if and only if $(u, \partial_x u) \in W^\star_\omega(0)$, $\star = s, u$. Hence we shall focus on the estimates of the sizes and the splitting distance between $W^u_\omega(0)$ and $W^s_\omega(0)$.

9.1. **Estimates on the local stable/unstable manifolds for $\omega \in I_k(\varepsilon_0)$.** For a semilinear PDE like (9.1), the standard theorems (see, for example, [14]) yield the existence of smooth local invariant manifolds $W^\star_\omega(0)$, $\star = s, u, c, cs, cu$, in the phase space $\mathbf{X}$ defined in (2.1). There are two issues, however. On the one hand, usually the sizes of the local invariant manifolds are generally determined by the gap between the real parts of the eigenvalues. While $\nu_n \geq k^{-\frac{1}{2}}$ for $|n| \leq k-1$, the weakest stable/unstable eigenvalues $\pm\nu_k = \mathcal{O}(\varepsilon k^{-\frac{1}{2}})$ of (9.1) are too small for the analysis of possible breathers of amplitude $\|u\|_{\ell_1} = O(k^{-\frac{1}{2}})$. On the other hand, the "angles" between the stable and unstable eigenfunctions in $\mathbf{X}$ of (9.1) can be rather small for $n \sim k$. In this subsection, we shall outline the construction of $W^\star_\omega(0)$, $\star = s, u$, with desired estimates based on the specific structure of (1.10), or equivalently (9.1). Essentially our strategy is to construct $W^s_\omega(0)$ as the union of strong stable fibers based on a weak stable manifold.

Observe

$$\mathcal{Z}_o = \left\{ (u_1, u_2) \in \mathbf{X} \mid u_j(\tau) = \sum_{n \in \mathbb{Z}} \left(-\frac{i}{2}\right) u_{j,kn} e^{ikn\tau} = \sum_{n \in \mathbb{N}} u_{j,kn} \sin(kn\tau),\ u_{j,kn} \in \mathbb{R},\ j = 1, 2 \right\},$$

is an invariant subspace under (1.10), or equivalently (9.1). Any such solution $u(x,\tau)$ is odd and actually $\frac{2\pi}{k}$-periodic in τ. Let

$$\tilde{\varepsilon} = k^{-\frac{1}{2}}\varepsilon \leq \varepsilon_0, \quad \tilde{\tau} = k\tau, \quad \tilde{\omega} = k\omega = (1+\tilde{\varepsilon}^2)^{-\frac{1}{2}}, \quad y = \tilde{\varepsilon}\tilde{\omega}x, \quad u = \tilde{\varepsilon}\tilde{\omega}v,$$

then $v(y, \tilde{\tau})$ is 2π-periodic and odd in $\tilde{\tau}$ and satisfies

$$\partial_y^2 v - \frac{1}{\tilde{\varepsilon}^2}\partial_{\tilde{\tau}}^2 v - \frac{1}{\tilde{\varepsilon}^2\tilde{\omega}^2} v + \frac{1}{3}v^3 + \frac{1}{\tilde{\varepsilon}^3\tilde{\omega}^3} f(\tilde{\varepsilon}\tilde{\omega}v) = 0.$$

Note that this is in the form of (1.15) with $k = 1$ (and note that $0 < \tilde{\varepsilon} \leq \varepsilon \leq \varepsilon_0$). Therefore for any $y_0 \in \mathbb{R}$, there exists $\varepsilon_0, M > 0$ (independent of k) such that, for $\varepsilon \in (0, \varepsilon_0]$, Theorem 2.1 applies to imply the existence of the unique odd-in-$\tilde{\tau}$ stable and unstable solutions $v^\star_{\mathrm{wk}}(y, \tilde{\tau})$ of (1.10) such that $(v^\star_{\mathrm{wk}}, \partial_y v^\star_{\mathrm{wk}}) \in \mathcal{Z}_o$ and

$$\left\| (1 - \partial_{\tilde{\tau}}^2) \left(\begin{pmatrix} v^\star_{\mathrm{wk}}(y,\tilde{\tau}) \\ \partial_y v^\star_{\mathrm{wk}}(y,\tilde{\tau}) \end{pmatrix} - \begin{pmatrix} v^h(y) \\ (v^h)'(y) \end{pmatrix} \sin\tilde{\tau} \right) \right\|_{\ell_1} \leq M\tilde{\varepsilon}^2 v^h(y), \quad \text{where } v^h(y) = \frac{2\sqrt{2}}{\cosh y}, \tag{9.4}$$

for $y \geq -y_0$ for $\star = s$ or $y \leq y_0$ for $\star = u$. One notices that we replaced $\partial_{\tilde{\tau}}^2$ in Theorem 2.1 which does not change the estimates as $v^\star_{\mathrm{wk}}(y, \tilde{\tau})$ are odd in $\tilde{\tau}$. Moreover $\partial_y \Pi_1[v^\star_{\mathrm{wk}}(0, \cdot)] = 0$ and they satisfy the exponentially small splitting estimate at $y = 0$

$$\left\| \Big(|-\partial_{\tilde{\tau}}^2 - \tfrac{1}{\tilde{\omega}^2}|^{\frac{1}{2}} (v^u_{\mathrm{wk}} - v^s_{\mathrm{wk}}) + i\tilde{\varepsilon}\partial_y (v^u_{\mathrm{wk}} - v^s_{\mathrm{wk}}) \Big)(0, \cdot) - \tfrac{4\sqrt{2}}{\tilde{\varepsilon}} C_{\mathrm{in}} e^{-\frac{\sqrt{2}\pi}{\tilde{\varepsilon}}} \sin 3\tilde{\tau} \right\|_{\ell_1} \leq \frac{M e^{-\frac{\sqrt{2}\pi}{\tilde{\varepsilon}}}}{\tilde{\varepsilon}\log(\tilde{\varepsilon})^{-1}}.$$

Moreover, there exist solution $v(y, \tilde{\tau})$ in $\mathcal{Z}_0$ homoclinic to either 0 or its center manifolds, which have bounds in terms of the values of their Hamiltonian $\mathcal{H}$. From the estimates on the splitting and the inf $\mathcal{H}$ in Theorem 2.1, these orbits satisfy

$$\tilde{\varepsilon}^{-1} \big\| |-\partial_{\tilde{\tau}}^2 - \tfrac{1}{\tilde{\omega}^2}|^{\frac{1}{2}} (v - v^\star_{\mathrm{wk}}) \big\|_{L^2_{\tilde{\tau}}(-\pi,\pi)} + \|\partial_y (v - v^\star_{\mathrm{wk}})\|_{L^2_{\tilde{\tau}}(-\pi,\pi)} \leq M\tilde{\varepsilon}^{-2} e^{-\frac{\sqrt{2}\pi}{\tilde{\varepsilon}}}, \tag{9.5}$$

for $y \geq -y_0$ with $\star = s$ and $y \leq y_0$ with $\star = u$. When $C_{\mathrm{in}} \neq 0$, a lower bound of the same order also holds.

We rescale and obtain the unique stable and unstable solutions

$$u^\star_{\mathrm{wk}}(x,\tau) = \sqrt{k}\varepsilon\omega v^\star_{\mathrm{wk}}(\sqrt{k}\varepsilon\omega x, k\tau).$$

of (1.10) such that $(u^\star_{\mathrm{wk}}, \partial_x u^\star_{\mathrm{wk}}) \in \mathcal{Z}_o$ for any $x \in \mathbb{R}$. For any $x_0 \in \mathbb{R}$, there exists $\varepsilon_0, M > 0$ independent of k such that, for $\varepsilon \in (0, \varepsilon_0]$,

$$\left\| (1 - \tfrac{1}{k^2}\partial_\tau^2) \left(\begin{pmatrix} u^\star_{\mathrm{wk}}(x,\tau) \\ \frac{\partial_x u^\star_{\mathrm{wk}}(x,\tau)}{\sqrt{k}\varepsilon\omega} \end{pmatrix} - \sqrt{k}\varepsilon\omega \begin{pmatrix} v^h(\varepsilon\sqrt{k}\omega x) \\ (v^h)'(\varepsilon\sqrt{k}\omega x) \end{pmatrix} \sin k\tau \right) \right\|_{\ell_1} \leq Mk^{-\frac{1}{2}}\varepsilon^3\omega v^h(\varepsilon\sqrt{k}\omega x), \tag{9.6}$$

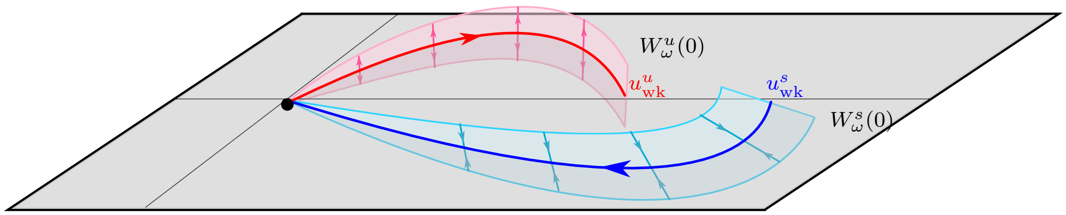

FIGURE 12. We construct the stable manifold $W^s_\omega(0)$ as the union of the strong stable fibers based on the weak stable manifold formed by the solution $u^s_{\rm wk}$ (and its τ-translations), which lives in the invariant subspace of $2\pi/k$-periodic-in-τ functions. We proceed analogously for the unstable manifold $W^u_\omega(0)$.

for $x \geq -\frac{x_0}{\sqrt{k}\varepsilon\omega}$ for $\star = s$ or $x \leq \frac{x_0}{\sqrt{k}\varepsilon\omega}$ for $\star = u$. Moreover $\partial_x \Pi_k[u^\star_{\rm wk}(0,\cdot)] = 0$ and they satisfy the exponentially small splitting estimate

$$\left\| \tfrac{1}{(k\omega)^2}\Big(|-\omega^2\partial_\tau^2 - 1|^{\frac12}(u^u_{\rm wk} - u^s_{\rm wk}) + i(\partial_x u^u_{\rm wk} - \partial_x u^s_{\rm wk})\Big)(0,\cdot) - 4\sqrt{2}C_{\rm in}e^{-\frac{\sqrt{2k}\pi}{\varepsilon}}\sin 3k\tau \right\|_{\ell_1} \leq \frac{Me^{-\frac{\sqrt{2k}\pi}{\varepsilon}}}{\frac12\log k - \log\varepsilon}.$$

Since $0 < 1-(k\omega)^2 < \frac{\varepsilon^2}{k}$, the stable and unstable solutions prove statements (2a-b) of Theorem 1.3. The existence and estimates of generalized breathers (in $\mathcal{Z}_o$) follow from the same rescaling and thus statements (2c) and (3a) of Theorem 1.3 are also proved. In fact, rather than going through Theorem 1.3 it is more direct to deduce statement (2) of Theorem 1.2 from (9.4) and (9.5).

We shall prove statement (3b) of Theorem 1.3 in the rest of the section. The translations (in τ) of these solutions $(u^\star_{\rm wk}(x,\cdot+\theta), \partial_x u^\star_{\rm wk}(x,\cdot+\theta))$ form locally invariant weakly stable/unstable 2-dim surfaces, parametrized by x and $\theta \in \mathbb{R}/(\frac{2\pi}{k}\mathbb{Z})$, of the nonlinear Klein-Gordon equation (1.10), or equivalently (9.1). It is worth pointing out that $(u^\star_{\rm wk}(x,\cdot), \partial_x u^\star_{\rm wk}(x,\cdot))$, $x \in \mathbb{R}$, corresponds to only one of the two branches of the 1-dim stable/unstable manifold of (1.10) in $\mathcal{Z}_o$, while the other branch corresponds to $(-u^\star_{\rm wk}(x,\cdot), -\partial_x u^\star_{\rm wk}(x,\cdot))$. When $x = \pm\infty$ is included, the 2-dim surface generated by the translation in τ does include the other branch, when $\theta = \frac{\pi}{k}$, and form an open neighborhood of 0 of the stable/unstable manifolds in $\mathcal{Z}_o$. Obviously they are submanifolds of the $(2k+1)$-dim stable/unstable manifolds and actually we shall construct the latter based on these weak ones (see Figure 12).

Proposition 9.1. *For any $y_0 \geq 0$, there exist $\varepsilon_0, \rho_1, M > 0$ independent of k and ω, and mappings for $\varepsilon \in (0,\varepsilon_0)$,*

$$\zeta^\star = \big(\zeta_1(s,\theta,\delta), \zeta_2(s,\theta,\delta)\big) \in \mathbf{X}, \quad \theta \in \mathbb{R}/(\tfrac{2\pi}{k}\mathbb{Z}), \quad \delta = (\delta_{1-k},\dots,\delta_{k-1}) \in \mathbb{C}^{2k-1}, \quad \delta_{-n} = -\overline{\delta_n}, \quad |\delta|_1 < \tfrac{\rho_1}{\sqrt{k}},$$

for $s \in \big(-\frac{y_0}{\varepsilon\sqrt{k\omega}}, +\infty\big]$, if $\star = s$ and $s \in \big[-\infty, \frac{y_0}{\varepsilon\sqrt{k\omega}}\big)$ if $\star = u$, such that

$$\Pi_n[\zeta_2^\star] \pm \nu_n \Pi_n[\zeta_1^\star] = 0, \ \forall |n| \leq k-1, \ \star = u,s; \quad \Pi_{-n}[\zeta^\star] = \overline{\Pi_n[\zeta^\star]}, \ \forall n \in \mathbb{Z}; \quad \zeta^\star(s,\theta,0) = 0,$$

$$\|\zeta_1^\star(s,\theta,\delta) - \zeta_1^\star(s,\theta,\tilde\delta)\|_{\ell_1} + \sqrt{k}\|\zeta_2^\star(s,\theta,\delta) - \zeta_2^\star(s,\theta,\tilde\delta)\|_{\ell_1} \leq M(\varepsilon^2 + k|\delta|_1^2 + k|\tilde\delta|_1^2)|\delta - \tilde\delta|_1,$$

and the images of $\xi^\star(s,\theta,\delta)$ is an open subset of $W^\star_\omega(0) \subset \mathbf{X}$ where

$$\xi^\star(s,\theta,\delta) = \big(\xi_1^\star(s,\theta,\delta), \xi_2^\star(s,\theta,\delta)\big) = \big(u^\star_{\rm wk}(s,\cdot+\theta), \partial_x u^\star_{\rm wk}(s,\cdot+\theta)\big) + \Xi^\star(\delta) + \zeta^\star(s,\theta,\delta),$$

$$\Xi^\star(\delta) = \sum_{|n|\leq k-1} \big(-\tfrac{i}{2}\big)\,\delta_n e^{in\tau}(1, \pm\nu_n), \ \ \star = u,s.$$

Moreover, the orbits on $W^\star_\omega(0)$ takes the form $\xi^\star\big(x+s,\theta,\delta(x)\big)$ with

$$\sum_{|n|\leq k-1} \big|\partial_x\delta_n \mp \nu_n\delta_n\big| \leq \frac{M}{\sqrt{k}}(\varepsilon^2 + k|\delta|_1^2)|\delta|_1, \ \ \star = u,s.$$

Remark 9.2. *By including $s = \pm\infty$, where $u^\star_{\rm wk}(\pm\infty,\cdot) = \partial_x u^\star_{\rm wk}(\pm\infty,\cdot) = 0$ for $\star = s,u$, the images of $\xi^\star$ do contain a whole open neighborhood of the zero solution in the stable/unstable manifolds $W^\star_\omega(0) \subset \mathbf{X}$. In fact, $\xi^\star(\pm\infty,\theta,\delta)$ become independent of θ and give the $(2k-1)$–dimensional strong stable/unstable manifolds corresponding to the eigenvalues $\pm\nu_n$, $|n| \leq k-1$. In the (s,δ) coordinates on the invariant manifolds $W^\star_\omega(0)$, the PDE (1.10) corresponds to a vector field whose s component is always 1 and the δ components depend on s and δ which is a small perturbation to $\pm\nu_n\delta_n$. The following proof could be carried out in the* *spaces with high regularity in τ such as $(1+|\partial_\tau|)^{-N}\mathbf{X}$ for any $N \geq 0$ and thus the local invariant manifolds $W_\omega^\star(0) \subset (1+|\partial_\tau|)^{-N}\mathbf{X}$ enjoy the same properties. The smoothness of $\zeta^\star$ in s and θ is also true, for which we refer the readers to, for example, [13] for details, while we focus on the quantitative estimates on the sizes and the Lipschitz constant in δ. Moreover, the term ε^2 in the estimates of $\partial_x\delta$ and the Lipschitz constant of $\zeta^\star$ in δ can be easily improved to be $\varepsilon^2 \sup_{\pm x\in[\pm s,\infty)} \cosh^{-1}(\varepsilon\sqrt{k}\omega x)$, $\star = s, u$. Alternatively, one may also work on the rescaled variables as in (1.14) and obtain equivalent estimates.*

Proof of Proposition 9.1. The proof follows the standard Lyapunov-Perron method which we shall only outline for the unstable case. Given parameters s and θ, we see solutions to (1.10) (or equivalently (9.1)) in the form of

$$u(x,\tau) = u_{\text{wk}}^u(s+x, \tau+\theta) + U(x,\tau), \quad x \leq 0,$$

which decay to 0 as $x \to -\infty$. The equation satisfied by U takes the form

$$\mathcal{L}_k U = \mathcal{F}_k(U) \tag{9.7}$$

where

$$\mathcal{L}_k U = \sum_{n\in\mathbb{Z}} \big((\partial_x^2 - \nu_n^2)U_n\big)e^{in\tau}, \quad \mathcal{F}_k(s,\theta,U) = g(u_{\text{wk}}^u + U) - g(u_{\text{wk}}^u), \;\; \text{for } U(x,\tau) = \sum_{n\in\mathbb{Z}} U_n(x)e^{in\tau}.$$

Here we used the fact u_{wk}^s is an exact solution. The decay property of $u(x,\tau)$ as $x \to +\infty$ is built into the Banach space which U belongs to

$$\mathcal{P} = \big\{U \in C^0\big((-\infty,0),\ell_1\big) \mid \|U\|_{\mathcal{P}} := \sup_{x\leq 0} e^{-\frac{2}{3}\nu_{k-1}x}\|U(x)\|_{\ell_1} < \infty\big\}.$$

To set up the Lyapunov-Perron integral equation, define the linear transformation

$$\big(\mathcal{G}_k(h)\big)(x,\tau) = \sum_{n\in\mathbb{Z}} \big(\mathcal{G}_{k,n}(h_n)\big)(x)e^{in\tau}, \;\; \text{where } h(x,\tau) = \sum_{n\in\mathbb{Z}} h_n(x)e^{in\tau}, \quad x \leq 0,$$

with

$$\big(\mathcal{G}_{k,n}(h_n)\big)(x) = \frac{1}{2\nu_n}e^{\nu_n x}\int_0^x e^{-\nu_n x'}h_n(x')dx' - \frac{1}{2\nu_n}e^{-\nu_n x}\int_{-\infty}^x e^{\nu_n x'}h_n(x')dx', \;\; |n| \leq k-1,$$

$$\big(\mathcal{G}_{k,n}(h_n)(x)\big) = \frac{1}{\nu_n}\int_{-\infty}^x \sinh\big(\nu_n(x-x')\big)h_n(x')dx', \;\; |n| \geq k,$$

which serves as an inverse of $\mathcal{L}_k$. Here we note that for $|n| > k$, $\nu_n = i\vartheta_n$ and $\vartheta_n \geq k^{-\frac{1}{2}}$ and thus $\sinh\big(\nu_n(x-x')\big) = i\sin\big(\vartheta_n(x-x')\big)$. We also define

$$\widetilde{\Xi}(\delta,x,\tau) = \sum_{|n|\leq k-1} \left(-\tfrac{i}{2}\right)\delta_n e^{\nu_n x + in\tau}.$$

The desired solution U satisfies the fixed point equation

$$U = \widetilde{\mathcal{F}}(s,\theta,\delta,U) := \widetilde{\Xi}(\delta) + \mathcal{G}_k\big(\mathcal{F}_k(s,\theta,U)\big).$$

Using (9.2), (9.3), and (9.6), it is straight forward to verify

$$\|\widetilde{\Xi}\|_{\mathcal{P}} \leq \frac{1}{2}|\delta|_1, \;\; \|\mathcal{G}_k(h)\|_{\mathcal{P}} \leq \frac{100}{\nu_{k-1}^2}\|h\|_{\mathcal{P}}, \;\; \|\mathcal{F}_k(s,\theta,U) - \mathcal{F}_k(s,\theta,\tilde{U})\|_{\mathcal{P}} \leq M\left(\frac{\varepsilon^2}{k} + \|U\|_{\mathcal{P}}^2 + \|\tilde{U}\|_{\mathcal{P}}^2\right)\|U - \tilde{U}\|_{\mathcal{P}}.$$

Therefore there exists $\rho_1 > 0$ independent of ε and $k \geq 1$, such that, for $|\delta|_1 \leq \frac{\rho_1}{\sqrt{k}}$, $\widetilde{\mathcal{F}}$ is a contraction on the ball of radius $\frac{2\rho_1}{\sqrt{k}}$ in $\mathcal{P}$. Let $U^u(s,\theta,\delta,x,\tau)$ be the unique fixed point of $\widetilde{\mathcal{F}}$,

$$\zeta_1^u(s,\theta,\delta) = U^u(s,\theta,\delta,0,\cdot) - \widetilde{\Xi}(\delta,0,\cdot), \quad \zeta_2^u(s,\theta,\delta) = \partial_x U^u(s,\theta,\delta,0,\cdot) - \partial_x\widetilde{\Xi}(\delta,0,\cdot),$$

and ξ^u accordingly. The desired estimates on ζ^u follow from straight forward calculations. The invariance of $W_\omega^u(0) = image(\xi^s)$ is a direct consequence of the uniqueness of the decaying solutions in $\mathcal{P}$, which implies that solutions on $W_\omega^u(0)$ are parametrized by $\delta(x)$ and take the following two forms

$$\begin{aligned} u(x,\cdot) =& u_{\text{wk}}^u(s+x,\cdot+\theta) + U^u\big(s,\theta,\delta(0),x,\cdot\big) \\ =& \xi_1^u\big(s+x,\theta,\delta(x)\big) = u_{\text{wk}}^u(s+x,\cdot+\theta) + U^u\big(s+x,\theta,\delta(x),0,\cdot\big). \end{aligned}$$

Hence

$$U^u\big(s+x,\theta,\delta(x),0,\cdot\big) = U^u\big(s,\theta,\delta(0),x,\cdot\big).$$

which in turn yields, for $|n| \leq k-1$,

$$\delta_n(x) = i\big(\Pi_n[U^u] + \nu_n^{-1}\Pi_n[\partial_x U^u]\big)\big|_{\left(s+x,\theta,\delta(x),0\right)} = i\big(\Pi_n[U^u] + \nu_n^{-1}\Pi_n[\partial_x U^u]\big)\big|_{\left(s,\theta,\delta(0),x\right)}.$$

Therefore, differentiating this identity and using (9.7),

$$\partial_x \delta_n(x) = \nu_n \delta_n(x) + i\nu_n^{-1}\Pi_n[\mathcal{F}_k(U^u)]\Big|_{\left(s,\theta,\delta(0),x\right)}.$$

Letting $x = 0$, we obtain the estimate on $\partial_x \delta$ straightforwardly and complete the proof of the proposition. □

9.2. **Single-bump homoclinic solutions.** In this subsection, we shall complete the proof of the Statement (3b) of Theorem 1.3. Assume that the nonlinearity $f(u)$ satisfies that the corresponding Stokes constant

$$C_{\mathrm{in}} \neq 0. \tag{9.8}$$

Fix $\sigma \in (0,1)$. We shall prove that there exist $\rho_2, \varepsilon_0 > 0$ depending only on $f(u)$ and σ such that, for any $k \geq 1$, $\omega \in I_k(\varepsilon_0)$, and $u(x,t)$ which is a nonzero $\frac{2\pi}{\omega}$-periodic-in-t solution to the nonlinear Klein-Gordon equation (1.2) such that it decays to zero in the energy norm in t as $|x| \to \infty$ (namely satisfying (1.5)) and satisfies

$$\sup_{x\in\mathbb{R}} \sqrt{k}\|u(x)\|_{\ell_1} \leq \rho_2, \tag{9.9}$$

then $u(x,t)$ must be multi-bump in the sense of Definition 1.1. In fact, since it decays to 0 as $|x| \to \infty$, we only need to find

$$x_2 < x_3 < x_4, \ \text{ s. t. } \ \|u(x_3,\cdot)\|_{\ell_1} \leq \sigma \min\big\{\|u(x_2,\cdot)\|_{\ell_1},\ \|u(x_4,\cdot)\|_{\ell_1}\big\}. \tag{9.10}$$

Let $y_0 > y_1 > 0$ satisfy

$$v^h(y_1) = \sigma, \quad v^h(y_0) = \tfrac{\sigma}{2},$$

which are independent of k and ε. Let ε_0, ρ_1 be determined by y_0 according to Proposition 9.1. We can actually choose ε_0 and ρ_1 even smaller such that that ε_0 also satisfies Theorem 2.1 and

$$\|\zeta_1^\star(s,\theta,\delta) - \zeta_1^\star(s,\theta,\tilde\delta)\|_{\ell_1} + \sqrt{k}\|\zeta_2^\star(s,\theta,\delta) - \zeta_2^\star(s,\theta,\tilde\delta)\|_{\ell_1} \leq \tfrac{1}{9}|\delta - \tilde\delta|_1, \tag{9.11}$$

and on $W_\omega^\star(0)$,

$$\sum_{|n|\leq k-1} |\partial_x \delta_n \mp \nu_n \delta_n| \leq \tfrac{1}{9\sqrt{k}}|\delta|_1, \ \ \star = u, s. \tag{9.12}$$

Recall $u_{\mathrm{wk}}^\star(x,\cdot)$ has Fourier modes supported only in $k\mathbb{Z}$ and $\Xi^\star$ supported in $|n| \leq k-1$. We have

$$\|\xi_1^\star(s,\theta,\delta)\|_{\ell_1} \geq \tfrac{1}{2}|\delta|_1 - \|\zeta_1^\star(s,\theta,\delta)\|_{\ell_1} \geq \tfrac{7}{18}|\delta|_1. \tag{9.13}$$

Let $W_\omega^\star(0)$, $\star = u, s$, be the (semilocal) unstable and stable manifolds parametrized by $\xi^\star(s,\theta,\delta)$ with parameter bounds y_0 and $\frac{\rho_1}{\sqrt{k}}$ through $\zeta^\star(s,\theta,\delta)$ provided by Proposition 9.1. Let

$$\rho_2 = \frac{1}{9}\rho_1, \quad \widetilde{W}_\omega^\star(0) = \left\{\xi^\star(s,\theta,\delta) \mid \theta \in \tfrac{2\pi}{k},\ \pm s \in \left[-\tfrac{y_1}{\varepsilon\sqrt{k}\omega}, +\infty\right],\ |\delta|_1 \leq \frac{3\rho_2}{\sqrt{k}}\right\}, \ \ \star = s, u,$$

which is a subset containing 0 in its interior, and

$$x_u = \sup\big\{x \mid \big(u(x',\cdot), \partial_x u(x',\cdot)\big) \in \widetilde{W}_\omega^u(0),\ \forall x' \in (-\infty, x]\big\}$$
$$x_s = \inf\big\{x \mid \big(u(x',\cdot), \partial_x u(x',\cdot)\big) \in \widetilde{W}_\omega^s(0),\ \forall x' \in [x, \infty)\big\}.$$

As the orbit of $u(x,\cdot)$ is assumed to be homoclinic to 0 in x, while the dynamics inside $\widetilde{W}_\omega^\star(0)$ is either purely growing or decaying, clearly both $x_\star \neq \pm\infty$, $\star = u, s$. Let

$$w_\star = \big(u(x_\star,\cdot), \partial_x u(x_\star,\cdot)\big) \in \widetilde{W}_\omega^\star(0), \ \ \star = s, u,$$

and we have

$$\exists\, \theta_\star,\ \delta_\star,\ s_\star, \ \text{ s. t. } \ |\delta_\star|_1 \leq \tfrac{2\rho_2}{\sqrt{k}}, \ \ \pm s_\star \in \big[-\tfrac{y_1}{\varepsilon\sqrt{k}\omega}, +\infty\big], \ \ w_\star = \xi^\star(s_\star, \theta_\star, \delta_\star).$$

We proceed in the following steps.

Claim A. It holds $s_\star = \pm\frac{y_1}{\varepsilon\sqrt{k}\omega}$, $\star = u, s$. In fact, assume it is false with, say $s_u \in \big[-\infty, \frac{y_1}{\varepsilon\sqrt{k}\omega}\big)$, then the definition of x_u yields $|\delta_u|_1 = \frac{3\rho_2}{\sqrt{k}}$. Hence we obtain from (9.13)

$$\|u(x_u,\cdot)\|_{\ell_1} = \|\xi_1^u(s_u,\theta_u,\delta_u)\|_{\ell_1} \geq \tfrac{7\rho_2}{6\sqrt{k}},$$

a contradiction to (9.9).

Claim B. It holds $|\delta_\star|_1 < \frac{\sigma\varepsilon}{9\sqrt{k}}$, $\star = s, u$. Again if we assume it is false with, say $|\delta_u|_1 \geq \frac{\sigma\varepsilon}{9\sqrt{k}}$. From Proposition 9.1, we can write, for $x \geq x_u$,

$$u(x,\cdot) = \xi_1^u\big(s_u + x - x_u, \theta_u, \delta^u(x)\big), \text{ as long as } \big(u(x,\cdot), \partial_x u(x,\cdot)\big) \in W_\omega^u(0),$$

where $\delta(x_u) = \delta_u$. According to (9.12),

$$|\delta(x)|_1 \geq |\delta_u|_1 e^{\frac{x-x_u}{2\sqrt{k}}} \geq \tfrac{\sigma\varepsilon}{9\sqrt{k}} e^{\frac{x-x_u}{2\sqrt{k}}}.$$

Therefore, by taking ε_0 and ρ_1 reasonably small depending only on σ,

$$\exists\, \tilde{x} \in \big[x_u, x_u + 2\sqrt{k}\log\tfrac{27\rho_2}{\sigma\varepsilon}\big] \subset \big[x_u, x_u + \tfrac{y_0-y_1}{\varepsilon\sqrt{k}\omega}\big], \text{ s. t. } |\delta(\tilde{x})|_1 = \tfrac{3\rho_2}{\sqrt{k}}.$$

The same argument as in the proof of Claim A. implies that $\|u(\tilde{x},\cdot)\|_{\ell_1} > \frac{\rho_2}{\sqrt{k}}$ and thus a contradiction.

Lemma 9.3. *It holds $x_s > x_u$ (See Figure 13).*

Proof. We prove it by contradiction. Assume it is false, namely $x_u \geq x_s$. In this case $w_u \in \widetilde{W}_\omega^s(0)$ and thus

$$\exists\,(\tilde{s},\tilde{\theta},\tilde{\delta}) \text{ s. t. } w_u = \xi^s(\tilde{s},\tilde{\theta},\tilde{\delta}),\ \tilde{s} \in \big[-\tfrac{y_1}{\varepsilon\sqrt{k}\omega}, \infty\big],\ |\tilde{\delta}|_1 \leq \frac{3\rho_2}{\sqrt{k}}.$$

Combining it with the coordinates of $w_u \in \widetilde{W}_\omega^u(0)$, we have

$$\begin{aligned} w_u =& \big(u_{\rm wk}^u(s_u, \cdot + \theta_u), \partial_x u_{\rm wk}^u(s_u, \cdot + \theta_u)\big) + \sum_{|n|\leq k-1} \big(-\tfrac{i}{2}\big)\, \delta_n^u e^{in\tau}(1,\nu_n) + \zeta^u(s_u,\theta_u,\delta_u) \\ =& \big(u_{\rm wk}^s(\tilde{s}, \cdot + \tilde{\theta}), \partial_x u_{\rm wk}^s(\tilde{s}, \cdot + \tilde{\theta})\big) + \sum_{|n|\leq k-1} \big(-\tfrac{i}{2}\big)\, \tilde{\delta}_n e^{in\tau}(1,-\nu_n) + \zeta^s(\tilde{s},\tilde{\theta},\tilde{\delta}). \end{aligned}$$

Since $u_{\rm wk}^\star$, $\star = u, s$, do not contain modes with $|n| < k$, applying Π_n and using Proposition 9.1, we obtain

$$\delta_n^u = i\big(\Pi_n[\zeta_1^s(\tilde{s},\tilde{\theta},\tilde{\delta})] + \nu_n^{-1}\Pi_n[\zeta_2^s(\tilde{s},\tilde{\theta},\tilde{\delta})]\big) \text{ and } \tilde{\delta}_n = i\big(\Pi_n[\zeta_1^u(s_u,\theta_u,\delta_u)] - \nu_n^{-1}\Pi_n[\zeta_2^u(s_u,\theta_u,\delta_u)]\big),$$

for any $|n| \leq k-1$. It follows from (9.3), (9.11) and the fact that $\zeta^\star(s,\theta,0) = 0$ (see Proposition 9.1), that $\delta_u = \tilde{\delta} = 0$ and thus

$$w_u = \big(u_{\rm wk}^u(s_u, \cdot + \theta_u), \partial_x u_{\rm wk}^u(s_u, \cdot + \theta_u)\big) = \big(u_{\rm wk}^s(\tilde{s}, \cdot + \tilde{\theta}), \partial_x u_{\rm wk}^s(\tilde{s}, \cdot + \tilde{\theta})\big),$$

which implies

$$u(x,\tau) \equiv u_{\rm wk}^u(x - x_u + s_u, \tau + \theta_u) \equiv u_{\rm wk}^s(x - x_u + \tilde{s}, \tau + \tilde{\theta}).$$

Due the approximation (9.6), $u_{\rm wk}^\star(x, \tau + \theta)$ is odd in τ if and only if $\theta = 0$ or $\frac{\pi}{k}$, but with opposite signs when θ takes these two values. Hence $\theta_u = \tilde{\theta}$ and we can assume them to be zero due to the translation invariance in τ of (1.10). Taking $x = x_u - s_u$, we obtain

$$\Pi_1[\partial_x u_{\rm wk}^s(\tilde{s} - s_u, \cdot)] = \Pi_1[\partial_x u_{\rm wk}^u(0,\cdot)] = 0, \quad s_u = \tfrac{y_1}{\varepsilon\sqrt{k}\omega},\ \ \tilde{s} \in \big[-\tfrac{y_1}{\varepsilon\sqrt{k}\omega}, \infty\big].$$

Due to (9.6) and Theorem 1.3, this is possible only if $s_u = \tilde{s}$ and thus

$$u(x,\tau) \equiv u_{\rm wk}^u(x - x_u + s_u, \tau) \equiv u_{\rm wk}^s(x - x_u + s_u, \tau).$$

However this is impossible as $C_{\rm in} \neq 0$ is assumed (see (9.8)). □

To end the proof that $u(x,\tau)$ satisfies (9.10) and thus multi-bump, let

$$x_2 = x_u - s_u, \quad x_3 = x_u > x_2, \quad x_4 = x_s - s_s > x_s > x_u = x_3.$$

We have

$$u(x_2,\cdot) = \xi_1^u\big(0,\theta_u,\delta^u(x_2)\big), \quad u(x_3,\cdot) = \xi_1^u(s_u,\theta_u,\delta_u), \quad u(x_4,\cdot) = \xi_1^s\big(0,\theta_s,\delta^s(x_4)\big),$$

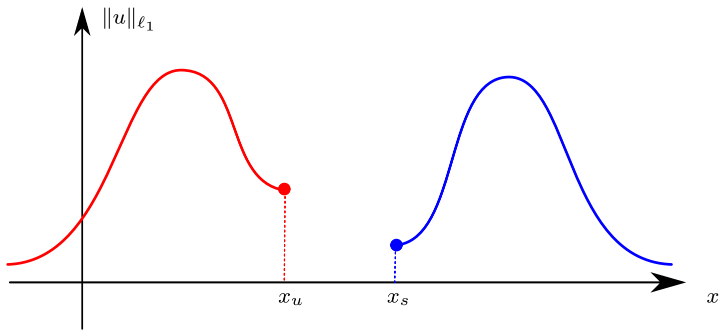

FIGURE 13. By Lemma 9.3, x_u is below x_s. We notice that, for some $x < x_u$ (resp. $x > x_s$), $W^u_\omega(0)$ (resp. $W^s_\omega(0)$) has left the σ-neighborhood of the origin and returned to it at $x = x_u$ (resp. $x = x_s$).

where the coordinates $\delta^\star(x)$, $\star = u, s$, satisfies (9.12) and $\delta^\star(x_\star) = \delta_\star$. Clearly (9.12) and Claim B imply

$$|\delta^u(x_2)|_1 < |\delta_u|_1 < \tfrac{\sigma\varepsilon}{9\sqrt{k}}, \quad |\delta^s(x_4)|_1 < |\delta_s|_1 < \tfrac{\sigma\varepsilon}{9\sqrt{k}}.$$

From these estimates, Claim A, the definition of y_1, (9.6), we have

$$\|u(x_2,\cdot)\|_{\ell_1},\ \|u(x_4,\cdot)\|_{\ell_1} \geq \tfrac{2\varepsilon}{\sqrt{k}}, \quad \|u(x_3,\cdot)\|_{\ell_1} \leq \tfrac{2\sigma\varepsilon}{\sqrt{k}},$$

and (9.10) follows immediately.

APPENDIX A. PROOF OF PROPOSITION 4.3

Stated otherwise, M denotes any constant independent of κ and ε. The proof of items (1) and (2) are straightforward using that (3.1) acts on the Fourier coefficients of ξ. To prove item (3), we consider $h \in \mathcal{E}_{m,\alpha}$ and we estimate $\mathcal{G}_1(h_1)$ and $\mathcal{G}_n(h_n)$ (see (4.2) and (4.3)). For $\mathcal{G}_n$, using Lemma 5.5 in [31], one can see that

$$\|\mathcal{G}_n(h)\|_{m,\alpha} \leq \frac{M\varepsilon^2}{\lambda_n^2}\|h\|_{m,\alpha},\ n \geq 2. \tag{A.1}$$

Now we estimate $\mathcal{G}_1$ given by

$$\mathcal{G}_1(h)(y) = -\zeta_1(y)\int_0^y \zeta_2(s)h(s)ds + \zeta_2(y)\int_{-\infty}^y \zeta_1(s)h(s)ds.$$

First, we bound $\mathcal{G}_1(h)(y)$ for values of y in $D^{\text{out},u}_\kappa \cap \{\text{Re}(y) \leq -1\}$. Notice that the functions $\zeta_1(y), \zeta_2(y)$ given in (4.4) satisfy

$$|\zeta_1(y)| \leq \frac{M}{|\cosh(y)|} \quad \text{and} \quad |\zeta_2(y)| \leq M|\cosh(y)|, \tag{A.2}$$

for every

$$y \in D^{\text{out},u}_\kappa \cap \{y \in \mathbb{C};\ |\text{Im}(y)| \leq -K\,\text{Re}(y)\} \qquad \text{where} \qquad K = \left(\tan(\beta) + \frac{\pi}{2} - \kappa\varepsilon\right).$$

The second integral in $\mathcal{G}_1$ satisfies that, for every $y \in D^{\text{out},u}_\kappa \cap \{\text{Re}(y) \leq -1\}$,

$$\left|\int_{-\infty}^y \zeta_1(s)h(s)ds\right| \leq M\|h\|_{m,\alpha}\int_{-\infty}^0 \frac{1}{|\cosh^{m+1}(s+y)|}ds \leq M\frac{\|h\|_{m,\alpha}}{|\cosh^{m+1}(y)|}.$$

Therefore

$$\left|\zeta_2(y)\int_{-\infty}^y \zeta_1(s)h(s)ds\right| \leq \frac{M\|h\|_{m,\alpha}}{|\cosh^m(y)|}. \tag{A.3}$$

Now, to estimate the first integral in $\mathcal{G}_1$, let y^* be the unique point in the segment of line between 0 and y such that $\text{Re}(y^*) = -1$. Hence, it follows from (A.2) that,

(1) If s is in the line between 0 and y^*, then

$$|\zeta_2(s)h(s)| \leq \frac{M\|h\|_{m,\alpha}|\cosh(s)|}{|s^2+\pi^2/4|^\alpha} \leq M\|h\|_{m,\alpha}.$$

(2) If s is in the line between y^* and y, then

$$|\zeta_2(s)h(s)| \leq \frac{M\|h\|_{m,\alpha}}{|\cosh^{m-1}(s)|}.$$

Thus since $m>1$, using the previous estimates, we have that

$$\left|\int_0^y \zeta_2(s)h(s)ds\right| \leq \left|\int_{y^*}^0 \zeta_2(s)h(s)ds\right| + \left|\int_y^{y^*} \zeta_2(s)h(s)ds\right| \leq M\|h\|_{m,\alpha},$$

and consequently

$$\left|\zeta_1(y)\int_0^y \zeta_2(s)h(s)ds\right| \leq \frac{M\|h\|_{m,\alpha}}{|\cosh(y)|}. \tag{A.4}$$

Now, from (4.2), (A.3) and (A.4), we obtain that

$$\sup_{y\in D_\kappa^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\leq -1\}} |\cosh(y)\mathcal{G}_1(h)(y)| \leq M\|h\|_{m,\alpha}. \tag{A.5}$$

For the region $D_\kappa^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}$, we consider a new set of fundamental solutions $\{\zeta_+,\zeta_-\}$ of $\mathcal{L}_1(\zeta)=0$ which has good properties at $\pm i\pi/2$. We rewrite the solutions $\zeta_1(y)$ and $\zeta_2(y)$ as linear combinations of $\zeta_+(y)$ and $\zeta_-(y)$ and use them to obtain a new expression of the operator $\mathcal{G}_1$. We emphasize that the operator $\mathcal{G}_1$ is already defined. We only express it in a different way.

Lemma A.1. *The functions*

$$\zeta_\pm(y) = \zeta_1(y)\int_{\pm i\frac{\pi}{2}}^y \frac{1}{\zeta_1^2(s)}ds = -\frac{\sqrt{2}}{4}\frac{1}{\cosh^2(y)}\left(\frac{3y\sinh(y)}{2} - \cosh(y) + \frac{1}{4}\sinh(y)\sinh(2y) \mp i\frac{3\pi}{4}\sinh(y)\right)$$

are solutions of equation $\mathcal{L}_1(\zeta)=0$ and have the following properties.

- *The Wronksian satisfies*

$$W(\zeta_+,\zeta_-) = \zeta_+\dot{\zeta}_- - \zeta_-\dot{\zeta}_+ = -i\frac{3\pi}{16}.$$

  and therefore $\zeta_\pm$ are linearly independent.
- *They can be written as*

$$\zeta_\pm(y) = \frac{(y\mp i\pi/2)^3}{(y\pm i\pi/2)^2}\eta_\pm(y), \tag{A.6}$$

  where $\eta_\pm$ are analytic functions in $D_\kappa^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}$ uniformly bounded (with respect to ε and κ).
- *The operator $\mathcal{G}_1$ given by* (4.2) *can be rewritten as*

$$\mathcal{G}_1(h) = i\frac{16}{3\pi}\left(-\zeta_+(y)\int_0^y \zeta_-(s)h(s)ds + \zeta_-(y)\int_0^y \zeta_+(s)h(s)ds\right) + \zeta_2(y)\int_{-\infty}^0 \zeta_1(s)h(s)ds,$$

  where ζ_1,ζ_2 are given in (4.4).

The proof of this lemma is a straightforward computation using the relation between $\zeta_\pm$ and ζ_1,ζ_2.

Using this lemma, we bound $\mathcal{G}_1(h)$ for $y\in D_\kappa^{\mathrm{out},u}$ satisfying $\mathrm{Re}(y)\geq -1$. First, notice that we can use (A.2) to see that

$$\begin{aligned}\left|\int_{-\infty}^0 \zeta_1(s)h(s)ds\right| &\leq M\|h\|_{m,\alpha}\left(\int_{-\infty}^{-1}\frac{1}{|\cosh^{m+1}(s)|}ds + \int_{-1}^0 \frac{1}{|\cosh(s)(s^2+\pi^2/4)^\alpha|}ds\right)\\ &\leq M\|h\|_{m,\alpha}.\end{aligned}$$

From the expression of $\zeta_2(y)$ in (4.4), we have that $\zeta_2(y)$ has poles of order 2 at $\pm i\pi/2 + i2k\pi$. Since $\alpha\geq 5$,

$$\sup_{y\in D_\kappa^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}}\left|(y^2+\pi^2/4)^{\alpha-2}\zeta_2(y)\int_{-\infty}^0 \zeta_1(s)h(s)ds\right| \leq M\|h\|_{m,\alpha}.$$

Now, we use that $\alpha \geq 5$ and equation (A.6) to see that

$$\begin{aligned}\left|\zeta_+(y)\int_0^y \zeta_-(s)h(s)ds\right| &\leq M\frac{|y-i\pi/2|^3}{|y+i\pi/2|^2}\int_0^y \frac{|s+i\pi/2|^3}{|s-i\pi/2|^2}|h(s)|ds\\ &\leq M\|h\|_{m,\alpha}\frac{|y-i\pi/2|^3}{|y+i\pi/2|^2}\int_0^y \frac{1}{|s+i\pi/2|^{\alpha-3}|s-i\pi/2|^{\alpha+2}}ds\\ &\leq \frac{M\|h\|_{m,\alpha}}{|y^2+\pi^2/4|^{\alpha-2}}.\end{aligned}$$

We conclude that

$$\sup_{y\in D_\kappa^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}}\left|(y^2+\pi^2/4)^{\alpha-2}\zeta_+(y)\int_0^y \zeta_-(s)h(s)ds\right| \leq M\|h\|_{m,\alpha}.$$

In a similar way, we can prove that

$$\sup_{y\in D_\kappa^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}}\left|(y^2+\pi^2/4)^{\alpha-2}\zeta_-(y)\int_0^y \zeta_+(s)h(s)ds\right| \leq M\|h\|_{m,\alpha}.$$

Therefore

$$\sup_{y\in D_\kappa^{\mathrm{out},u}\cap\{\mathrm{Re}(y)\geq -1\}}\left|(y^2+\pi^2/4)^{\alpha-2}\mathcal{G}_1(h)(y)\right| \leq M\|h\|_{m,\alpha}. \tag{A.7}$$

Hence, using (4.1), (A.1), (A.5) and (A.7) , one obtains Item 3 of Proposition 4.3.

To prove the estimates on $\partial_\tau\mathcal{G}(h)$ and $\partial_\tau^2\mathcal{G}(h)$ it is sufficient to use (A.1) and

$$\Pi_n[\partial_\tau^2\mathcal{G}(h)] = -n^2\Pi_n[\mathcal{G}(h)].$$

Finally, for item (5), notice that

$$\partial_y\circ\mathcal{G}_n(h) = \frac{1}{2}e^{i\frac{\lambda_n}{\varepsilon}y}\int_{-\infty}^y e^{-i\frac{\lambda_n}{\varepsilon}s}h(s)ds + \frac{1}{2}e^{-i\frac{\lambda_n}{\varepsilon}y}\int_{-\infty}^y e^{i\frac{\lambda_n}{\varepsilon}s}h(s)ds,\ n\geq 2,$$

and thus, one can easily obtain

$$\|\partial_y\circ\mathcal{G}_n(h)\|_{m,\alpha} \leq \frac{M\varepsilon}{\lambda_n}\|h\|_{m,\alpha},\ n\geq 2.$$

The decay of $\partial_y\circ\mathcal{G}_n(h)$ for $n\geq 2$ also implies that

$$\partial_y\circ\mathcal{G}_n(h) = \mathcal{G}(\partial_y h)$$

and thus we also have

$$\|\partial_y\circ\mathcal{G}_n(h)\|_{m,\alpha} \leq \frac{M\varepsilon^2}{\lambda_n^2}\|\partial_y h\|_{m,\alpha},\ n\geq 2.$$

For the first mode, since

$$\begin{aligned}\partial_y\circ\mathcal{G}_1(h) =& i\frac{16}{3\pi}\left(-\zeta_+'(y)\int_0^y \zeta_-(s)h(s)ds + \zeta_-'(y)\int_0^y \zeta_+(s)h(s)ds\right) + \zeta_2'(y)\int_{-\infty}^0 \zeta_1(s)h(s)ds\\ =& -\zeta_1'(y)\int_0^y \zeta_2(s)h(s)ds + \zeta_2'(y)\int_{-\infty}^y \zeta_1(s)h(s)ds,\end{aligned}$$

one has $\|\partial_y\circ\mathcal{G}_1(h)\|_{1,\alpha-1} \leq M\|h\|_{m,\alpha}$.

Appendix B. Proof of Proposition 3.11

From (3.17) and Proposition 3.9, we have that, for each $k\geq 1$,

$$\begin{aligned}\dot\Gamma_{2k+1} &= \lambda_{2k+1}\Xi_{2k+1} + i\varepsilon\ddot\Delta_{2k+1}\\ &= \lambda_{2k+1}\Xi_{2k+1} + i\varepsilon\left(-\frac{\lambda_{2k+1}^2}{\varepsilon^2}\Delta_{2k+1} + \Pi_{2k+1}\left[\eta_3(y,\tau)\Delta\right]\right)\\ &= -i\frac{\lambda_{2k+1}}{\varepsilon}\Gamma_{2k+1} + i\varepsilon\Pi_{2k+1}\left[\eta_3(y,\tau)\Delta\right].\end{aligned} \tag{B.1}$$

Analogously, for each $k \geq 1$,

$$\dot{\Theta}_{2k+1} = i\frac{\lambda_{2k+1}}{\varepsilon}\Theta_{2k+1} - i\varepsilon\Pi_{2k+1}\left[\eta_3(y,\tau)\Delta\right]. \tag{B.2}$$

Moreover, for the variable Ξ_1, by (3.1) and Proposition 3.9, we have that

$$\dot{\Xi}_1 = \left(1 - \frac{3(v^h)^2}{4}\right)\Delta_1 + \Pi_1\left[\eta_1(y,\tau)\Delta_1\sin(\tau) + \eta_2(y,\tau)\widetilde{\Pi}[\Delta]\right].$$

Using (3.15) for Δ_1 and $\left(1 - \frac{3(v^h)^2}{4}\right)\dot{v}^h = \dddot{v}^h$, we obtain

$$\begin{aligned}
\dot{\Xi}_1 &= \frac{\dddot{v}^h}{\ddot{v}^h}\Xi_1 + \left(1 - \frac{3(v^h)^2}{4}\right)\left(A(\Xi) + B(\widetilde{\Pi}[\Delta])\right) \\
&\quad + \Pi_1\left[\eta_1(y,\tau)\left(\frac{\dot{v}^h}{\ddot{v}^h}\Xi_1 + A(\Xi) + B(\widetilde{\Pi}[\Delta])\right)\sin(\tau) + \eta_2(y,\tau)\widetilde{\Pi}[\Delta]\right] \\
&= \frac{\dddot{v}^h}{\ddot{v}^h}\Xi_1 + \left(1 - \frac{3(v^h)^2}{4}\right)A(\Xi_1\sin(\tau)) + \Pi_1\left[\eta_1(y,\tau)\left(\frac{\dot{v}^h}{\ddot{v}^h}\Xi_1 + A(\Xi_1\sin(\tau))\right)\sin\tau\right] \\
&\quad + \left(1 - \frac{3(v^h)^2}{4}\right)\left(A(\widetilde{\Pi}[\Xi]) + B(\widetilde{\Pi}[\Delta])\right) \\
&\quad + \Pi_1\left[\eta_1(y,\tau)\left(A(\widetilde{\Pi}[\Xi]) + B(\widetilde{\Pi}[\Delta])\right)\sin\tau + \eta_2(y,\tau)\widetilde{\Pi}[\Delta]\right].
\end{aligned}$$

Finally, using (3.17),

$$\begin{aligned}
\dot{\Xi}_1 &= \frac{\dddot{v}^h}{\ddot{v}^h}\Xi_1 + \left(1 - \frac{3(v^h)^2}{4}\right)A(\Xi_1\sin\tau) + \Pi_1\left[\eta_1(y,\tau)\left(\frac{\dot{v}^h}{\ddot{v}^h}\Xi_1 + A(\Xi_1\sin\tau)\right)\sin\tau\right] \\
&+ \left(1 - \frac{3(v^h)^2}{4}\right)\left(\frac{1}{2i\varepsilon}A(\Gamma - \Theta) + B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1} + \Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right) \\
&+ \Pi_1\left[\eta_1(y,\tau)\left(\frac{1}{2i\varepsilon}A(\Gamma - \Theta)\right)\sin(\tau) + \eta_1(y,\tau)B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1} + \Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\sin\tau\right. \\
&\left. + \eta_2(y,\tau)\left(\sum_{n\geq 2}\frac{\Gamma_{2k+1} + \Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right].
\end{aligned}$$

For the other components, as

$$\begin{aligned}
i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\Delta\right] &= i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\left(\left(\frac{\dot{v}^h}{\ddot{v}^h}\Xi_1 + A(\Xi) + B(\widetilde{\Pi}[\Delta])\right)\sin(\tau) + \widetilde{\Pi}[\Delta]\right)\right] \\
&= i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\left(\frac{\dot{v}^h}{\ddot{v}^h}\Xi_1 + A(\Xi_1\sin(\tau))\right)\sin(\tau)\right] \\
&\quad + i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\left(A(\widetilde{\Pi}[\Xi])\sin(\tau) + B(\widetilde{\Pi}[\Delta])\sin(\tau) + \widetilde{\Pi}[\Delta]\right)\right] \\
&= i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\left(\frac{\dot{v}^h}{\ddot{v}^h}\Xi_1 + A(\Xi_1\sin(\tau))\right)\sin(\tau)\right] \\
&\quad + i\varepsilon\widetilde{\Pi}\left[\frac{\eta_3(y,\tau)}{2i\varepsilon}A(\Gamma - \Theta)\sin(\tau) + \eta_3(y,\tau)B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1} + \Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\sin(\tau)\right. \\
&\quad \left. + \eta_3(y,\tau)\sum_{k\geq 1}\frac{\Gamma_{2k+1} + \Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right]
\end{aligned}$$

the proof is concluded by using (B.1) and (B.2) and taking

$$\begin{aligned}
m_W(y)\Xi_1 &= \left(1-\frac{3(v^h)^2}{4}\right)A(\Xi_1\sin\tau)+\Pi_1\left[\eta_1(y,\tau)\left(\frac{\dot v^h}{\ddot v^h}\Xi_1+A(\Xi_1\sin\tau)\right)\sin\tau\right]\\
\mathcal{M}_W(\Gamma,\Theta) &= \left(1-\frac{3(v^h)^2}{4}\right)\left(\frac{1}{2i\varepsilon}A(\Gamma-\Theta)+B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right)\\
&\quad+\Pi_1\left[\frac{\eta_1(y,\tau)}{2i\varepsilon}A(\Gamma-\Theta)\sin(\tau)+\eta_1(y,\tau)B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\sin\tau\right.\\
&\quad\left.+\eta_2(y,\tau)\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right]\\
m_{\mathrm{osc}}(y,\tau)\Xi_1 &= i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\left(\frac{\dot v^h}{\ddot v^h}\Xi_1+A(\Xi_1\sin\tau)\right)\sin\tau\right]\\
\mathcal{M}_{\mathrm{osc}}(\Gamma,\Theta) &= i\varepsilon\widetilde{\Pi}\left[\eta_3(y,\tau)\left(\frac{1}{2i\varepsilon}A(\Gamma-\Theta)\sin\tau+B\left(\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\sin\tau\right.\right.\\
&\quad\left.\left.+\sum_{k\geq 1}\frac{\Gamma_{2k+1}+\Theta_{2k+1}}{2\lambda_{2k+1}}\sin((2k+1)\tau)\right)\right],
\end{aligned}$$

and using the bounds for the functions η_j, $j=1,2,3$ and the operators A and B provided in Propositions 3.9 and 3.10.

Appendix C. Generic non-vanishing of the Stokes constant

In Theorem 2.1, we have seen that the invariant manifolds $W^u(0)$ and $W^s(0)$ split provided $C_{\mathrm{in}}\neq 0$. In fact, we have seen that the distance between them at the first time they intersect the transversal section Σ depends on the so-called Stokes constant C_{in}.

The Stokes constant C_{in} depends analytically on the function f introduced in (1.2) (see 3.5). This appendix is devoted to show that C_{in} generically does not vanish, i.e. there exists an open dense set $\mathcal{R}$ of the space of analytic (odd) germs of functions at 0, such that $C_{\mathrm{in}}=C_{\mathrm{in}}(f)\neq 0$, for every $f\in\mathcal{R}$.

Consider the following family of partial differential equations depending on a parameter $\mu\in\mathbb{R}$

$$\partial_t^2 u-\partial_x^2 u+\frac{1}{\sqrt{2}}\sin(\sqrt{2}u)+\mu\Delta(u)=0, \tag{C.1}$$

where

$$\Delta(u)=-\frac{1}{\sqrt{2}}\sin(\sqrt{2}u)+u-\frac{1}{3}u^3-f(u), \tag{C.2}$$

and f is a real-analytic odd function such that $f(u)=\mathcal{O}(u^5)$ for $|u|\ll 1$. The corresponding Stokes constant $C_{\mathrm{in}}(\mu)$ is analytic in μ (Lemma 3.5).

Notice that, if $\mu=0$, then (C.1) becomes the sine-Gordon equation. On the other hand, if $\mu=1$, then (C.1) is the Klein-Gordon equation considered in (1.2). Moreover, (C.1) can be rewritten as

$$\partial_t^2 u-\partial_x^2 u+u-\frac{1}{3}u^3-g(u;\mu)=0,$$

where

$$g(u;\mu)=-\left((1-\mu)\left(\frac{1}{\sqrt{2}}\sin(\sqrt{2}u)-u+\frac{1}{3}u^3\right)-\mu f(u)\right)$$

is an odd real-analytic function and $g(u;\mu)=\mathcal{O}(u^5)$ with a linear dependence on the parameter μ.

Now, considering the scalings employed in this paper

$$u(x,t)=\varepsilon v(\varepsilon x,\omega t),\quad \varepsilon=\sqrt{1-\omega^2},\qquad \omega<1,$$

where we take $k=1$ in (1.13) and the above ε corresponds to the $\varepsilon\omega$ in (1.13), we have that $u(x,t)$ satisfies (C.1) if, and only if, $v(y,\tau)$ satisfies

$$\frac{\omega^2}{\varepsilon^2}\partial_\tau^2 v - \partial_y^2 v + \frac{1}{\varepsilon^3\sqrt{2}}\sin(\sqrt{2}\varepsilon v) + \frac{\mu}{\varepsilon^3}\Delta(\varepsilon v) = 0,$$

or equivalently

$$\frac{\omega^2}{\varepsilon^2}\partial_\tau^2 v - \partial_y^2 v + \frac{1}{\varepsilon^2}v - \frac{1}{3}v^3 - \frac{1}{\varepsilon^3}g(\varepsilon v;\mu) = 0.$$

Therefore, all the results in Section 3 also hold for (C.1). Furthermore, all the solutions of (C.1) obtained in such results depend analytically on the parameter μ.

C.1. **A perturbative approach in μ.** In this section, we obtain information on the Stokes constant $C_{\rm in} = C_{\rm in}(f;\mu)$ associated to (C.1) based on a perturbative in μ analysis which corresponds to applying the classical Melnikov Theory to the inner equation (corresponding to (C.1)).

Consider the inner variable (3.6) and the inner scaling (3.7), the equation (C.1) writes as

$$\omega^2\partial_\tau^2\phi - \partial_z^2\phi + \frac{1}{\sqrt{2}}\sin(\sqrt{2}\phi) + \mu\Delta(\phi) = 0.$$

Taking the singular limit $\varepsilon = 0$, we obtain the *inner equation* associated to (C.1)

$$\partial_\tau^2\phi_0 - \partial_z^2\phi_0 + \frac{1}{\sqrt{2}}\sin(\sqrt{2}\phi_0) + \mu\Delta(\phi_0) = 0. \tag{C.3}$$

From Theorem 3.3, equation (C.3) admits two one-parameter family of solutions $\phi_\mu^{0,\star} : D_{\theta,\kappa}^{\star,\rm in}\times\mathbb{T}\to\mathbb{C}$, $\star = u,s$, where the parameter μ belongs to some compact interval $K\subset\mathbb{C}$ containing the origin. In addition $\phi^{0,\star}(z,\tau;\mu) = \phi_\mu^{0,\star}(z,\tau)$ is analytic in the variables z and μ. Also,

$$\lim_{|z|\to\infty}\phi^{0,\star}(z,\tau;\mu) = 0,\ \forall\tau\in\mathbb{T},\ \forall\mu\in K.$$

For $\mu = 0$, the inner equation (C.3) admits a breather solution

$$\phi_b^0(z,\tau) = \frac{4}{\sqrt{2}}\arctan\left(-\frac{i\sin(\tau)}{z}\right). \tag{C.4}$$

Now, μ as a perturbative parameter in (C.3), we shall obtain an asymptotic formula for the Stokes constant $C_{\rm in}(\mu)$ through a Melnikov analysis. In order to do this, write

$$\phi^{0,\star}(z,\tau;\mu) = \phi_b^0(z,\tau) + \mu\psi^\star(z,\tau) + \mu^2 R^\star(z,\tau;\mu), \tag{C.5}$$

where $R^\star(z,\tau;\mu)$ is analytic in the variables z and μ. A simple computation shows that $\psi^\star$ satisfies the linear equation

$$\partial_\tau^2\psi^\star - \partial_z^2\psi^\star + \cos(\sqrt{2}\phi_b^0)\psi^\star - \Delta(\phi_b^0) = 0. \tag{C.6}$$

Lemma C.1. *The homogeneous linear partial differential equation*

$$\partial_\tau^2\xi - \partial_z^2\xi + \cos(\sqrt{2}\phi_b^0)\xi = 0 \tag{C.7}$$

has a family of solutions given by

$$\xi_n^\pm(z,\tau) = \frac{2}{\mu_n^2}\left(\chi_n^\pm(z,\tau) - \chi_{-n}^\pm(z,\tau)\right), \tag{C.8}$$

where $n\in\mathbb{N}$, $n\geq 2$, $\mu_n = \sqrt{n^2-1}$ and, for each $l\in\mathbb{Z}$, $\chi_l^\pm$ is the function given by

$$\begin{aligned}\chi_l^\pm(z,\tau) = {}& e^{\pm i\mu_l z + il\tau}\left(1 - \frac{\sin^2(\tau)}{z^2}\right)^{-1}\\ &\times\left\{\pm\frac{\mu_l}{2z} - \frac{l}{2}\frac{\cos(\tau)\sin(\tau)}{z^2} - \frac{i}{4}\mu_l^2 + i\frac{(l^2+1)}{4}\frac{\sin^2(\tau)}{z^2}\right\}.\end{aligned} \tag{C.9}$$

The proof of this lemma is a direct consequence of Lemma 4 of [18]. See also (C.17) below for the original formulas in [18]. In fact these linear solutions along with $\partial_z\phi_b^0$ and additional one for $n=1$ form a basis of all the solutions of the linear equation (C.7).

From Theorem 3.3, we have that

$$\phi^{0,u}(z,\tau;\mu)-\phi^{0,s}(z,\tau;\mu)=e^{-i\mu_3 z}\left(C_{\mathrm{in}}(\mu)\sin(3\tau)+\chi(z,\tau;\mu)\right) \tag{C.10}$$

for each $z\in\mathcal{R}^{\mathrm{in},+}_{\theta,\kappa}=D^{u,\mathrm{in}}_{\theta,\kappa}\cap D^{s,\mathrm{in}}_{\theta,\kappa}\cap\{z;\ z\in i\mathbb{R}\text{ and }\mathrm{Im}(z)<0\}$, where C_{in} in an analytic function in the variable μ, and $\chi_\mu(z,\tau)=\chi(z,\tau;\mu)$ is an analytic function in the variables z and μ such that

$$\|\chi_\mu\|_{\ell_1}(z),\|\partial_\tau\chi_\mu\|_{\ell_1}(z)\le\frac{M_2}{|z|}\quad\textit{and}\quad\|\partial_z\chi_\mu\|_{\ell_1}(z)\le\frac{M_2}{|z|^2},\ \forall z\in\mathcal{R}^{\mathrm{in},+}_{\theta,\kappa}.$$

Proposition C.2. *The function C_{in} in* (C.10) *satisfies*

$$C'_{\mathrm{in}}(0)=\frac{1}{2\pi i\mu_3}\int_{-\infty}^{\infty}\int_0^{2\pi}\Delta(\phi_b^0(z+s,\tau))\xi_3^+(z+s,\tau)d\tau ds, \tag{C.11}$$

which is independent of $z\in\mathcal{R}^{\mathrm{in},+}_{\theta,\kappa}$, where ξ_3^+ is given in (C.9).

Proof. Consider ξ_3^+ given in (C.9). Since ψ^u satisfies (C.6), and multiplying it by ξ_3^+, we obtain

$$\xi_3^+\left(\partial_\tau^2\psi^u-\partial_z^2\psi^u+\cos(\sqrt{2}\phi_b^0)\psi^u\right)=\xi_3^+\Delta(\phi_b^0). \tag{C.12}$$

Let $z\in D^{u,\mathrm{in}}_{\theta,\kappa}$. Integrating (C.12) by parts and using that ξ_3^+ satisfies (C.7) and

$$\xi_3^+(z,\tau)=e^{i\mu_3 z}\left(\sin(3\tau)+\mathcal{O}_{\ell_1}(z^{-1})\right),\quad\text{as }|z|\to\infty,$$

we obtain

$$\int_{-\infty}^0\int_0^{2\pi}\Delta(\phi_b^0(s+z,\tau))\xi_3^+(s+z,\tau)d\tau ds=\int_0^{2\pi}\left[\psi^u(z,\tau)\partial_z\xi_3^+(z,\tau)-\partial_z\psi^u(z,\tau)\xi_3^+(z,\tau)\right]d\tau. \tag{C.13}$$

Analogously, if $z\in D^{s,\mathrm{in}}_{\theta,\kappa}$, we obtain

$$\int_{+\infty}^0\int_0^{2\pi}\Delta(\phi_b^0(s+z,\tau))\xi_3^+(s+z,\tau)d\tau ds=\int_0^{2\pi}\left[\psi^s(z,\tau)\partial_z\xi_3^+(z,\tau)-\partial_z\psi^s(z,\tau)\xi_3^+(z,\tau)\right]d\tau. \tag{C.14}$$

Recall that, if $\mu=0$, then $C_{\mathrm{in}}(0)=0$ and $\chi(z,\tau;0)\equiv0$, since $\phi^{0,u}(z,\tau;0)=\phi^{0,s}(z,\tau;0)$. Now, using (C.5) and (C.10), expanding C_{in} and χ around $\mu=0$ and taking $\mu\to0$, it follows that

$$\psi^u(z,\tau)-\psi^s(z,\tau)=e^{-i\mu_3 z}\left(C'_{\mathrm{in}}(0)\sin(3\tau)+\partial_\mu\chi(z,\tau;0)\right). \tag{C.15}$$

Hence, using (C.13), (C.14) and (C.15), a straightforward computation shows that, for each $z\in\mathcal{R}^{\mathrm{in},+}_{\theta,\kappa}$,

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_0^{2\pi}\Delta(\phi_b^0(s+z,\tau))\xi_3^+(s+z,\tau)d\tau ds&=\int_0^{2\pi}(\psi^u-\psi^s)(z,\tau)\partial_z\xi_3^+(z,\tau)d\tau\\&\quad-\int_0^{2\pi}\partial_z(\psi^u-\psi^s)(z,\tau)\xi_3^+(z,\tau)d\tau\\&=2\pi i\mu_3C'_{\mathrm{in}}(0)+R(z,\tau),\end{aligned} \tag{C.16}$$

where $R(z,\tau)=\mathcal{O}_{\ell_1}(z^{-1})$. Since the left-hand side of (C.16) does not depend on z, it follows that $R\equiv0$, and thus (C.11) holds. □

In order to analyze the integral

$$\mathcal{R}_3(\Delta)=\int_{-\infty}^{+\infty}\int_0^{2\pi}\Delta(\phi_b^0(s+z,\tau))\xi_3^+(s+z,\tau)d\tau ds,$$

we shall discuss some results obtained by Denzler in [18] and relate them with our problem. First, we notice that (C.1) has a family of breathers given by

$$u_b(x,t;m,\omega)=\frac{4}{\sqrt{2}}\arctan\left(\frac{m}{\omega}\frac{\sin(\omega t)}{\cosh(mx)}\right),\ m^2+\omega^2=1.$$

In Lemma 4 of [18], the author has proved that, for every $l \in \mathbb{Z}$ and $\lambda_l = \sqrt{l^2\omega^2 - 1}$,

$$\begin{aligned} \chi_l^{\pm,D}(x,t;m,\omega) &= e^{\pm i\lambda_l x + il\omega t}\left(1 + \frac{m^2\sin^2(\omega t)}{\omega^2\cosh^2(mx)}\right)^{-1} \\ &\times\left\{\pm\frac{\lambda_l}{2m}\tanh(mx) + \frac{l}{2}\frac{\cos(\omega t)\sin(\omega t)}{\cosh^2(mx)}\right. \\ &\left. -i\left(\left(\frac{\lambda_l}{2m}\right)^2 - \frac{1}{4} - i\frac{l^2+1}{4}\frac{\sin^2(\omega t)}{\cosh^2(mx)}\right)\right\} \end{aligned} \tag{C.17}$$

is a solution of the homogeneous linear equation

$$\partial_t^2\xi - \partial_x^2\xi + \cos(\sqrt{2}u_b(x,t;m,\omega))\xi = 0.$$

In the remaining paper [18], Denzler has analyzed the equations

$$\mathcal{R}_l^{\pm,D}\left(\Delta, \frac{m}{\omega}\right) := \int_{-\infty}^{+\infty}\int_0^{\frac{2\pi}{\omega}} \chi_l^{\pm,D}(x,t)\Delta(\sqrt{2}u_b(x,t;m,\omega))dtdx = 0,\ l \in \mathbb{Z}. \tag{C.18}$$

It is worth to mention that, since Δ is odd, it follows that $\mathcal{R}_{-l}^{\pm,D} = -\mathcal{R}_l^{\pm,D}$, for every $l \in \mathbb{Z}$ (see [18]).

A simple computation shows that the function $\chi_l^{\pm}$ given by (C.9) satisfies that

$$\chi_l^{\pm}(z,\tau) = \lim_{\varepsilon\to 0}\varepsilon^2 e^{\pm\frac{\lambda_l\pi}{2\varepsilon}}\chi_l^{\pm,D}\left(\frac{i\pi}{2\varepsilon} + z, \frac{\tau}{\sqrt{1-\varepsilon^2}}; \varepsilon, \sqrt{1-\varepsilon^2}\right). \tag{C.19}$$

Moreover, the solution ϕ_b^0 given by (C.4) satisfies that

$$\phi_b^0(z,\tau) = \lim_{\varepsilon\to 0} u_b\left(\frac{i\pi}{2\varepsilon} + z, \frac{\tau}{\sqrt{1-\varepsilon^2}}; \varepsilon, \sqrt{1-\varepsilon^2}\right). \tag{C.20}$$

Considering the function

$$\widetilde{\Delta}(u) = \Delta(u/\sqrt{2}), \tag{C.21}$$

where Δ is given by (C.2), and the change of variables

$$x = \frac{i\pi}{2\varepsilon} + z + s,\ t = \frac{\tau}{\sqrt{1-\varepsilon^2}},$$

in (C.18), it follows from (C.19) and (C.20) that, for every $l \in \mathbb{Z}$,

$$\int_{-\infty}^{+\infty}\int_0^{2\pi}\Delta(\phi_b^0(s+z,\tau))\chi_l^+(s+z,\tau)d\tau ds = \lim_{\varepsilon\to 0}\varepsilon^2 e^{\frac{\lambda_l\pi}{2\varepsilon}}\mathcal{R}_l^{+,D}\left(\widetilde{\Delta}, \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right).$$

Observe $\chi_{-l}^{\pm}(z,\tau) = \chi_l^{\pm}(z,-\tau)$. Hence, using (C.8) and that $\widetilde{\Delta}$ is odd, we obtain

$$\mathcal{R}_3(\Delta) = \lim_{\varepsilon\to 0}\frac{4\varepsilon^2 e^{\frac{\lambda_3\pi}{2\varepsilon}}}{\mu_3^2}\mathcal{R}_3^{+,D}\left(\widetilde{\Delta}, \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right). \tag{C.22}$$

Combining (C.22) and Lemmas 5 and 6 of [18], we obtain the following result which ensures the splitting of the invariant manifolds $W^u(0)$ and $W^s(0)$ of (C.1) for a generic choice of Δ.

Proposition C.3. *Let* $(\Delta_p)_{p=1}^{\infty}$ *be the sequence given by the Taylor expansion of the analytic function*

$$\Theta(z) = \frac{\Delta(2\sqrt{2}\arctan(z))}{1+z^2} = \sum_{p=1}^{\infty}\Delta_p z^p,\ \forall |z| \le \rho < 1,$$

and define $\Delta_0 = \Delta_{-1} = 0$*, then*

$$C_{\rm in}'(0) = i4\sqrt{2}\pi S(\Delta), \quad \text{where } S(\Delta) = \sum_{q=0}^{\infty}\frac{(-2)^q}{q!(3+q)!}A_{3+2q} \neq 0, \tag{C.23}$$

where $A_p = B_{p+2} - B_p$*,* $B_p = p(\Gamma_p - \Gamma_{p-2})$*, and* $\Gamma_p = (-1)^{[p/2]}\dfrac{\Delta_p}{(p+1)^2}$.

Clearly $C_{\rm in}'(0) \neq 0$ provided $S(\Delta) \neq 0$.

Proof. From [18] (see Lemma 5), we have that $\mathcal{R}_3^{+,D}$ given by (C.18) is written as

$$\mathcal{R}_3^{+,D}\left(\widetilde{\Delta}, \frac{m}{\omega}\right) = \left(i\frac{m}{\omega}\right)^2 \frac{\pi^2}{2m^2\omega^2} E_3\left(\frac{m}{\omega}\right) \widetilde{\mathcal{R}}_3^{+,D}\left(\widetilde{\Delta}, \frac{m}{\omega}\right),\ m^2+\omega^2=1$$

where $\widetilde{\Delta}$ is given by (C.21), E_3 is a non-vanishing function given by

$$E_3\left(\frac{m}{\omega}\right) = \frac{2\left(m^{-2}-1\right)}{\cosh(\pi\lambda_3/(2m))},$$

and $\widetilde{\mathcal{R}}_3^{+,D}(\widetilde{\Delta}, z)$ is a function which can be continued to an analytic function for $|z|<\rho$ and such that

$$\begin{aligned} \widetilde{\mathcal{R}}_3^{+,D}\left(\widetilde{\Delta}, \frac{m}{\omega}\right) &= \sum_{q=0}^{\infty} \left(\frac{m}{\omega}\right)^{2q} \binom{3+2q}{q} [(2+2q)!]^{-1} \prod_{r=0}^{q-1}\left[r(3+r)+\frac{2}{m^2}\right] \\ &\quad \times \left\{\Delta_{3+2q}\left(5\omega^2-1-\frac{9\omega^2-1}{3+2q}\right) + \Delta_{1+2q}\left(5\omega^2+\frac{9\omega^2}{3+2q}\right)\right\}. \end{aligned} \tag{C.24}$$

From Lemma 6 in [18], we have that the function $\widetilde{\mathcal{R}}_3^{+,D}$ given by (C.24) can be expressed in terms of A_p as

$$\widetilde{\mathcal{R}}_3^{+,D}\left(\widetilde{\Delta}, \frac{m}{\omega}\right) = C\sum_{q=0}^{\infty} \frac{A_{3+2q}}{q!(3+q)!}\left(-\frac{1}{\omega^2}\right)^q \prod_{r=0}^{q}\left[2+r(3+r)m^2\right],$$

where $C=16\omega^2$.

A simple computation shows that

$$\lim_{\varepsilon\to 0} \frac{4\varepsilon^2 e^{\frac{\lambda_3\pi}{2\varepsilon}}}{\mu_3^2} E_3\left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right) = 2, \tag{C.25}$$

and

$$\lim_{\varepsilon\to 0} \widetilde{\mathcal{R}}_3^{+,D}\left(\widetilde{\Delta}, \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right) = 32S(\Delta), \tag{C.26}$$

where $S(\Delta)$ is given by (C.23). Therefore, the result follows from (C.11), (C.22), (C.25) and (C.26). □

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