Symbolic dynamics and oscillatory motions in the 3 body problem

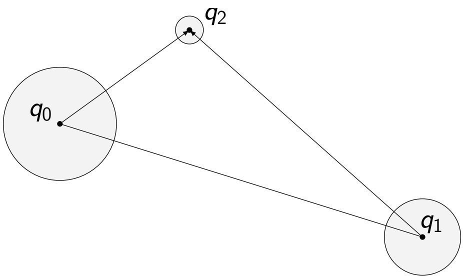
Marcel Guàrdia, Pau Martín, Tere M. Seara

The planar three body problem

- Three co-planar bodies of masses $m_0, m_1, m_2 > 0$ under Newtonian gravitational force:

$$\frac{d^2 q_i}{dt^2} = \sum_{j=0, j\neq i}^{2} m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}, \qquad q_0, q_1, q_2 \in \mathbb{R}^2$$

- Two strongly related problems:
  - Chaotic motions (Smale horseshoe)
  - Final motions



Chazy (1922): Final motions for the 3 body problem

Call r_i the mutual distances between bodies.

Final motions: Possible behaviors of the 3BP when $t \to \pm\infty$.

- Hyperbolic ($H^\pm$): $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$.
- Parabolic ($P^\pm$): $|r_i| \to \infty$, $|\dot{r}_i| \to 0$.
- Bounded ($B^\pm$): $\sup_{t \geq t_0} |r_i| < \infty$.
- Hyperbolic-Parabolic ($HP_k{}^\pm$): $|r_i| \to \infty \ \forall i$, $|\dot{r}_k| \to 0$, $|\dot{r}_i| \to c_i > 0$, $i \neq k$.
- Hyperbolic-Elliptic ($HE_k{}^\pm$): $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$, $i \neq k$, $\sup_{t \geq t_0} |r_k| < \pm\infty$.
- Parabolic-Elliptic ($PE_k{}^\pm$): $|r_i| \to \infty$, $|\dot{r}_i| \to 0$, $i \neq k$, $\sup\limits_{t \geq t_0} |r_k| < \infty$.
- Oscillatory ($OS^\pm$): $\limsup_{t \to \pm\infty} \sup_i |r_i| = \infty$, $\liminf\limits_{t \to \pm\infty} \sup\limits_i |r_i| < \infty$.

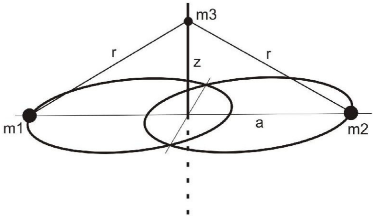
Final motions for the 3BP

- In the limit $m_1, m_2 \to 0$: Two uncoupled two 2BP.
- Only H, P, HP_k, HE_k, PE_k, B (All except oscillatory motions!).

  $\to$ Past and future final motions must coincide.
- Questions by Chazy (1922) for the 3BP:
  - Do oscillatory motions exist?
  - Can one combine different past and future final motions?
- Long literature on oscillatory motions and on chaotic motions for the 3BP. But:
  - Most for the Restricted 3BP: $m_2 = 0$ so q_0, q_1 perform a 2BP.
  - Quite strict assumptions on the masses of the bodies.

Oscillatory motions: First result

- Sitnikov (1960) considered the Restricted Spatial Elliptic 3BP:
  - $m_1 = m_2 = 1/2$
  - q_1, q_2 move on ellipses of small eccentricity.
  - q_3 ($m_3 = 0$) moves on the (invariant) vertical axis.



He obtained:

- Oscillatory motions
- Free combination of past and future final motions.

Oscillatory motions: Past results

- Moser (1973): New proof of Sitnikov results via chaotic motions.
- Restr. Planar Circular 3BP: q_0, q_1 (masses μ and $1-\mu$, $\mu \in (0, \frac{1}{2}]$) perform circular motion and q_2 is coplanar:
  - Simó and Llibre (1980): Oscillatory motions for $0 < \mu \ll 1$.
  - G.-Martín-Seara (2016): Oscillatory motions for all masses: $\mu \in (0, \frac{1}{2}]$.
- Other approaches, other results for the Restricted 3BP: Kaloshin-Galante, G.-Martín-Sabbagh-Seara, Xia, Seara-Zhang...
- Oscillatory motions for the full 3BP ($m_2 > 0$):
  - Alexeev (1969) for a Sitnikov-like model: $m_0 = m_1 \gg m_2$.
  - Moeckel (2007): For a "large" (non-generic) choice of masses. The proof relies on the passage close to triple collision.

Abundance of the Final motions

- Measure of each $X^- \cap Y^+$?
- It is known for each $X^- \cap Y^+$ whether it has positive or zero measure except for $OS^- \cap OS^+$.
- (Wide open) Conjecture (Kolmogorov, Alexeev): Lebesgue measure of $OS^- \cap OS^+$ is zero.
- Kaloshin–Gorodetski (2011): The Hausdorff dimension of oscillatory motions is maximal for both the Sitnikov problem and the RPC3BP.
- Recall that most existence results deal with Restricted models/narrow ranges of parameters!

Main result: Final motions

Theorem (G.–Martín–Seara)

Consider the three body problem with masses $m_0, m_1, m_2 > 0$ such that $m_0 \neq m_1$. Then,

$$X^- \cap Y^+ \neq \emptyset \qquad \textit{with} \qquad X, Y = OS, B, PE_2, HE_2.$$

In particular, $OS^- \cap OS^+ \neq \emptyset$.

- We can combine all possible negative energy final motions.
- The bodies of masses m_0 and m_1 perform (approximately) circular motions. That is, $|q_0 - q_1|$ is aproximately constant.
- The third body may have radically different behaviors: oscillatory, bounded, hyperbolic or parabolic.

Main result: Chaotic motions

Theorem (G.–Martín–Seara)

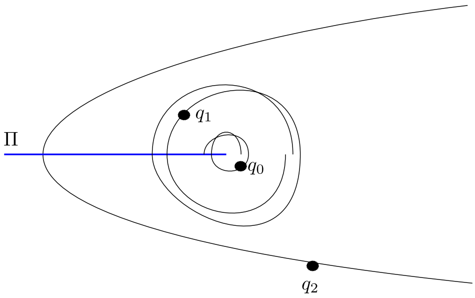
Consider the three body problem with masses $m_0, m_1, m_2 > 0$ such that $m_0 \neq m_1$ and denote by Φ_t its flow. Then, there exists a section Π transverse to Φ_t such that the induced Poincaré map

$$\mathcal{P} : \mathcal{U} = \mathring{\mathcal{U}} \subset \Pi \to \Pi$$

has an invariant set $\mathcal{X}$ which is homeomorphic to $\mathbb{N}^{\mathbb{Z}}$ such that $\mathcal{P}|_{\mathcal{X}}$ is topologically conjugated to the shift.

- The set $\mathcal{X}$ is a hyperbolic set once the 3BP is reduced by its classical first integrals $\to$ Positive topological entropy.
- Previous results also in other regions of phase space (but for quite strict mass choices): Bolotin-McKay, Bolotin, Marco-Niederman, Arioli, Wilczak-Zgliczynski, Capinski,...

Main result: Chaotic motions



- q_0 and q_1 evolve at a bounded distance from their center of mass.
- q_2 makes excursions close to a parabolic motion.

- Let N_k be the number of complete revolutions of q_0, q_1 between two consecutives passages of q_2 through Π.
- There exist integers $L \gg 1$ such that for any $\omega \in \mathbb{N}^{\mathbb{Z}}$, there exists a solution of the 3BP such that $N_k = L + \omega_k$.

Moser's approach for the Sitnikov problem

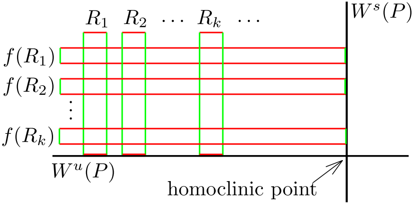
- Sitnikov Problem $\to$ 2 dimensional Poincaré map
- It has a fixed point at infinity P.
- Step 1: Check that P has stable and unstable invariant manifolds

  Not obvious! Its linearization is not hyperbolic but parabolic, i.e. linearization=Identity (McGehee 1972).

- Step 2: The invariant manifolds intersect transversally (Melnikov Method).
- If P were hyperbolic: Smale Theorem would lead to a Smale horseshoe.

Moser's approach for the Sitnikov problem

- Step 3: Parabolic Lambda lemma for the local dynamics.



- Step 4: Isolating blocks (plus cone fields) $\to$ Smale horseshoe.
- Key point: The model can be reduced to a 2D area preserving map.
- Same happens for the RPC3BP and the Alexeev model!

How can one implement Moser approach for the 3BP?

- Reduction by the classical first integrals + Poincaré map:
  4 dimensional symplectic map

  Instead of a fixed point at infinity: a disk of fixed points $\mathbb{D} \subset \mathbb{R}^2$.

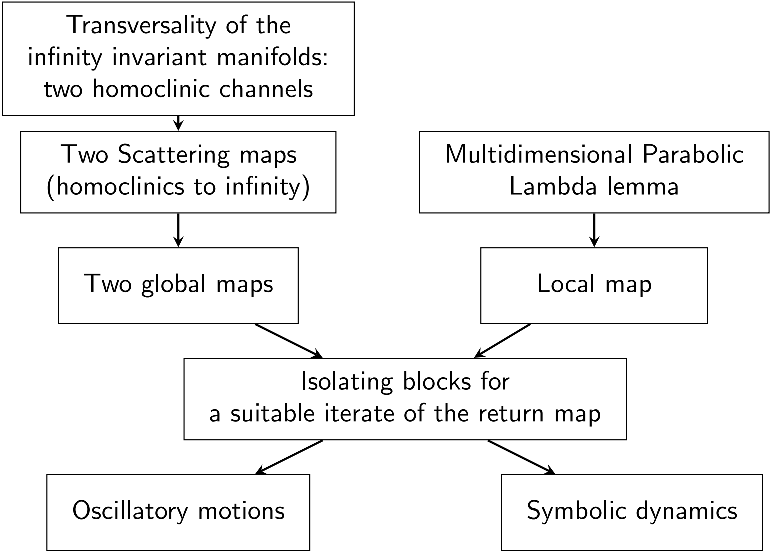
  There are "center" directions: harder to build hyperbolic sets.

- We want results for all values of the masses $m_0, m_1, m_2 > 0$ $(m_0 \neq m_1)$.

  Transversality of the invariant manifolds of infinity?

  We need a close to integrable regime.

Moser Approach: Scheme



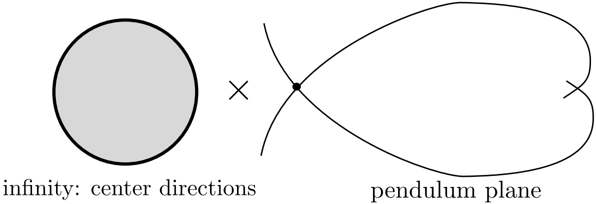
A different nearly integrable regime

- To prove transversality between the invariant manifolds of infinity we must be in a perturbative regime.
- Regime: $|q_2 - q_0|, |q_2 - q_1| \gg |q_1 - q_0|$
- At first order, the third body only "sees" one body at the center of mass of the other two:

$$H = \text{Two uncoupled Kepler problems} + \text{Small perturbation}.$$

- Two time scales:
  - q_0 and q_1 on ellipses with fast rotation.
  - q_2 on a parabola with slow motion.
- The transversality between the invariant manifolds is exponentially small with respect to the distances ratio.

How to construct the isolating blocks



- We need a 4 dimensional isolating block for $\mathcal{P}$.
- Pendulum–plane: Proceed as Moser for the Sitnikov problem
- How do we construct an isolating block for the "center" directions?
- The scattering maps capture the heteroclinic orbits between different points at infinity.
- Combining two scattering maps one can construct an isolating block.

An isolating block for (a suitable iterate of) the return map

- They are the "first order" of a return map in the center directions.
- Applying the return map a large number of times one can construct the isolating block.