Exponentially small splitting of separatrices for L_3 in the Restricted Planar Circular 3-Body Problem

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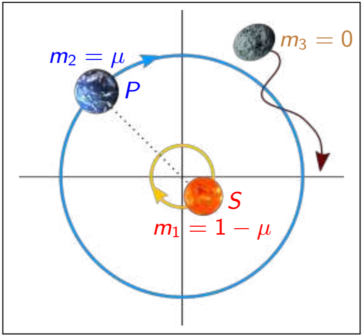
Restricted Planar Circular 3-Body Problem



- **Restricted**: one body is massless, $m_3 = 0$.
- **Planar**: the bodies move on the same plane.
- **Circular**: the primaries (m_1 and m_2) perform a circular motion.

We study the motion of the **massless body**:

$$(q(t), p(t)) \in \mathbb{R}^4.$$

We normalize:

- $m_1 = 1 - \mu$ and $m_2 = \mu$ with $\mu \in \left(0, \frac{1}{2}\right]$.

Approach: Perturbative study for $0 < \mu \ll 1$.

- Unperturbed system ($\mu = 0$) corresponds to a 2-Body Problem.

Rotating framework and Lagrange points

Rotating framework: the primaries are fixed at $(\mu, 0)$ and $(\mu - 1, 0)$.

- The system is autonomous.
- h_0 is integrable (2-BP).
- μh_1 is a perturbation away from collision.

Hamiltonian: $h = h_0 + \mu h_1$

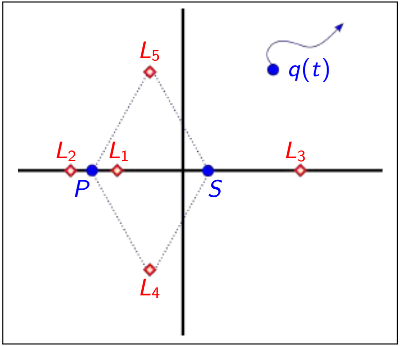
$$h_0(q,p) = \frac{\|p\|^2}{2} - q^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{1}{\|q\|},$$

$$\mu h_1(q;\mu) = \frac{1}{\|q\|} - \frac{(1-\mu)}{\|q-(\mu,0)\|} - \frac{\mu}{\|q-(\mu-1,0)\|}.$$

Five critical points: Lagrange points.

For $\mu > 0$ small:

- L_1, L_2 and L_3 saddle-center.
- L_4 and L_5 center-center.



We start studying the dynamics surrounding L_3 and its invariant manifolds

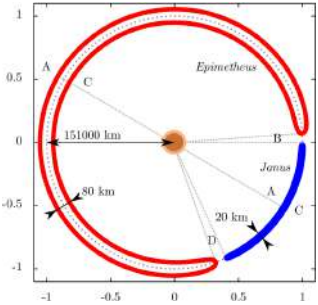
Dynamics on a neighborhood of L_3 and its invariant manifolds

- **Horseshoe-shaped orbits**: quasi-periodic orbits encompassing L_3, L_4 and L_5.

- The interest of these orbits arises when modeling the motion of **co-orbital satellites** (Janus and Epimetheus, for example).

  J. Cors, J. Palacián, P. Yanguas. 2019.

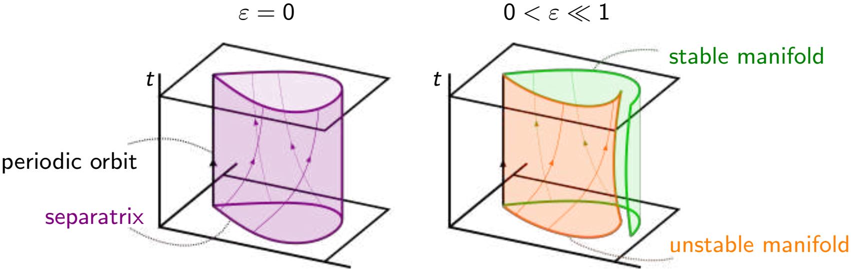
  L. Niederman, A. Pousse and P. Robutel. 2020.



- The center stable and center unstable manifolds of L_3 act as boundaries of **regions of effective stability** around L_4 and L_5.

  C. Simó, P. Sousa-Silva and M. Terra. 2013.

Splitting of separatrices - Classical approach

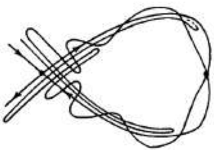


The Smale-Birkhoff homoclinic theorem

Transversal intersections between the unstable and stable manifolds $\Longrightarrow$ chaotic motions.

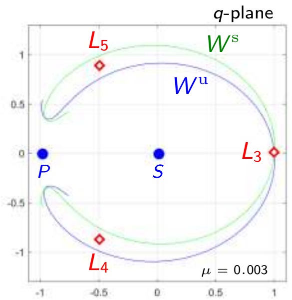
The Poincaré-Melnikov method

$$\mathrm{dist} = \varepsilon M + \mathcal{O}(\varepsilon^2).$$



The invariant manifolds of L_3

- L_3 has **one dimensional stable and unstable** manifolds W^{s} and W^{u}.
- We focus on the upper branches. The lower branches are symmetric.
- The stable and unstable manifolds **either coincide or do not intersect**.
- The breakdown of W^{u} and W^{s} does not lead to chaos.



Goal:

For $0 < \mu \ll 1$, to **measure the distance** between the stable and unstable manifolds of L_3 .

- To prove the existence of chaotic dynamics around L_3 (Future work):
  - Study Lyapunov periodic orbits close to L_3.
  - Look for transversal intersections of their 2-dimensional invariant manifolds.

Exponentially small splitting of separatrices

L_3 is a critical point of saddle-center type. For $0 < \mu \ll 1$ it has eigenvalues:

$$\pm\sqrt{\mu\frac{21}{8}}\,(1+\mathcal{O}(\mu))\,, \qquad \pm i(1+\mathcal{O}(\mu)).$$

Eigenvalues at different time-scales: **singular perturbation** problem.

- Following a Poincaré-Melnikov method approach is not possible:

$$\mathrm{dist} = \mu M + \mathcal{O}(\mu^2), \qquad \text{with} \qquad M = \mathcal{O}(e^{-\frac{1}{\sqrt{\mu}}}).$$

Exponentially small phenomenon:

Due to the **rapidly rotating dynamics**, the invariant manifolds are **exponentially close** to each other with respect to $\sqrt{\mu}$.

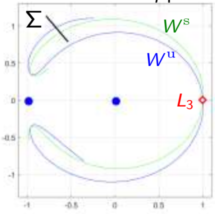
- **Beyond all orders phenomenon**: The difference cannot be detected by expanding the invariant manifolds into a series $\sqrt{\mu}$.

Main Result

Main Theorem:

Take the section Σ (see Figure) and let $(q^{\mathrm{u,s}}, p^{\mathrm{u,s}})$ be the first intersection of $W^{\mathrm{u,s}}$ with Σ. For $0 < \mu \ll 1$,

$$\|q^{\mathrm{u}} - q^{\mathrm{s}}\| + \|p^{\mathrm{u}} - p^{\mathrm{s}}\| = \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left(\Theta + \mathcal{O}\left(\frac{1}{|\log \mu|} \right) \right)$$



- The **constant A** is given by

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}\, dx \approx 0.177744.$$

Computed by J. Font (1984), C. Simó, P. Sousa-Silva and M. Terra (2013).

- The **constant Θ** corresponds to a Stokes constant. It does not have a closed formula, but it can be computed numerically by the inner equation.
- We have seen $\boldsymbol{\Theta \approx 1.63}$. One could implement a computer assisted proof to obtain rigorous estimates.

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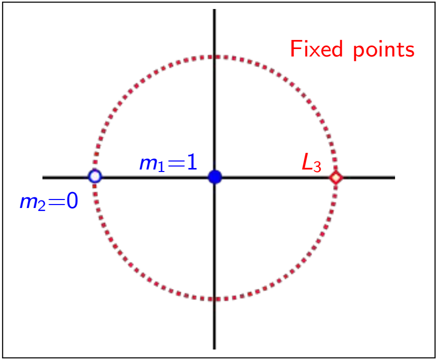
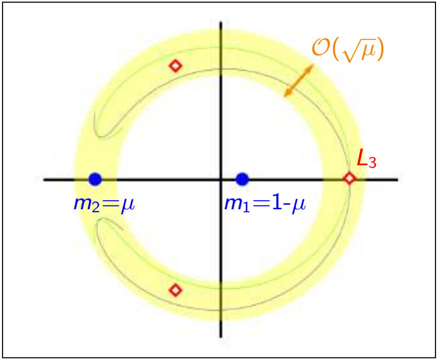
Unperturbed system

$\mu = 0$ $\qquad\qquad$ $\mu > 0$





- The eigenvalues of L_3 are $\pm\sqrt{\frac{21}{8}\mu}(1+\mathcal{O}(\mu))$ and $\pm i(1+\mathcal{O}(\mu))$.

Goal:

Apply a singular change of coordinates to obtain a **new first order** of the system **with a saddle and separatrices**.

New "unperturbed system"

Hamiltonian in "good" coordinates and scaled time:

$$H = H_{\rm p} + H_{\rm osc} + \mathcal{O}(\mu^{\frac{1}{4}}),$$

$$H_{\rm p}(\lambda,\Lambda) = -\frac{3}{2}\Lambda^2 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}}, \qquad H_{\rm osc}(x,y;\mu) = \frac{xy}{\mu^{\frac{1}{2}}},$$

with symplectic form $d\lambda \wedge d\Lambda + i\, dx \wedge dy$.

- The linearization of the vector field at L_3 is

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ -\frac{7}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\mu^{\frac{1}{2}}} & 0 \\ 0 & 0 & 0 & -\frac{i}{\mu^{\frac{1}{2}}} \end{pmatrix} + \mathcal{O}(\mu^{\frac{1}{4}}).$$

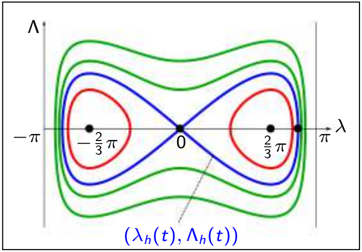
New "unperturbed system":

A pendulum-like Hamiltonian $H_{\rm p}$ with two homoclinic connections plus a fast oscillator $H_{\rm osc}$.

Problem:

Terms $\mathcal{O}(\mu^{\frac{1}{4}})$ are not explicit.

Hamiltonian $H_{\mathrm{p}}(\lambda, \Lambda)$



- The Hamiltonian

$$H_{\mathrm{p}}(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 + V(\lambda),$$

with

$$V(\lambda) = -\cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}}$$

has two homoclinic connections or **separatrices**.

- Parametrization of the right separatrix:

$$(\lambda_h(t), \Lambda_h(t)), \quad t \in \mathbb{R}.$$

- There is no explicit expression for (λ_h, Λ_h).

We expect the perturbed invariant manifolds to be close to $(\lambda_h, \Lambda_h, 0, 0)$.

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Heuristic idea of the proof

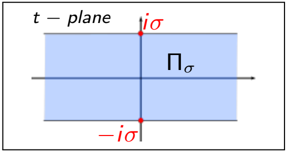
Recall the Hamiltonian:

$$H = -\frac{3}{2}\Lambda^2 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}} + \frac{xy}{\sqrt{\mu}} + \mathcal{O}(\mu^{\frac{1}{4}}).$$

Let $(\Delta x, \Delta y) = (x^{\mathrm{u}} - x^{\mathrm{s}}, y^{\mathrm{u}} - y^{\mathrm{s}})$ the difference between the stable and the unstable manifolds. Then

$$\dot{\Delta x} \approx \frac{i}{\sqrt{\mu}}\Delta x \quad \Longrightarrow \quad \Delta x(t) \approx C_x e^{\frac{i}{\sqrt{\mu}}t},$$

$$\dot{\Delta y} \approx -\frac{i}{\sqrt{\mu}}\Delta y \quad \Longrightarrow \quad \Delta y(t) \approx C_y e^{-\frac{i}{\sqrt{\mu}}t}.$$



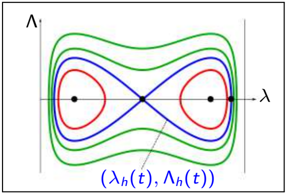
Proving that $(\Delta x, \Delta y)$ is analytic in Π_σ:

$$|C_x| \lesssim e^{-\frac{\sigma}{\sqrt{\mu}}} |\Delta x(-i\sigma)|,$$

$$|C_y| \lesssim e^{-\frac{\sigma}{\sqrt{\mu}}} |\Delta y(i\sigma)|.$$

$(\Delta x, \Delta y)$ bounded in a complex domain $\Longrightarrow$ exponentially small bound in $\mathbb{R}$.

Analytical continuation of the separatrix

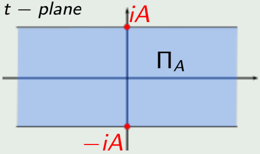


- Parametrization of the right separatrix:

$$(\lambda_h(t), \Lambda_h(t)), \quad t \in \mathbb{R},$$

- **Goal**: analytically extend (λ_h, Λ_h) to the complex plane.

Theorem A:



$(A \approx 0.177744)$

Let:

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}} dx.$$

- (λ_h, Λ_h) analytically extends to the complex strip

$$\Pi_A = \{t \in \mathbb{C} \ : \ |\mathrm{Im}\, t| < A\},$$

- (λ_h, Λ_h) has two singularities $t = \pm iA$ at $\partial \Pi_A$.

Strategy of the proof

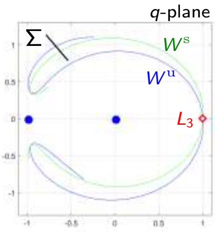
- We **analytical extend** parametrizations of the stable and unstable manifolds to Π_σ with $\sigma := A - \sqrt{\mu}$.
- To capture the first order of the difference we give its dominant terms close to $\pm i\sigma$.
- The **inner equation** is given by the first order of the Hamiltonian close to $\pm i\sigma$ and it is independent of μ.
- By using **matching techniques**, we prove that, close to $\pm i\sigma$, the solutions of the inner equation are "good" approximations" of the manifolds.
- We obtain an **asymptotic formula** for the difference by proving that the dominant term comes from the solutions of the inner equation.

Final comments

Main Theorem:

Take the section Σ (see Figure) and let $(q^{\mathrm{u,s}}, p^{\mathrm{u,s}})$ be the first intersection of $W^{\mathrm{u,s}}$ with Σ. For $0 < \mu \ll 1$,

$$\|q^{\mathrm{u}} - q^{\mathrm{s}}\| + \|p^{\mathrm{u}} - p^{\mathrm{s}}\| = \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left(\Theta + \mathcal{O}\left(\frac{1}{|\log \mu|} \right) \right)$$



- The Hamiltonian we deal with has **not an explicit formula**.
- The constant $A \approx 0.177744$ has an explicit formula.
- The constant Θ is given by the **inner equation**. Numerically we have seen $\Theta \approx 1.63$. One could implement a computer assisted proof to obtain rigorous estimates for Θ.

Thanks for your attention!