Breakdown of small amplitude breathers for the nonlinear Klein-Gordon equation

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Breathers and the Klein-Gordon equation

- Klein-Gordon equation

$$u_{tt} = u_{xx} - u + f(u), \qquad f(u) = \mathcal{O}(u^2), \qquad x \in \mathbb{R}$$

- Breathers: Periodic in time spatially localized solutions.
- Breathers for the Sine-Gordon equation $u_{tt} = u_{xx} - \sin u$:

$$u(x,t) = 4 \arctan\left(\frac{m}{\omega}\frac{\sin(\omega t)}{\cosh(mx)}\right), \quad m, \omega > 0, \ m^2 + \omega^2 = 1.$$

- What about other nonlinearities f?
- Families of breathers should be unlikely to happen.

Spatial dynamics: Breathers as homoclinic orbits

$$u_{tt} = u_{xx} - u + f(u), \qquad f(u) = \mathcal{O}(u^2)$$

- Dynamical system with x as time: phase space is space of $2\pi/\omega$-periodic functions in t for some $\omega > 0$.
- Breathers $\equiv$ Homoclinic orbits to the steady state $u = 0$.
- $u = 0$ has finite dimensional stable and unstable eigenspaces: the stable/unstable invariant manifolds unlikely to intersect.
- But this is hard to prove in general...
- Breathers do exist for Hamiltonian systems on lattices (McKay, Aubry,...).

Non-existence of breathers for the Klein Gordon eq.

Global results:

- Kowalczyk, Martel and Muñoz (2016): Nonexistence of odd (in x) breathers for any odd f.
- The breathers of the Sine-Gordon equation are even in x.

Perturbative results:

- Birnir–McKean–Weinstein and Denzler (1990's): Perturbed Sine-Gordon equation

$$u_{tt} = u_{xx} - \sin u + \epsilon\Delta(u), \quad \epsilon \ll 1, \quad \Delta \ \text{analytic}$$

- Persistence of the family of breathers implies $\Delta(u)$ is a trivial perturbation.

Small amplitude breathers for the odd Klein Gordon eq.

- What about (families of) small amplitude breathers?
- Equivalent to small homoclinic loops to $u = 0$.
- Simplest setting: Odd Klein-Gordon equation

$$\partial_t^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0, \quad f(u) = \mathcal{O}(u^5), \text{ odd}$$

- Soffer–Weinstein, 1999 (also Bambusi–Cuccagna, 2011): Non-existence of breathers if one adds a potential (under some hypotheses).

Kruskal and Segur

- Kruskal–Segur (1987): Formal arguments for the ϕ^4 model to indicate the breakdown of breathers with

$$\text{frequency } \omega : 0 < 1 - \omega \ll 1 \quad \text{and} \quad \text{amplitude } \sim \sqrt{1-\omega^2}.$$

- Questions:

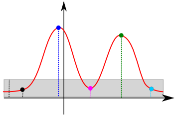
  1. How to make rigorous the formal arguments to prove the breakdown of breathers (and extend the proof to all possible ω's).
  2. Do small amplitude breathers with exponentially small (with respect to the amplitude) tails exist? $\leftarrow$ Generalized breathers

Small amplitude breathers for the odd Klein Gordon eq.

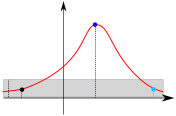
- Goal: For "typical" f, small amplitude breathers do not exist.
- But we need to impose certain restrictions...
- Let $\sigma \in (0,1)$ and $\omega > 0$. A $\frac{2\pi}{\omega}$-periodic-in-t function $u(x,t)$ is σ-multi-bump in x if there exist $x_1 < x_2 < x_3 < x_4 < x_5$ such that

$$\|u(x_j,\cdot)\|_{H^1_t} \leq \sigma \|u(x_i,\cdot)\|_{H^1_t}, \quad \forall j \in \{1,3,5\},\ i \in \{2,4\}.$$

Otherwise, it is said to be σ-single-bump.



Multi-bump



Single-bump

Main result: Non-existence of breathers

$$\partial_t^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0, \quad f(u) = \mathcal{O}(u^5), \text{ odd}$$

Theorem (G.-Gomide-Seara-Zeng)

There exists $\Theta \in \mathbb{C}$, depending analytically on f, such that if $\Theta \neq 0$: For any $\sigma \in (0,1)$, there exists $\rho^* > 0$ such that there does not exist any solution $u(x,t)$ which:

1. *is $\frac{2\pi}{\omega}$-periodic in t for some $\omega > 0$,*
2. *satisfies*

$$\|u(x,\cdot)\|_{H_t^1\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} + \|\partial_x u(x,\cdot)\|_{L_t^2\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} \to 0, \quad \text{as} \quad |x| \to +\infty,$$

3. *satisfies $\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{H_t^1\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} < \rho^* \min\{1, \omega^{\frac{1}{2}}\}$,*
4. *is σ-single-bump.*

Some remarks

- Θ depends analytically on f $\to$ For "typical" f, $\Theta_f \neq 0$.
- So, for typical f, small amplitude breathers do not exist provided:
  - We restrict to single-bump breathers,
  - We admit the smallness to depend on ω:

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{H^1_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} < \rho^* \min\{1,\omega^{\frac{1}{2}}\}.$$

- Typically, multi-bump breathers should not exist either.
- One should be able to rule out breathers such that

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{H^1_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} < \rho^*.$$

Generalized breathers

- Nan Lu (2014): There exist breathers with exponentially small tails for some periods.
- Fix the frequency $\omega = \sqrt{1-\epsilon^2}$ with $0 < \epsilon \ll 1$.
- There exist solutions u such that are $2\pi/\sqrt{1-\epsilon^2}$ – periodic and

$$\frac{\epsilon}{2} \leq \sup \|u(x,\cdot)\|_{H^1_t} \leq 2\epsilon \text{ and } \limsup_{x\to\pm\infty} \|u(x,\cdot)\|_{H^1_t} \lesssim e^{-c/\epsilon}, \quad c > 0.$$

- Groves and Schneider (2000's): "modulated pulse" solutions with small (beyond all orders) tails for the nonlinear Klein Gordon equations (and quasilinear wave equations).

Main results: Generalized breathers

Theorem (G.-Gomide-Seara-Zeng)

Fix the frequency $\omega = \sqrt{1-\epsilon^2}$ *with* $0 < \epsilon \ll 1$.

- *There exist* $2\pi/\omega$ *-periodic-in-*t *solutions* u *such that*

$$\frac{\epsilon}{2} \leq \sup \|u(x,\cdot)\|_{H^1_t} \leq 2\epsilon \qquad \text{and}$$

$$\lim \sup_{|x|\to\infty} \left(\|u(x,\cdot)\|_{H^1_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} + \|\partial_x u(x,\cdot)\|_{L^2_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} \right) \leq M e^{-\frac{\sqrt{2}\pi}{\epsilon}}.$$

- *If* $\Theta \neq 0$, *they also satisfy*

$$\lim \inf_{|x|\to\infty} \left(\|u(x,\cdot)\|_{H^1_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} + \|\partial_x u(x,\cdot)\|_{L^2_t\left(-\frac{\pi}{\omega},\frac{\pi}{\omega}\right)} \right) \geq M^{-1} e^{-\frac{\sqrt{2}\pi}{\epsilon}}.$$

Some ideas about the proof

- The proofs of all the results rely on spatial dynamics tecnhiques.
- We need to analyze the stable/unstable invariant manifolds associated to the steady state $u = 0$.
- For the generalized breathers: center-stable and center-unstable invariant manifolds.
- In this talk, we focus on the proof of the breakdown of breathers.
- We need to deal with exponentially small phenomena.

Breather breakdown from spatial dynamics point of view

$$\partial_{tt} u - \partial_{xx} u + u - \frac{1}{3}u^3 - f(u) = 0, \quad f(u) = \mathcal{O}(u^5), \text{ odd}$$

- Fix periodicity in $t = 2\pi/\omega$.
- We rule out the existence of single-bump small homoclinic loops in $H^1_t\left(-\frac{\pi}{\omega}, \frac{\pi}{\omega}\right)$.
- Eigenvalues of the linearized equation at $u = 0$:

$$\lambda_n^\pm = \pm\sqrt{1 - n^2\omega^2}, \quad n \in \mathbb{Z}.$$

- The number of hyperbolic eigenvalues is always finite and increases when $\omega \to 0$.

Breather breakdown from spatial dynamics point of view

- Eigenvalues: $\lambda_n^\pm = \pm\sqrt{1 - n^2\omega^2}$.
- Bifurcations: At $\omega = \frac{1}{k}$, $k \in \mathbb{N}$, a new pair of (weakly) hyperbolic eigenvalues appears.
- Two settings:
  - Close to bifurcation: $0 < \frac{1}{k} - \omega \ll 1$, $k \in \mathbb{N}$.
  - Far from bifurcation: Otherwise (including $\omega = 1/k$, $k \in \mathbb{N}$).

Far from bifurcation

- Far from bifurcation: All hyperbolic eigenvalues are "strong".
- All orbits in the stable/unstable invariant manifolds of $u = 0$ escape "far away" from $u = 0$.
- If homoclinic loops exist, they must be large.
- Small homoclinic loops may only appear when ω is close to bifurcation.

Close from bifurcation

- Kruskal-Segur setting: Close to the first bifurcation i.e. periodicity in $t = 2\pi/\omega$ with

$$0 < 1 - \omega \ll 1.$$

- Key setting: Close to the first bifurcation in the odd in t setting:

$$u(x,t) = \sum_{n\geq 1} u_n(x) \sin(n\omega t).$$

- The other cases can be proven using the results for this setting.
- For the first bifurcation: Take

$$\omega = \sqrt{1-\epsilon^2} \qquad \text{with} \qquad 0 < \epsilon \ll 1.$$

First bifurcation in the odd in t setting

$$\partial_{tt}u - \partial_{xx}u + u - \frac{1}{3}u^3 - f(u) = 0, \quad f(u) = \mathcal{O}(u^5), \text{ odd}$$

- Eigenvalues: $\lambda_1^\pm = \pm\epsilon$ and $\lambda_n^\pm = \pm i\sqrt{n^2(1-\epsilon^2)-1}$, $n \geq 2$.
- Spatial dynamics (x as evolution variable): one dimensional (weak) stable and unstable invariant manifolds.
- Weakness $\lambda_1^\pm \to$ The invariant manifolds have "size" $\mathcal{O}(\epsilon)$.
- Scaling: $u = \epsilon v$, $y = \epsilon x$ and $\tau = \omega t$:

$$\partial_y^2 v - \frac{\omega^2}{\epsilon^2}\partial_\tau^2 v - \frac{1}{\epsilon^2}v + \frac{1}{3}v^3 + \frac{1}{\epsilon^3}f(\epsilon v) = 0,$$

Equation for the Fourier coefficients (odd setting)

- Equation for the Fourier coefficients $v(y,\tau) = \sum_{n\geq 1} v_n(y)\sin(n\tau)$:

$$\begin{cases} \ddot{v}_1 = v_1 - \Pi_1\left[\frac{v^3}{3} + \mathcal{O}(\epsilon^2)\right], \\ \ddot{v}_n = -\frac{\lambda_n^2}{\epsilon^2} v_n - \Pi_n\left[\frac{v^3}{3} + \mathcal{O}(\epsilon^2)\right], \ n \geq 2, \end{cases}$$

with $\cdot = d/dy$ and $\lambda_n = \sqrt{n^2(1-\epsilon^2)-1}$.

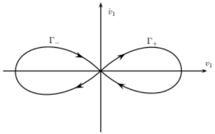
- Equivalently, for v_n:

$$\epsilon^2 \ddot{v}_n = -\lambda_n^2 v_n - \Pi_n\left[\epsilon^2 \frac{v^3}{3} + \mathcal{O}(\epsilon^4)\right], \ n \geq 2.$$

- Singular limit $\epsilon \to 0$: the plane $\mathcal{M} = \{v_n = \dot{v}_n = 0, \ n \geq 2\}$ is the slow manifold with dynamics $\ddot{v}_1 = v_1 - \frac{v_1^3}{4}$ (Duffing equation).

Homoclinic breakdown

- The Duffing equation $\ddot{v}_1 = v_1 - \frac{v_1^3}{4}$ has two homoclinic loops.



$$v_1 = \pm \frac{2\sqrt{2}}{\cosh(y)}$$

- Do these homoclinic loops persist for the full problem?
- Singular perturbation problem:

$$\begin{array}{r} \text{Fast rotation versus} \\ \text{weak hyperbolicity} \end{array} \longrightarrow \text{Exponentially small phenomena.}$$

- How to measure the distance between the perturbed invariant manifolds?
- Classical perturbative methods (Melnikov Theory) cannot be applied.

Formal series expansions

- Look for parameterizations of $W^{\mathrm{st}}(0)$ and $W^{\mathrm{uns}}(0)$ (one dimensional)

$$v_n^{\mathrm{uns}}(y,\epsilon),\ \ v_n^{\mathrm{st}}(y,\epsilon),\ \ n\geq 1.$$

- Look for formal solutions as a power series of ϵ:

$$v_n^{\alpha}(y,\epsilon)=v_{n,0}(y)+\epsilon v_{n,1}^{\alpha}(y)+\epsilon^2 v_{n,2}^{\alpha}(y)+\dots\quad \text{for } \alpha=\mathrm{uns},\mathrm{st}$$

- One can check:

$$v_{n,m}^{\mathrm{uns}}(y)=v_{n,m}^{\mathrm{st}}(y)\ \ \forall n,m\in\mathbb{N}$$

- Thus: their difference is beyond all orders:

$$v_n^{\mathrm{uns}}(y,\epsilon)-v_n^{\mathrm{st}}(y,\epsilon)=\mathcal{O}(\epsilon^m)\ \ \forall m\in\mathbb{N}.$$

- Typically: the power series in ϵ are divergent and the difference between manifolds is flat with respect to ϵ.

Main result

- Take a section transversal to the solutions

$$\Sigma = \{(v, \partial_y v);\ \mathcal{H}(v, \partial_y v) = 0 \quad \text{and} \quad \Pi_1 [\partial_y v] = 0\}$$

Theorem (G.-Gomide-Seara-Zeng)

Call $P^{uns,st}$ the first intersection points of $W^{uns,st}$ with Σ. Then, there exists a constant $\Theta \in \mathbb{R}$ such that, for $\epsilon \ll 1$, the distance $d(\epsilon) = P^{uns} - P^{st}$ satisfies

$$\Pi_3[d(\epsilon)] = \frac{2}{\epsilon} e^{-\frac{\pi\sqrt{2}}{\epsilon}} \left(\Theta + \mathcal{O}\left(\frac{1}{\log(\epsilon)}\right)\right)$$

and

$$\Pi_n[d(\epsilon)] = \frac{2}{\epsilon} e^{-\frac{\pi\sqrt{2}}{\epsilon}} \mathcal{O}\left(\frac{1}{\log(\epsilon)}\right) \qquad n > 3.$$

Implication on breathers (if $\Theta \neq 0$)

- The constant Θ is the one appearing in the main theorems.
- If $\Theta \neq 0$, then the invariant manifolds $W^{\text{uns}}(0)$ and $W^{\text{st}}(0)$ do not intersect the first time they reach Σ.
- It rules out the existence of single-bump homoclinic loops.
- Even if $\Theta \neq 0$, $W^{\text{uns}}(0)$, $W^{\text{st}}(0)$ may still coincide after more rounds.
- It implies non-existence of single–bump breathers with period $2\pi/\sqrt{1-\epsilon^2}$ but not of multi–bump breathers.

Some ideas about the proof

- Exponentially small splitting of separatrices: We follow the ideas by Lazutkin for the homoclinic breakdown for the Standard Map (also Kruskal and Segur).
- Mostly been applied to
  - 2 dimensional area preserving maps
  - Invariant manifolds of periodic orbits or invariant tori at resonances of nearly integrable Hamiltonian systems (Arnold diffusion)
  - Local bifurcations for Hamiltonian/Reversible/Volume preserving systems

Analytic continuation to complex domains

- Homoclinic for the singular limit: $v_h(y,\tau) = \dfrac{2\sqrt{2}}{\cosh(y)} \sin\tau$.
- Look for solutions v^{uns} and v^{st} of Klein-Gordon eq. with $y \to \pm\infty$.
- $v^{\mathrm{uns}}, v^{\mathrm{st}}$ are ϵ-close to v_h and exponentially close to each other.
- v_h has singularities at $y = \pm i\pi/2$.
- Extend $v^{\mathrm{uns}}, v^{\mathrm{st}}$ to complex y up to $y \sim \pm i\pi/2$.
- v_h blows up at $y = \pm i\pi/2 \to v^{\mathrm{uns}}, v^{\mathrm{st}}$ should be large for $y \sim \pm i\pi/2$
- Its difference is easier to measure at $y \sim \pm i\pi/2$.

The inner equation

- When $y \mp i\pi/2 \sim \epsilon$, $v^{\mathrm{uns}}, v^{\mathrm{st}}$ are not well approximated by the unperturbed homoclinic v_h.
- Lazutkin and Kruskal & Segur: Look for the first order of the perturbed invariant manifolds when $y \mp i\pi/2 \sim \epsilon$.
- Singular change: $z = \epsilon^{-1}\left(y - i\frac{\pi}{2}\right)$ and $\phi(z,\tau) = \epsilon v\left(i\frac{\pi}{2} + \epsilon z, \tau\right)$.
- Let $\epsilon \to 0$ and we get a new equation for the first order: the inner equation

$$\partial_z^2\phi - \partial_\tau^2\phi - \phi + \frac{1}{3}\phi^3 + f(\phi) = 0$$

The inner equation

- We look for analytic solutions ϕ^{uns}, ϕ^{st}, of the inner equation which are the first order of the stable/unstable manifold.
- Both are asymptotic to the same (divergent) series

$$\phi^{\text{uns,st}}(z,\tau) \sim \sum_{k\geq 0} \frac{a_k(\tau)}{z^{2k+1}}$$

(in different sectorial domains $\subset \mathbb{C}$).

- $\phi^{\text{uns}} - \phi^{\text{st}} = e^{-i2\sqrt{2}z}\left(\Theta \sin(3\tau) + \mathcal{O}\left(\frac{1}{z}\right)\right) \quad$ as $\Im z \to -\infty$.
- Θ is the Stokes constant (Borel Resummation, Resurgence Theory) which appeared in the main theorem.
- Θ depends on the full jet of the nonlinearity f.

Last step

- Show the difference of the solutions of inner equations gives an asymptotic formula for the difference between the invariant manifolds real values of y.

- This analysis is the starting point to deal with all bifurcations.

- In the k bifurcation in the general (non-odd) setting, the invariant manifolds have dimension $2k + 1$.

- Two eigenvalues are weak and the other strong.

Thank you for your attention