Symbolic dynamics and oscillatory motions in the 3 body problem

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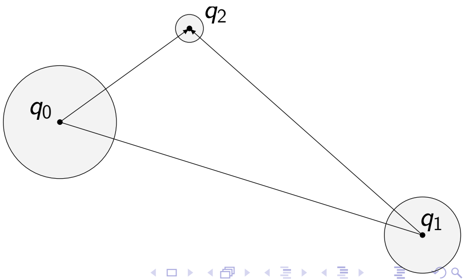
The planar three body problem

- Three co-planar bodies of masses $m_0, m_1, m_2 > 0$ under Newtonian gravitational force:

$$\frac{d^2 q_i}{dt^2} = \sum_{j=0, j\neq i}^{2} m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}, \qquad q_0, q_1, q_2 \in \mathbb{R}^2$$

- Long term dynamics?

- Two strongly related problems:
  - Chaotic motions
  - Final motions



Chaotic motions

- (One of) the paradigmatic chaotic dynamics is the shift – the Smale horseshoe.
- Take either $X = \{1, \dots, M\}$ or $X = \mathbb{N}$ and $X^{\mathbb{Z}}$.
- Shift $\sigma : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ defined as

$$(\sigma\omega)_k = \omega_{k+1}$$

- Properties:
  - Sensitive dependence on initial conditions.
  - Topologically transitive: dense orbits in $X^{\mathbb{Z}}$.
  - Periodic orbits are dense in $X^{\mathbb{Z}}$.
  - Positive topological entropy
- Can we embed this dynamics in the 3BP?

Chazy (1922): Final motions for the 3 body problem

Call r_i the mutual distances between bodies.

Final motions: Possible behaviors of the 3BP when $t \to \pm\infty$.

- Hyperbolic ($H^\pm$): $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$.
- Hyperbolic-Parabolic ($HP_k{}^\pm$): $|r_i| \to \infty\ \forall i$, $|\dot{r}_k| \to 0$, $|\dot{r}_i| \to c_i > 0$, $i \neq k$.
- Hyperbolic-Elliptic ($HE_k{}^\pm$): $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$, $i \neq k$, $\sup_{t \geq t_0} |r_k| < \pm\infty$.
- Parabolic-Elliptic ($PE_k{}^\pm$): $|r_i| \to \infty$, $|\dot{r}_i| \to 0$, $i \neq k$, $\displaystyle\sup_{t \geq t_0} |r_k| < \infty$.
- Parabolic ($P^\pm$): $|r_i| \to \infty$, $|\dot{r}_i| \to 0$.
- Bounded ($B^\pm$): $\sup_{t \geq t_0} |r_i| < \infty$.
- Oscillatory ($OS^\pm$): $\limsup_{t \to \pm\infty} \sup_i |r_i| = \infty$, $\displaystyle\liminf_{t \to \pm\infty} \sup_i |r_i| < \infty$.

Final motions for the 3BP

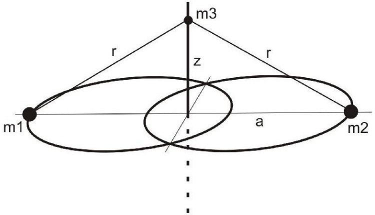
- In the limit $m_1, m_2 \to 0$: Two uncoupled two 2BP.
- Only H, P, HP_k, HE_k, PE_k, B (All except oscillatory motions!).

  $\to$ Past and future final motions must coincide,
- Questions by Chazy (1922) for the 3BP:
  - Do oscillatory motions exist?
  - Can one combine different past and future final motions?
- Long literature on oscillatory motions (and also on chaotic motions) for the 3BP. But:
  - Most for the Restricted 3BP: $m_2 = 0$ so q_0, q_1 perform a 2BP.
  - Quite strict assumptions on the masses of the bodies.

Oscillatory motions: Past results

First result:

- Sitnikov (1960) considered the Restricted spatial elliptic 3BP.
- Existence of oscillatory motions when
    - $m_1 = m_2 = 1/2$ and q_1, q_2 move on ellipses of small eccentricity.
    - q_3 ($m_3 = 0$) moves on the (invariant) vertical axis.
- Free combination of past and future final motions.



Oscillatory motions: Past results

- Moser (1973): New proof of Sitnikov results via chaotic motions.
- Restr. Planar Circular 3BP: q_0, q_1 (masses μ and $1-\mu$, $\mu \in (0, \frac{1}{2}]$) perform circular motion and q_2 is coplanar:
  - Simó and Llibre (1980): Oscillatory motions for $0 < \mu \ll 1$.
  - G.-Martín-Seara (2016): Oscillatory motions for all masses: $\mu \in (0, \frac{1}{2}]$.
- Other approaches, other results for the Restricted 3BP: Kaloshin-Galante, G.-Martín-Sabbagh-Seara, Xia, Seara-Zhang...
- Oscillatory motions for the full 3BP ($m_2 > 0$):
  - Alexeev (1969) for a Sitnikov-like model: $m_0 = m_1 \gg m_2$.
  - Moeckel (2007): For a "large" (non-generic) choice of masses. The proof relies on the passage close to triple collision.

Abundance of the Final motions

- Measure of each $X^- \cap Y^+$?
- It is known for each $X^- \cap Y^+$ whether it has positive or zero measure except for $OS^- \cap OS^+$.
- (Wide open) Conjecture (Kolmogorov, Alexeev): Lebesgue measure of $OS^- \cap OS^+$ is zero.
- According to Arnold this is the central problem of Celestial Mechanics.
- Kaloshin–Gorodetski (2011): The Hausdorff dimension of oscillatory motions is maximal for both the Sitnikov problem and the RPC3BP.
- Recall that most existence results deal with Restriced models/narrow ranges of parameters!

Main result: Final motions

Theorem (G.–Martín–Seara)

Consider the three body problem with masses $m_0, m_1, m_2 > 0$ such that $m_0 \neq m_1$. Then,

$$X^- \cap Y^+ \neq \emptyset \qquad \textit{with} \qquad X, Y = OS, B, PE_2, HE_2.$$

In particular, $OS^- \cap OS^+ \neq \emptyset$.

- The bodies of masses m_0 and m_1 perform (approximately) circular motions. That is, $|q_0 - q_1|$ is aproximately constant.
- The third body may have radically different behaviors: oscillatory, bounded, hyperbolic or parabolic.
- We can combine all possible negative energy final motions.

Main result: Chaotic motions

Theorem (G.–Martín–Seara)

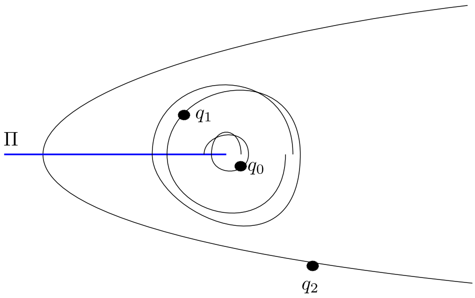
Consider the three body problem with masses $m_0, m_1, m_2 > 0$ such that $m_0 \neq m_1$ and denote by Φ_t its flow. Then, there exists a section Π transverse to Φ_t such that the induced Poincaré map

$$\mathcal{P} : \mathcal{U} = \mathring{\mathcal{U}} \subset \Pi \to \Pi$$

has an invariant set $\mathcal{X}$ which is homeomorphic to $\mathbb{N}^{\mathbb{Z}}$ such that $\mathcal{P}|_{\mathcal{X}}$ is topologically conjugated to the shift.

- The set $\mathcal{X}$ is a hyperbolic set once the 3BP is reduced by its classical first integrals.
- Previous results also in other regions of phase space (but for quite strict mass choices): Bolotin-McKay, Bolotin, Marco-Niederman, Arioli, Wilczak-Zgliczynski, Capinski,...

Main result: Chaotic motions



- q_0 and q_1 evolve at a bounded distance from their center of mass.
- q_2 makes excursions close to a parabolic motion.

- Let N_k be the number of complete revolutions of q_0, q_1 between two consecutives passages of q_2 through Π.
- There exist integers $L \gg 1$ such that for any $\omega \in \mathbb{N}^{\mathbb{Z}}$, there exists a solution of the 3BP such that $N_k = L + \omega_k$.

The proof of the two theorems

- We follow the Moser approach for the Sitnikov Problem
- Its implementation for the 3BP presents several difficulties
- Plan for the rest of the talk:
  - Moser's proof for the Sitnikov model
  - Differences between the Sitnikov model and the planar 3BP
  - Proof for the planar 3BP.

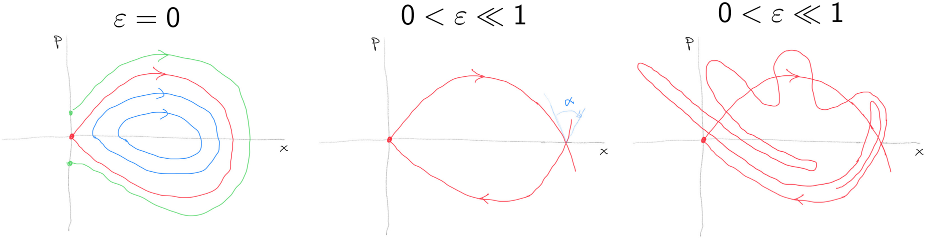
Moser's approach for the Sitnikov problem

- Sitnikov model: $H(p, q, t) = \frac{p^2}{2} - \frac{1}{\sqrt{q^2 + R(t)}}$
- $R(t)$ is the radius of the ellipses: $R(t) = \frac{1}{2} + \frac{\epsilon}{2} \cos t + \mathcal{O}(\epsilon^2)$.
- $P_\pm = (q, p, t) = (\pm\infty, 0, t),\ t \in \mathbb{T}$ are periodic orbits at infinity
- Compactification (McGehee) change of coordinates $q = f(x)$ so that both become $P = (0, 0, t)$.
- Step 1: Check that P has stable and unstable invariant manifolds

  Not obvious! Its linearization is not hyperbolic but parabolic, i.e. degenerate linearization (McGehee 1972).

Moser's approach for the Sitnikov problem

- Take the stroboscopic Poincaré map $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$,

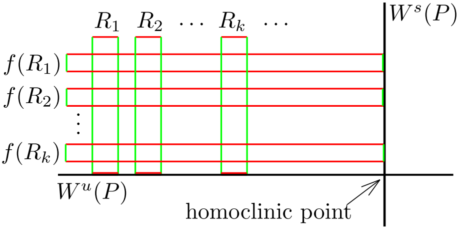


- Step 2: Check that the invariant manifolds split for $0 < \epsilon \ll 1$.
- Melnikov Method: $\alpha(\varepsilon) = M\varepsilon + \mathcal{O}(\varepsilon)$ with $M \neq 0$.
- If P were hyperbolic: Smale Theorem would lead to a Smale horseshoe which contains P on its closure.

Moser approach for the Sitnikov problem

- P is a topological saddle:
  Locally

$$\dot{x} = -(x+y)^3 x(1 + \text{h.o.t})$$
$$\dot{y} = (x+y)^3 y(1 + \text{h.o.t})$$
$$\dot{t} = 1$$



- Step 3: Parabolic Lambda lemma for the local dynamics.
- Step 4: Construct an isolating block (plus cone fields) to build the Smale horseshoe.
- Key point: The model can be reduced to a 2D area preserving map.
- Same happens for the RPC3BP and the Alexeev model.

How can one implement Moser approach for the 3BP?

- After reducing by the classical first integrals: 3 degrees of freedom (dimension 6).

  Analog of infinity in a fixed energy level is: $\mathbb{D} \times \mathbb{T} \subset \mathbb{R}^2 \times \mathbb{T}$.

  2 parameter family of periodic orbits at infinity.

  There are "center" directions: much harder to build hyperbolic sets.

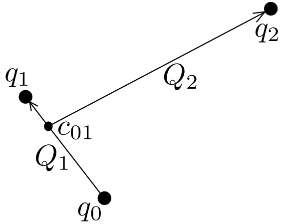
- We want results for all values of the masses $m_0, m_1, m_2 > 0$ $(m_0 \neq m_1)$.

  Transversality of the invariant manifolds of infinity?

  We need a close to integrable regime.

A good system of coordinates

- Jacobi coordinates to reduce conservation of linear momentum:



$$Q_1 = q_1 - q_0$$

$$Q_2 = q_2 - c_{01}, \; c_{01} = \frac{m_0}{m_0 + m_1} q_0 + \frac{m_1}{m_0 + m_1} q_1$$

- Q_1 performs approximate ellipses $\to$ Delaunay coordinates (ℓ, g, L, Γ) (Action-angle coordinates for the 2BP with negative energy)
  - ℓ mean anomaly
  - L square of the semi-major axis
  - g argument of the perihelion
  - Γ angular momentum
- Q_2 performs approximate parabolas $\to$ Polar coordinates (r, θ, y, G)
- y radial momentum and G angular momentum.

A good system of coordinates

- Variables: (ℓ, g, L, Γ) and (r, θ, y, G)
- Invariance by rotation $\Theta = \Gamma + G$ is a first integral
- The Hamiltonian depends on g and θ through $\phi = g - \theta$.
- Reducing by the rotations: 3 d.o.f Hamiltonian depending on the parameter Θ.

$$H = H(\ell, \phi, r, L, \Gamma, y; \Theta)$$

- Delaunay coordinates are not well suited for close to circular motion.
- Poincaré variables: $(\lambda, \alpha, r, L, \beta, y)$ with

$$\lambda = \ell + \phi, \quad \alpha = \sqrt{2(L-\Gamma)}e^{i\phi}, \quad \alpha = \sqrt{2(L-\Gamma)}e^{-i\phi}.$$

- New Hamiltonian $H = H(\lambda, \alpha, r, L, \beta, y; \Theta)$.

Parabolic infinity and its manifolds

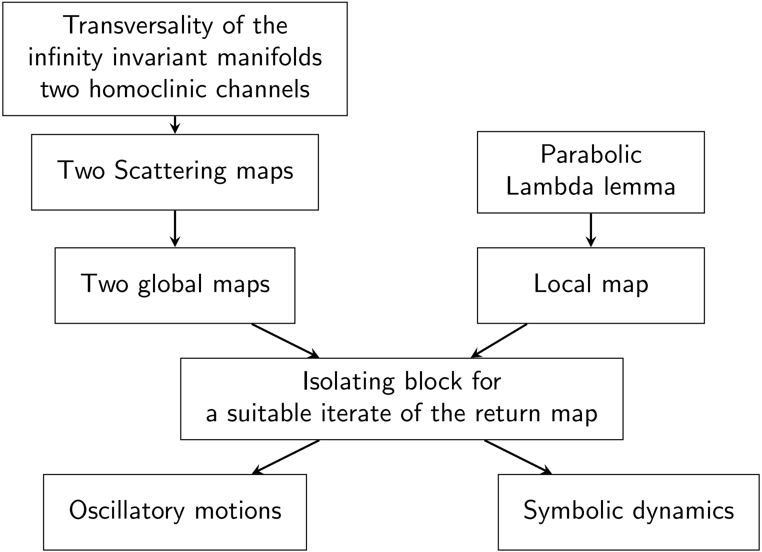
$$H = H(\lambda, \alpha, r, L, \beta, y; \Theta)$$

- Reaching infinity with zero velocity: $r = \infty$ and $y = 0$.
- At $(r, y) = (\infty, 0)$: Hamiltonian is $H = -\frac{1}{2L^2}$
- Fixing energy $\equiv$ Fixing the semimajor axis of the primaries.
- Dynamics at an energy level: $\lambda \in \mathbb{T}$ and $\alpha = \overline{\beta} \in \mathbb{D} \subset \mathbb{C}$ and

$$\dot{\lambda} = \frac{1}{L^3}, \quad \dot{\alpha} = \dot{\beta} = 0.$$

- It is foliated by periodic orbits $\mathbb{T}_{\alpha_0,\beta_0} = \{\alpha = \alpha_0, \beta = \beta_0\}$.
- Baldomá-Fontich-Martín: $\mathbb{T}_{\alpha_0,\beta_0}$ have invariant manifolds.

Moser Approach: Scheme



A different nearly integrable regime

- To prove transversality between the invariant manifolds of infinity we must be in a perturbative regime.
- Take $|Q_2| \gg |Q_1| \iff r \gg L$.
- Equivalently: Fix $L \sim 1$ (energy level) and take $\Theta \gg 1$.
- From the third body, the first two are "very close to each other".
- At first order, the third body only "sees" one body at the center of mass of the other two.
- Conclusion:

$$H = \text{Two uncoupled Kepler problems} + \text{Small perturbation.}$$

- Two time scales: Motion on the far away parabola is much slower than the motion on the ellipses.

Transversality of the invariant manifolds

- Fix energy $H = -1/2$. Parabolic infinity:

$$\Lambda_\infty = \left\{L = 1, r = +\infty, y = 0, \lambda \in \mathbb{T}, \alpha = \overline{\beta} \in \mathbb{D} \subset \mathbb{C}\right\}$$

- The parameter $\Theta \gg 1$ measures the distance between the third body and the other two.

Theorem

Take any $m_0, m_1, m_2 > 0$, $m_0 \neq m_1$. There exist $z_0^0, z_0^1 \in \mathbb{D}$ such that:

- $\mathbb{T}_{z_0^0}$ *and* $\mathbb{T}_{z_0^1}$ *have homoclinic connections.*
- *The invariant manifolds of Λ_∞ intersect transversally along them.*
- *The points $z_0^0 = (\alpha_0^0, \beta_0^0), z_0^1 = (\alpha_0^1, \beta_0^1)$ satisfy*

$$\alpha_0^1 - \alpha_0^0 = -\frac{6i}{\sqrt{\pi}}\Theta^{9/2} e^{-\frac{\Theta^3}{3}} \left(1 + \mathcal{O}\left(\Theta^{-1/2}\right)\right).$$

Transversality between the invariant manifolds

- Hidden in the theorem: The transversality between the invariant manifolds is exponentially small (in some directions)

$$\text{angle} \sim \Theta^{-1/2} e^{-\frac{\Theta^3}{3}}, \qquad \Theta \gg 1$$

(Exponentially small splitting of separatrices)

- Why? Fast rotation coupled with slow motion on the invariant manifolds.
- Classical perturbative methods (Melnikov Theory) cannot be applied to construct these transverse homoclinic orbits.
- Deep analysis of the analytic extension of the parameterizations of the perturbed invariant manifolds in complex domains.

How to construct an isolating block

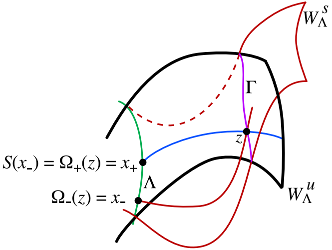
- Take λ-Poincaré map and reduce by the energy: 4 dimensional symplectic map $\mathcal{P}$.
- We need a 4 dimensional isolating block for $\mathcal{P}$.
- In the (r, y)–plane: Proceed as in the Sitnikov problem (Moser construction following the Smale Theorem).
- How do we construct an isolating block for the "center" (α, β) variables?
- Main tool: The scattering map.

The scattering map (after Delshams – de la Llave – Seara)

- Normally hyperbolic (parabolic) invariant manifold Λ
- Its stable and unstable manifolds intersect along a homoclinic channel Γ.
- Scattering map associated to the homoclinic channel Γ

$$S : \Lambda \to \Lambda, \quad x_+ = S(x_-)$$

when
$\emptyset \neq W^s(x_+) \cap W^u(x_-) \in \Gamma$



- Delshams– de la Llave – Seara: Scattering map is symplectic.

Two scattering maps

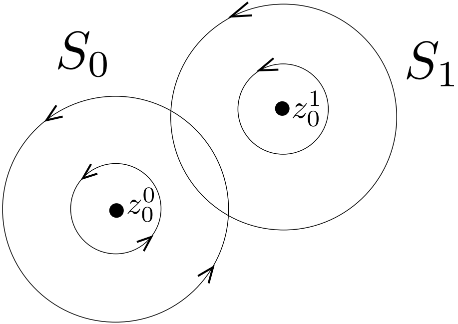
- In $\mathbb{D}$ we can define two scattering maps given by two homoclinic channels

$$\mathcal{S}_i : \mathbb{D}_i \subset \mathbb{D} \to \mathbb{C}.$$

- The homoclinic points z_0^0 and z_0^1 satisfy

$$z_0^i = S_i(z_0^i)$$

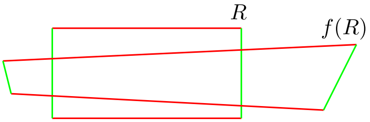
- Close to them, $\mathcal{S}_i$ are close to integrable twist maps.



- They are elliptic fixed points by the scattering maps.

Isolating block for (a suitable iterate of) the scattering map

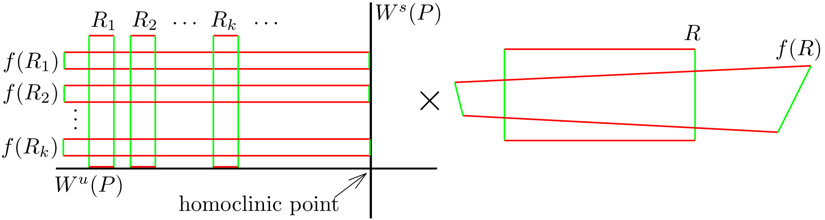
- There exists $M > 0$ such that $S_1^M \circ S_0$ has an isolating block.
- Tools: Birkhoff normal form, KAM Theorem...
- The Scattering maps encodes heteroclinic orbits to infinity.



- We want an isolating block for the "true" Poincaré map!
- Return map from a neighborhood of infinity to itself:
  - (r, y) plane: it behaves as the one for the Sitnikov problem.
  - (α, β) plane: it can be "approximated" by the scattering maps.
- Extra difficulty: One has to control the dynamics close to infinity by a Parabolic Lambda Lemma.

An isolating block for (a suitable iterate of) the return map

- Applying the return map a large number of times one can construct the isolating block using:
  - The techniques by Moser in the (r, y) plane.
  - The isolating block of the scattering map in the (α, β) plane.



Thank you for your attention