Breakdown of homoclinic orbits to L_3 in the Restricted Planar Circular 3-Body Problem

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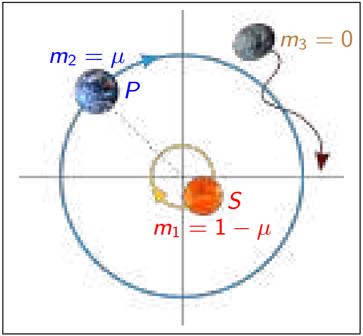
Restricted Planar Circular 3-Body Problem



- **Restricted**: one body is massless, $m_3 = 0$.
- **Planar**: the bodies move on the same plane.
- **Circular**: the primaries (m_1 and m_2) perform a circular motion.

We study the motion of the **massless body**:

$$(q(t), p(t)) \in \mathbb{R}^4.$$

We normalize:

- $m_1 = 1 - \mu$ and $m_2 = \mu$ with $\mu \in \left(0, \frac{1}{2}\right]$.

Approach: Perturbative study for $0 < \mu \ll 1$.

- Unperturbed system ($\mu = 0$) corresponds to a 2-Body Problem.

Rotating framework and Lagrange points

Rotating framework: the primaries are fixed at $(\mu, 0)$ and $(\mu - 1, 0)$.

Hamiltonian: $h = h_0 + \mu h_1$

$$h_0(q,p) = \frac{\|p\|^2}{2} - q^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{1}{\|q\|} \qquad \text{2-Body Problem (integrable),}$$

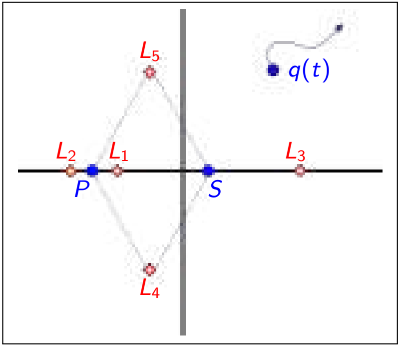
$$\mu h_1(q;\mu) = \frac{1}{\|q\|} - \frac{(1-\mu)}{\|q-(\mu,0)\|} - \frac{\mu}{\|q-(\mu-1,0)\|} \qquad \text{Perturbation (far from collision)}$$

Five critical points: Lagrange points.

For $\mu > 0$ small:

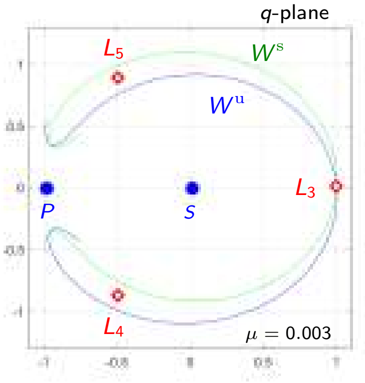
- L_1, L_2 and L_3 saddle-center.
- L_4 and L_5 center-center.

We start studying the dynamics surrounding L_3 and its invariant manifolds



The invariant manifolds of L_3

- L_3 is a **saddle-center** equilibrium point.
- It has **one dimensional stable and unstable** manifolds W^{s} and W^{u}.

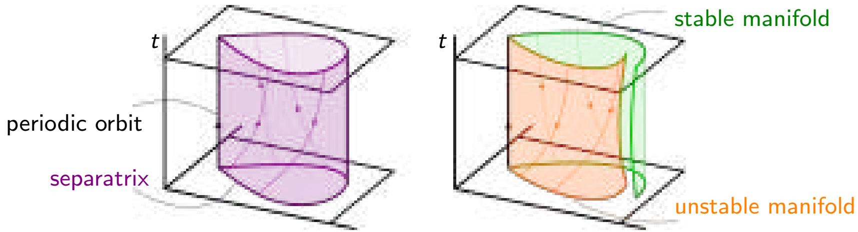


Goal:

To prove the existence of **chaotic dynamics around $\mathbf{L_3}$** by applying the methods used for the **spliting of separatrices** phenomenon.

Splitting of separatrices - Classical approach

$\varepsilon = 0$ $\qquad$ $0 < \varepsilon \ll 1$

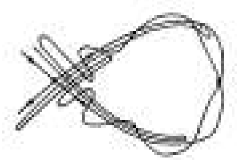


The Smale-Birkhoff homoclinic theorem

Transversal intersections between the unstable and stable manifolds $\Longrightarrow$ chaotic motions.

The Poincaré-Melnikov method

$$\mathrm{dist} = \varepsilon M + \mathcal{O}(\varepsilon^2).$$

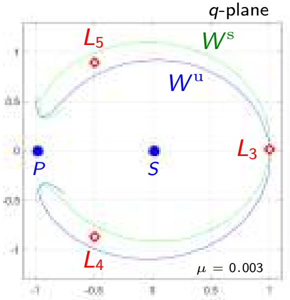


Main Result

Main result:

For $0 < \mu \ll 1$, we have **obtained an asymptotic formula for the distance** between the stable and unstable manifolds of L_3.

- We focus on the upper branches. The lower branches are symmetric.
- The stable and unstable manifolds **either coincide or do not intersect**.
- The breakdown of W^{u} and W^{s} does not lead to chaos.

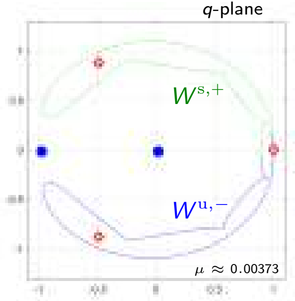


Future work

- To prove the **existence of chaotic dynamics** around L_3:
  - Study Lyapunov periodic orbits close to L_3.
  - Look for transversal intersections of their 2-dimensional invariant manifolds.

- To prove the existence of a sequence $\mu_k \to 0$ such that, for these parameters, there exist **secondary homoclinic connections**.

  E. Barrabés, J. M. Mondelo and M. Ollé. (2008).

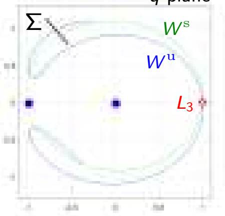


Main Result

Main Theorem:

Take the section Σ (see Figure) and let $(q^{u,s}, p^{u,s})$ be the first intersection of $W^{u,s}$ with Σ. For $0 < \mu \ll 1$,

$$\|q^u - q^s\| + \|p^u - p^s\| = C_\Sigma\, \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left(\Theta + \mathcal{O}\left(\frac{1}{|\log \mu|}\right)\right)$$



- The **constant A** is given by

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x}\sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}\,dx \approx 0.177744.$$

Computed by Font (1984) and Simó, Sousa-Silva and Terra (2013).

- The **constant Θ** corresponds to a Stokes constant. (It does not have a closed formula). We have seen $\mathbf{\Theta \approx 1.63}$.
- **Beyond all orders phenomenon**: The difference cannot be detected by expanding the invariant manifolds into a series of μ.

Exponentially small splitting of separatrices

- Following a Poincaré-Melnikov method approach is not possible:

$$\mathrm{dist} = \mu^{\frac{1}{3}} M + \mathcal{O}(\mu^{\frac{2}{3}}), \qquad \text{with} \qquad M = \mathcal{O}(e^{-\frac{A}{\sqrt{\mu}}}).$$

- For $\mu > 0$ small, L_3 has eigenvalues:

$$\pm\sqrt{\mu \frac{21}{8}}\,(1+\mathcal{O}(\mu)), \qquad \pm i(1+\mathcal{O}(\mu)).$$

Eigenvalues at different time-scales $\Longrightarrow$ **singular perturbation** problem.

Exponentially small phenomenon:

Due to the **rapidly rotating dynamics**, the invariant manifolds are **exponentially close** to each other with respect to $\sqrt{\mu}$.

I. Baldomá, E. Fontich, M. Guardia, T. M. Seara. (2012).

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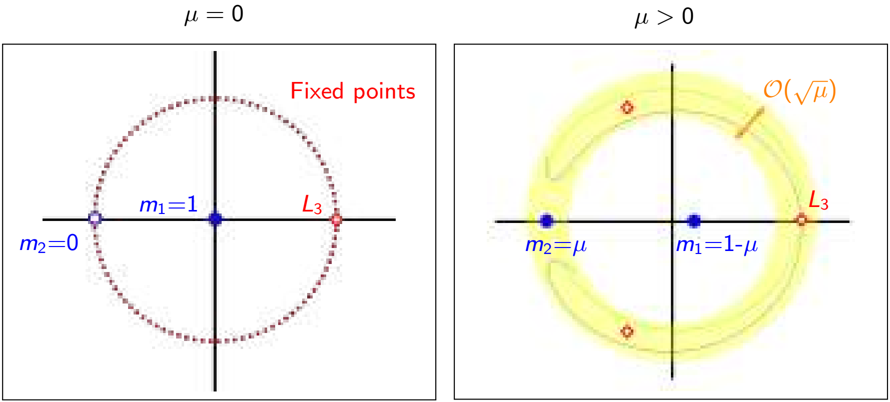
Unperturbed system



- The eigenvalues of L_3 are $\pm\sqrt{\frac{21}{8}\mu}(1+\mathcal{O}(\mu))$ and $\pm i(1+\mathcal{O}(\mu))$.

Goal:

Apply a singular change of coordinates to obtain a **new first order** of the system **with a saddle and separatrices**.

How do we obtain this new first order?

The eigenvalues of L_3 are $\pm\sqrt{\frac{21}{8}\mu}(1+\mathcal{O}(\mu))$ and $\pm i(1+\mathcal{O}(\mu))$.

- Step 1: Performing an **action-angle** change of coordinates, to decouple on a first order the saddle and center behaviour. (**Poincaré elements**)

  The linearized part of the vector field associated to L_3 becomes:

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} + \mathcal{O}(\mu).$$

- Step 2: **Scaling** the coordinates and the time to capture the slow-fast dynamics.

  The linearized part of the vector field associated to L_3 becomes:

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ -\frac{7}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\sqrt{\mu}} & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{\mu}} \end{pmatrix} + \mathcal{O}(\mu^{\frac{1}{4}}).$$

New "unperturbed system"

Hamiltonian in "good" coordinates and scaled time:

$$H = H_{\mathrm{p}} + H_{\mathrm{osc}} + \mathcal{O}(\mu^{\frac{1}{4}}),$$

$$H_{\mathrm{p}}(\lambda,\Lambda) = -\frac{3}{2}\Lambda^2 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}}, \qquad H_{\mathrm{osc}}(x,y;\mu) = \frac{xy}{\sqrt{\mu}},$$

with symplectic form $d\lambda \wedge d\Lambda + i\, dx \wedge dy$.

- The linearization of the vector field at L_3 is

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ -\frac{7}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\sqrt{\mu}} & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{\mu}} \end{pmatrix} + \mathcal{O}(\mu^{\frac{1}{4}}).$$

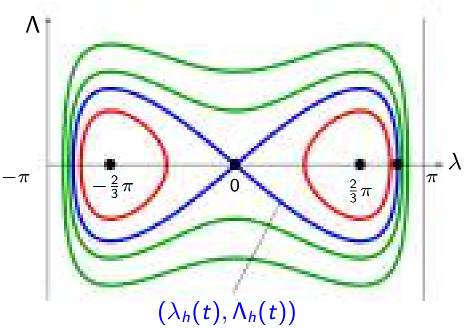
New "unperturbed system":

A pendulum-like Hamiltonian H_{p} with two homoclinic connections plus a fast oscillator H_{osc}.

Problem:

Terms $\mathcal{O}(\mu^{\frac{1}{4}})$ are not explicit.

Hamiltonian $H_{\mathrm{p}}(\lambda, \Lambda)$



- The Hamiltonian $H_{\mathrm{p}}(\lambda, \Lambda)$ has two homoclinic connections or **separatrices**.
- Parametrization of the right separatrix: $(\lambda_h(t), \Lambda_h(t))$.

We expect the perturbed invariant manifolds to be close to $(\lambda_h, \Lambda_h, 0, 0)$.

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Heuristic idea of the proof

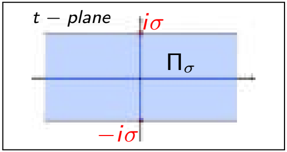
Recall the Hamiltonian:

$$H = -\frac{3}{2}\Lambda^2 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}} + \frac{xy}{\sqrt{\mu}} + \mathcal{O}(\mu^{\frac{1}{4}}).$$

Let $(\Delta x, \Delta y) = (x^{\mathrm{u}} - x^{\mathrm{s}}, y^{\mathrm{u}} - y^{\mathrm{s}})$ the difference between the stable and the unstable manifolds. Then

$$\dot{\Delta x} \approx \frac{i}{\sqrt{\mu}}\Delta x \quad \Longrightarrow \quad \Delta x(t) \approx C_x e^{\frac{i}{\sqrt{\mu}}t},$$

$$\dot{\Delta y} \approx -\frac{i}{\sqrt{\mu}}\Delta y \quad \Longrightarrow \quad \Delta y(t) \approx C_y e^{-\frac{i}{\sqrt{\mu}}t}.$$



Proving that $(\Delta x, \Delta y)$ is analytic in Π_σ:

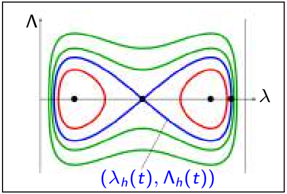
$$|C_x| \lesssim e^{-\frac{\sigma}{\sqrt{\mu}}} |\Delta x(-i\sigma)|,$$

$$|C_y| \lesssim e^{-\frac{\sigma}{\sqrt{\mu}}} |\Delta y(i\sigma)|.$$

$(\Delta x, \Delta y)$ bounded in a complex domain $\Longrightarrow$ exponentially small bound in $\mathbb{R}$.

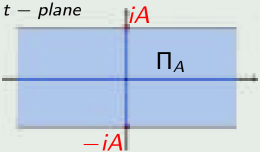
Analytical continuation of the separatrix

We expect the perturbed invariant manifolds to be close to $(\lambda_h, \Lambda_h, 0, 0)$.



- Parametrization of the right separatrix (λ_h, Λ_h).
- **Goal**: analytically extend (λ_h, Λ_h) to the complex plane.

Theorem:



$(A \approx 0.177744)$

Let:

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}\, dx.$$

- (λ_h, Λ_h) analytically extends to the complex strip

$$\Pi_A = \{ t \in \mathbb{C} \ : \ |\mathrm{Im}\, t| < A \},$$

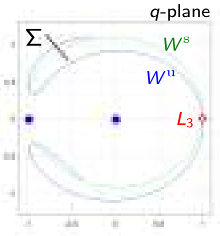
- (λ_h, Λ_h) has two singularities $t = \pm iA$ at $\partial \Pi_A$.

Conclusion

Main Theorem:

Take the section Σ (see Figure) and let $(q^{\mathrm{u,s}}, p^{\mathrm{u,s}})$ be the first intersection of $W^{\mathrm{u,s}}$ with Σ. For $0 < \mu \ll 1$,

$$\|q^{\mathrm{u}} - q^{\mathrm{s}}\| + \|p^{\mathrm{u}} - p^{\mathrm{s}}\| = C_\Sigma\, \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left(\Theta + \mathcal{O}\left(\frac{1}{|\log \mu|}\right)\right)$$



Thanks for your attention!