Symbolic Dynamics in the Restricted Elliptic Isosceles 3 Body Problem

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joint work with

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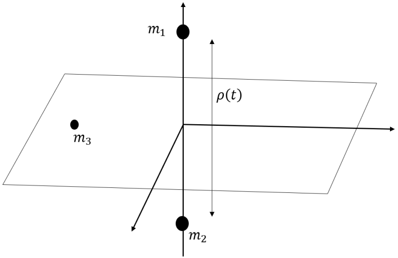
The Restricted Elliptic Isosceles 3-Body Problem (REI3BP)

- Primaries masses $m_1 = m_2$.
- They move along a degenerate ellipse (a segment).
- The massless body ($m_3 = 0$) moves on the plane perpendicular to the primaries orbit.



Two results concerning the movement of the massless particle $q(t)$:

- Existence of chaotic dynamics.
- Classification of the asymptotic ($t \to \pm\infty$) motions.

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Chaotic Dynamics

Theorem (Guardia, P., Seara, Vidal)

There exists a section Σ transverse to the flow of the REI3BP such that the Poincaré map ψ associated to Σ presents chaotic dynamics when restricted to a certain invariant set $S \subset \Sigma$.

Namely ψ possesses a *Smale horseshoe*: the dynamics of $\psi : S \to S$ is topologically conjugated to the shift acting on the space of doubly infinite sequences

$$\sigma : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$$
$$\{a_n\}_{n\in\mathbb{Z}} \mapsto \{a_{n-1}\}_{n\in\mathbb{Z}}.$$

- Topological transitivity.
- Sensitive dependence on initial conditions.
- Dense set of periodic orbits.

Final motions as $t \to \infty$

Classification of the possible final motions in the R3BP (Chazy 1922)

- $H^{\pm}$(hyperbolic):

$$\|q(t)\| \to \infty \text{ and } \|\dot{q}(t)\| \to c > 0 \text{ as } t \to \pm\infty.$$

- $P^{\pm}$(parabolic):

$$\|q(t)\| \to \infty \text{ and } \|\dot{q}(t)\| \to 0 \text{ as } t \to \pm\infty.$$

- $B^{\pm}$(bounded):

$$\limsup_{t\to\pm\infty} \|q(t)\| < \infty.$$

- $OS^{\pm}$(oscillatory):

$$\limsup_{t\to\pm\infty} \|q(t)\| = \infty, \ \liminf_{t\to\pm\infty} \|q(t)\| < \infty.$$

Final motions

Theorem (Guardia, P., Seara, Vidal)

Denote by X^+ (respectively Y^-) either H^+, P^+, B^+ or OS^+ (respectively H^-, P^-, B^- or OS^-). Then, for the REI3BP we have

$$X^+ \cap Y^- \neq \emptyset$$

for all possible combinations of X^+ and Y^-.

- The existence of all types except oscillatory motions in the R3BP was already known by Chazy.
- The existence of oscillatory motions is a non trivial question for each model in Celestial Mechanics.

Final motions

Examples of oscillatory motions:

- Sitnikov provides the first one in what is now called the *Sitnikov example*.
- Moser uses the existence of a *Smale horseshoe* close to the transversal intersection of certain invariant manifolds to construct oscillatory orbits in the Sitnikov example.
- Llibre and Simó show their existence in the RPC3BP for $\mu \ll 1$ using Moser's approach.
- Guardia, Martín and Seara extend the previous result to $\mu \in (0, 1/2]$.

Measure of the set of oscillatory motions:

- Conjectured by Alekseev (or Kolmogorov) to have Lebesgue meausure zero.
- Gorodetski and Kaloshin study their Hausdorff dimension in the RPC3BP and Sitnikov's example.

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The Hamiltonian setting

In polar coordinates (r, y, α, G) the system is Hamiltonian with respect to

$$H(r, y, t; G) = \frac{y^2}{2} + \frac{G^2}{2r^2} - \frac{1}{\sqrt{r^2 + \frac{\rho^2(t)}{4}}} \qquad (r, y, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}.$$

where $\rho(t)$ is a 2π-periodic function which gives the distance between the primaries.

- The system is invariant under rotation of the angle α so G is a conserved quantity.
- The original $2 + \frac{1}{2}$ d.o.f. system is reduced to a $1 + \frac{1}{2}$ d.o.f. where G is a parameter.
- We autonomize the system

$$\hat{H}(r, y, t, E) = H(r, y, t) - E$$

where $E \in \mathbb{R}$ is the conjugate momentum to t.

Nearly Integrable setting

- We perform the scaling

$$r = G^2 \tilde{r} \qquad y = G^{-1} \tilde{y} \qquad E = G \tilde{E},$$

- New Hamiltonian

$$\tilde{H}\left(\tilde{r}, \tilde{y}, t, \tilde{E}; G\right) = \frac{\tilde{y}^2}{2} + \frac{1}{2\tilde{r}^2} - \frac{1}{\sqrt{\tilde{r}^2 - \frac{\rho^2(t)}{4G^4}}} - G^3 \tilde{E}, \qquad \dot{t} = G^3.$$

Perturbative setting: For large G the REI3BP is a *small and rapidly oscillating perturbation* of the two body problem. We can write

$$\tilde{H}\left(\tilde{r}, \tilde{y}, t, \tilde{E}; G\right) = H_{2bp}\left(\tilde{r}, \tilde{y}\right) - G^3 \tilde{E} + U_{\mathrm{pert}}(\tilde{r}, t; G)$$

where

$$U_{\mathrm{pert}}(\tilde{r}, t; G) = \frac{1}{\tilde{r}} - \frac{1}{\sqrt{\tilde{r}^2 - \frac{\rho^2(t)}{4G^4}}} = \mathcal{O}(G^{-4})$$

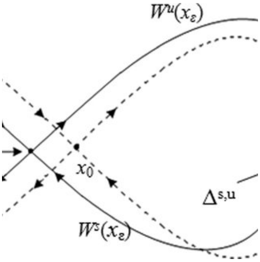
The system possesses a parabolic periodic orbit (infinity)

$$\Lambda = \left\{ \left(\tilde{r}, \tilde{y}, t, \tilde{E} \right) = (\infty, 0, t, 0) : t \in \mathbb{T} \right\}$$

with associated invariant manifolds $\mathcal{W}^{s,u}$.

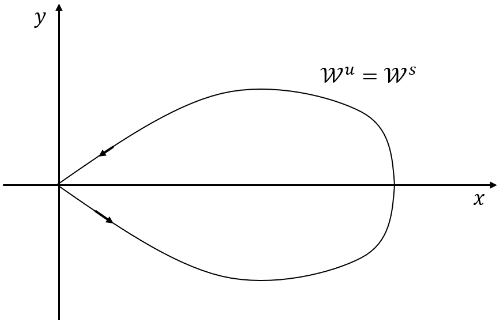
- In order to study neighbourhoods of Λ one usually introduces the McGehee transformation $\tilde{r} = 2/x^2$.
- In the McGehee coordinates

$$\Lambda = \left\{ \left(x, \tilde{y}, t, \tilde{E} \right) = (0, 0, t, 0) : t \in \mathbb{T} \right\}$$



Manifolds of Infinity

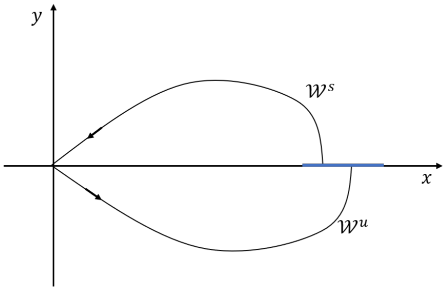
- For $G \to \infty$ the system is integrable.
- Dimension counting argument: $\mathcal{W}^{s,u}$ coincide along a 2 dimensional homoclinic manifold which is foliated by Keplerian parabolic orbits.



Nearly Integrable setting and Manifolds of Infinity

For $1 \ll G < \infty$:

- We prove that this homoclinic manifold breaks down but $\mathcal{W}^{s,u}$ intersect transversally along two homoclinic orbits to Λ.
- The invariant orbit Λ is degenerate (parabolic) so the classical stable manifold theorem does not hold.
- The splitting between $\mathcal{W}^{s,u}$ is exponentially small so Melnikov theory does not apply.



Existence of transversal intersections between $\mathcal{W}^s$ and $\mathcal{W}^u$

Fix the section $\{y = 0\}$. Then, $\mathcal{W}^{u,s} \cap \{y = 0\}$ become invariant curves $\gamma^{u,s}(t)$.

Proposition

There exists $G^* < \infty$ such that for $G \geq G^* \gg 1$ the distance between $\gamma^{u,s}$ is given by

$$d(t) = J_1(1)\sqrt{2\pi}G^{-\frac{5}{2}}e^{-\frac{G^3}{3}}\left(\sin(t+\varphi) + \mathcal{O}(G^{-\frac{1}{2}})\right)$$

with $J_1(1) \neq 0$ and a certain phase shift $\varphi \in \mathbb{R}$ which is explicit.

Proposition

There exists $G^* < \infty$ such that for $G \geq G^* \gg 1$ the invariant curves $\gamma^{u,s}$ intersect transversally.

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From transversal intersections to oscillatory orbits

Define the section

$$\Sigma = \left\{(x, \tilde{y}, t, \tilde{E}) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} \times \mathbb{R} \colon \tilde{y} = 0,\ \dot{\tilde{y}} > 0,\ \tilde{H} = 0\right\}$$

and the Poincaré map (we use coordinates (x, t) on Σ)

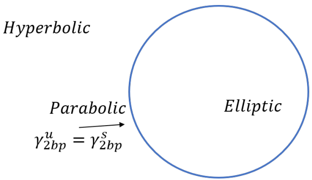
$$\begin{aligned} \psi : \Sigma &\to \Sigma \\ (x_0, t_0) &\mapsto (x_1, t_1). \end{aligned}$$

and set $t_1 = \infty$ for points which do not intersect Σ anymore. Iterating ψ, to a given point (x_0, t_0) we associate the sequence $\{a_n\}_{n\in\mathbb{Z}}$ with

$$a_n = \left[\frac{t_n - t_{n-1}}{2\pi}\right] \qquad a_n \in \mathbb{N}$$

whenever is defined and $a_n = \infty$ otherwise.

From transversal intersections to oscillatory orbits



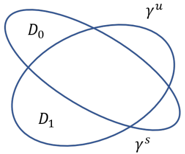
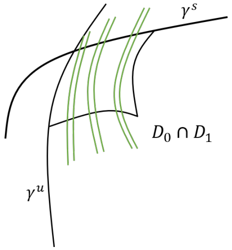


Figure: We write $z = x(\cos t, \sin t)$. On the left, the 2BP case. On the right the REI3BP.

- $D_0 = \mathrm{Dom}(\psi) \subset \Sigma$. We have that $\gamma^s = \partial D_0$.
- $D_1 = \mathrm{Range}(\psi) \subset \Sigma$. We have that $\gamma^u = \partial D_1$.
- The curves $\gamma^{u,s}$ **intersect transversally**$\rightarrow$ Smale horseshoe.



From transversal intersections to oscillatory orbits

Theorem

There exists a set $S \subset \psi^n(D_0 \cap D_1)$ $\forall n \in \mathbb{Z}$ *invariant under* ψ *and such that* $\psi|_S$ *is topologically conjugated to the shift*

$$\sigma\left(\{a_n\}_{n\in\mathbb{Z}}\right) = \{a_{n-1}\}_{n\in\mathbb{Z}}$$

acting on the space A *of doubly infinite sequences* $a = (\dots, a_{-1}, a_0, a_1, \dots)$.

- $a(p) = (\dots, a_{-1}, a_0, a_1, \dots)$ with $a_n \in \mathbb{N}$ $\forall n \in \mathbb{Z}$. Correspond to bounded ($\sup_{n\in\mathbb{Z}} a_n < \infty$) and oscillatory ($\sup_{n\in\mathbb{Z}} a_n = \infty$) orbits.

- Combinations with parabolic and hyperbolic motions are obtained working on the compactification $\mathcal{A} = \overline{A}$.

Thanks for your attention!