Time quasi-periodic traveling gravity water waves in infinite depth

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Result

Existence and linear stability of small amplitude **time quasi-periodic** water wave solutions of a bi-dimensional ocean with infinite depth under the action of pure gravity.

Quasi-periodic function

We say that $u(t)$ is a quasi-periodic function of frequency $\omega \in \mathbb{R}^d$ with values in a Hilbert space $\mathcal{H}$ if there exists a function $U : \mathbb{T}^d \to \mathcal{H}$ such that $u(t) = U(\omega t)$ and ω is irrational, namely

$$\omega \cdot k \neq 0 \qquad \forall k \in \mathbb{Z}^d \setminus \{0\}.$$

We are interested in studying the time-evolution of an *incompressible* and *irrotational* perfect fluid, under the action of gravity, on a 2-dimensional (1-d interface) time-dependent domain

$$\mathcal{D}_t := \big\{(x,y) \in \mathbb{R}\times\mathbb{R}\,;\ -h < y < \eta(t,x)\big\}$$

- **infinite depth** $h = \infty$;
- **periodic** in the horizontal variable $x \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$.

Bernoulli equation

$$\partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + g\eta = 0 \qquad \text{at} \quad y = \eta(t,x)$$

$$\Delta\Phi = 0 \text{ in } \mathcal{D}_t\,, \quad \Phi = \psi \quad \text{on} \quad \{y = \eta(t,x)\}.$$

$\Phi(t,x,y)$ velocity potential , $\eta(t,x)$ profile of the ocean (or free interface) .

Boundary conditions

$$\partial_y \Phi \to 0\,, \quad \text{as} \quad y = -\infty$$

$$\partial_t \eta = \partial_y \Phi - \eta_x(\partial_x \Phi) \quad \text{at} \quad y = \eta(t,x)$$

Zakharov-Craig/Sulem formulation

Zahkarov and Craig-Sulem observed that from the trace of the velocity potential to the free interface

$$\psi(t,x) := \Phi(t,x,\eta(t,x))$$

and the profile $\eta(t,x)$ one can recover the velocity potential as the unique solution of the elliptic problem

$$\Delta\Phi = 0 \text{ in } \mathcal{D}_\eta\,, \quad \partial_y\Phi \to 0 \text{ as } y \to -\infty\,, \quad \Phi = \psi \quad \text{on} \quad \{y = \eta(t,x)\}.$$

Gravity water waves system (WW equations)

In the (η,ψ) variable the gravity water waves system reads as

$$\begin{cases} \partial_t\eta = G(\eta)\psi \\ \partial_t\psi = -g\eta - \dfrac{1}{2}\psi_x^2 + \dfrac{1}{2}\dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{1+\eta_x^2} \end{cases}$$

Dirichlet-Neumann operator

The Dirichlet-Neumann operator is defined by

$$G(\eta)\psi := \sqrt{1+\eta_x^2}\,(\partial_{\vec{n}}\Phi)|_{y=\eta(t,x)} = (\partial_y\Phi - \eta_x\partial_x\Phi)(t,x,\eta(t,x))$$

where $\vec{n}$ is the outward unit normal to the free interface.

Fully nonlinear equations

The WW system is fully nonlinear and singular: the nonlinearity contains derivatives of order 1, the linear part is of order 1/2.

Zakharov noticed that this system can be written as an infinite-dimensional Hamiltonian system

$$H(\eta,\psi) := \underbrace{\frac{1}{2}\int_{\mathbb{T}} \psi\, G(\eta)\psi\, dx}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}\int_{\mathbb{T}} \eta^2\, dx}_{\text{potential energy}} \quad .$$

The equations of motion are

$$\begin{pmatrix} \dot\eta \\ \dot\psi \end{pmatrix} = X_H(\eta,\psi) := \begin{pmatrix} \nabla_\psi H(\eta,\psi) \\ -\nabla_\eta H(\eta,\psi) \end{pmatrix}$$

The gradients are L^2-gradients.

Constants of motion

- mass: $\int_{\mathbb{T}} \eta \, dx$;
- momentum: $\int_{\mathbb{T}} \eta_x \, \psi \, dx$.

Invariant subspace

The subspace of zero spatial-average

$$\left\{ (\eta, \psi) : \int_{\mathbb{T}} \eta \, dx = 0, \quad \int_{\mathbb{T}} \psi \, dx = 0 \right\}$$

is invariant, because $h = \infty$. We shall look for solutions on this subspace.

Symmetries

- *Reversibility*: w.r.t the involution $\rho(\eta, \psi)(x) = (\eta, -\psi)(-x)$, $X_H \circ \rho = -\rho \circ X_H$.
- *Parity preserving*: the subspace of even in x functions is preserved by X_H.

We look for small amplitude solutions, hence we study the linearized problem at the origin

$$\begin{cases} \dot\eta = G(0)\psi, \qquad G(0) = |D| = -\mathrm{i}\partial_x, \\ \dot\psi = \eta. \end{cases}$$

Complex coordinates: $z := \frac{1}{\sqrt{2}}\big(|D|^{1/4}\eta + \mathrm{i}|D|^{-1/4}\psi\big)$

Linear solutions

$$\mathrm{i}\dot z = |D|^{1/2} z$$

$$z(t,x) = \sum_{j\in\mathbb{Z}\setminus\{0\}} z_j(0)\, e^{\mathrm{i}(\lambda(j)t + jx)}, \quad z_j(0) := \frac{1}{\sqrt{2}}\big(|j|^{-1/4}\eta_j(0) + |j|^{1/4}\psi_j(0)\big)$$

Dispersion relation

$$j \mapsto \lambda(j) := \sqrt{|j|}$$

Complete resonance

Let $S \subset \mathbb{Z} \setminus \{0\}$ be a finite set. The linear solutions obtained by exciting the modes $\{e^{\mathrm{i}jx}\}_{j \in S}$

$$\sum_{j \in S} c_j \, e^{\mathrm{i}(\lambda(j)t + jx)}$$

are periodic or quasi-periodic.

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Quasi-periodic solutions

We look for solutions of the form

$$z(t,x) = \sum_{j\in S} \sqrt{\zeta_j} \, e^{\mathrm{i}(\omega(j)t+jx)} + r(t,x), \quad \text{S } \textbf{tangential set}$$

with $\omega(j) = \lambda(j) + O(|\zeta|)$, $r(t,x)$ is small in some Sobolev norm.

We shall impose several non-resonance conditions as the following ones.

Melnikov non-resonance conditions

$$|\omega \cdot \ell| \geq \frac{\gamma}{\langle \ell \rangle^{\tau}} \qquad \ell \in \mathbb{Z}^d \setminus \{0\}$$

$$|\omega \cdot \ell + \sqrt{|j|} - \sqrt{|k|}| \geq \frac{\gamma}{\langle \ell \rangle^{\tau}} \qquad j, k \in \mathbb{Z} \setminus \{0\}, \quad \ell \in \mathbb{Z}^d \setminus \{0\}$$

for some small constant $\gamma > 0$.

Sublinear dispersion

The set $\{\sqrt{n} - \sqrt{m} : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

We need parameters that modulate ω!

Comparison with other water waves systems

Gravity-capillary

$$H(\eta,\psi) := \frac{1}{2}\int_{\mathbb{T}} \psi\, G(\eta)\psi\, dx + \frac{1}{2}\int_{\mathbb{T}} \eta^2\, dx + \kappa \int_{\mathbb{T}} \sqrt{1+\eta_x^2}\, dx$$

Finite depth

The Dirichlet-Neumann operator depends also on the depth of the ocean $\mathtt{h}$.

$$G(\eta) = G(\eta, \mathtt{h}).$$

Dispersion relation

$$\lambda(\kappa, \mathtt{h}; j) := \sqrt{|j| \tanh(\mathtt{h}|j|)(g + \kappa j^2)}$$

The physical parameters $\mathtt{h}$ and κ can be used to modulate the linear frequencies of oscillation and impose **non-resonance conditions**.
Moreover, in the gravity-capillary case the dispersion law is $\lambda(\kappa; j) \sim \kappa |j|^{3/2}$:

superlinear $\Rightarrow$ stronger separation of eigenvalues

Literature: quasi-periodic solutions

Previous results

- Berti-Montalto: Standing waves, Gravity-Capillary, infinite depth , *Memoirs of AMS*, 2020.
- Baldi-Berti-Haus-Montalto : Standing waves, Pure Gravity, finite depth, *Invent. Math.*, 2018.

More recently

- Berti-Franzoi-Maspero: Traveling waves, Gravity-capillary with vorticity, *ARMA*, 2020.

The existence of quasi-periodic solutions is provided for some asymptotically full-measure set of the parameters of the problem, capillarity, depth and/or vorticity.

Pure gravity, infinite depth

- We fix the equation and we extract parameters directly from the equation.
- We find a larger class of solutions: we do not use assumptions of parity on $x, t, (t, x)$.

Some literature: periodic solutions

- Traveling waves: stationary w.r.t. a moving reference frame.
  - Stokes (Trans. Cambridge Phil, 1847) approximate flows...
  - Levi-Civita (Math. Ann. 1925), Struik (Math. Ann. 1926) $\rightsquigarrow$ $2d$-traveling gravity WW
  - Craig-Nicholls (SIAM J. Math. Anal. 2000) $\rightsquigarrow$ $2/3d$-*gravity-capillary* WW
  - Ioss-Plotnikov (Mem. Amer. Math. Soc. 2009)-(Arch. Rat. Mech. Anal. 2011) $\rightsquigarrow$ $3d$-*gravity* WW $\leftrightarrow$ small divisors
  - Lokharu-Seth-Wahlen (Arch. Rat. Mech. Anal. 2020) $\rightsquigarrow$ $3d$-*gravity-capillary*+vorticity
  - *many other* $\rightsquigarrow$ approach with variational methods.... (see Buffoni, Grooves, Sun, Wahlen...)
- Standing waves: even in x
  - Plotnikov-Toland (Arch. Rat. Mech. Anal. 2001)$\rightsquigarrow$ $2d$-*gravity*, *finite* depth
  - Iooss-Plotnikov-Toland (Arch. Ration. Mech. Anal. 2005) $\rightsquigarrow$$2d$-*gravity*, ∞ depth $\leftrightarrow$completely resonant
  - Iooss-Plotnikov (J. math. fluid mech. 2005) $\rightsquigarrow$$3d$-*gravity*, ∞ depth $\leftrightarrow$completely resonant
  - Alazard-Baldi (Arch. Ration. Mech. Anal. 2015) $\rightsquigarrow$ $2d$-*gravity-capillary*, ∞ depth $\leftrightarrow$almost all κ

Main result

Quasi-periodic solutions

We say that a function

$$(\eta(t,x),\psi(t,x)) : \mathbb{R}\times\mathbb{T}\to\mathbb{R}^2$$

is a quasi-periodic solution of the WW equations with irrational frequency vector $\omega\in\mathbb{R}^d$ (i.e. $\omega\cdot\ell\neq 0$ for all $\ell\in\mathbb{Z}^d\setminus\{0\}$) if there is a smooth embedding

$$\begin{aligned}\mathbb{T}^d &\to H^1_0(\mathbb{T};\mathbb{R})\times H^1_0(\mathbb{T};\mathbb{R})\\ \varphi &\mapsto U(\varphi,x) := (\eta(\varphi,x),\psi(\varphi,x))\end{aligned}$$

such that $(\eta(t,x),\psi(t,x)) = U(\omega t,x)$ solves the WW.

Traveling quasi-periodic function

A quasi-periodic solution is traveling with *velocity vector* $\mathtt{v}\in\mathbb{Z}^d$ if there is a function $\widetilde{U}:\mathbb{T}^d\to\mathbb{R}^2$ such that

$$(\eta(t,x),\psi(t,x)) = U(\omega t,x) = \widetilde{U}(\omega t-\mathtt{v}x)\,.$$

Hamiltonian Transport equation

$$\mathtt{v} \cdot \partial_\varphi U + X_M(U) = 0$$

$$\textbf{Momentum:} \quad M(U) := \int_{\mathbb{T}} \eta_x \, \psi \, dx, \qquad X_M(U) = (\eta_x, \psi_x)$$

The solutions have the form

$$U(\omega t, x) = \sum_{\ell \in \mathbb{Z}^d, j \in \mathbb{Z} \setminus \{0\}} U_{\ell j} \, e^{\mathrm{i} \ell \cdot \varphi} e^{\mathrm{i} j x} = \sum_{\substack{\ell \in \mathbb{Z}^d, j \in \mathbb{Z} \setminus \{0\}, \\ \mathtt{v} \cdot \ell + j = 0}} \widetilde{U}_\ell e^{\mathrm{i} \Theta \cdot \ell} = \widetilde{U}(\Theta)$$

$$\Theta := \varphi - \mathtt{v} x \in \mathbb{R}^d$$

and they are supported on the d-dimensional time-space lattice

$$\{(\ell, j) \in \mathbb{Z}^d \times \mathbb{Z} : \mathtt{v} \cdot \ell + j = 0\}$$

Frequency and velocity vectors

The prototype of a quasi-periodic traveling waves is the superposition of finitely many periodic traveling waves with rationally independent frequencies .

Our solutions will be close to the following approximate solutions

$$\eta_0(t,x) = \sum_{j\in S} \sqrt{2\zeta_j}|j|^{1/4}\cos(\omega(j)t - jx)$$

$$\psi_0(t,x) = -\sum_{j\in S} \sqrt{2\zeta_j}|j|^{-1/4}\sin(\omega(j)t - jx)$$

Tangential set: $S := \{j_1, \dots, j_d\}$.

Frequency vector

$$\omega(j) = \omega(\zeta; j) = \sqrt{|j|} + O(|\zeta|)$$

The amplitudes ζ_j will play the role of inner parameters to impose non-resonant conditions on the frequency.

Velocity vector

$$\mathtt{v} := (j_1, \dots, j_d) \in \mathbb{Z}^d$$

Tangential set

$$S := S^+ \cup S^-, \quad S^+ := \{j_1, \dots, j_m\} \subset \mathbb{N}, \quad S^- := \{j_{m+1}, \dots, j_d\} \subset -\mathbb{N}$$

Assumption: if $j \in S$ then $-j \notin S$.

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Quasi-periodic traveling gravity waves. (Feola - G, under review)

For a *generic* choice of the velocity vector $\mathtt{v}$ there exist $s \gg 1$ and a Cantor-like set $\mathfrak{A}$ of small amplitudes $\zeta \in \mathbb{R}^d_+$ with asymptotically full measure such that the following holds. The WW equation possesses a small amplitude, linearly stable, quasi-periodic solution

$$U(\omega t, x; \zeta) = (\eta, \psi)(t, x; \zeta) = (\eta_0, \psi_0)(t, x; \zeta) + o(\sqrt{|\zeta|})$$

which is a traveling wave with velocity vector $\mathtt{v}$, $\omega := \omega(\zeta) \in \mathbb{R}^d$ is a diophantine frequency vector and $U(\varphi, x) \in H^s(\mathbb{T}^{d+1}, \mathbb{R}^2)$.

- Generic choice means that $(j_1, \dots, j_d)$ have to be chosen out of the zero set of a polynomial $P \colon \mathbb{C}^d \to \mathbb{C}$.

Comments

1. Due to the complete resonance the diophantine constants are small as the amplitudes. We set $\zeta = O(\varepsilon^2)$ then

$$|\omega \cdot \ell| \geq \frac{\gamma}{\langle \ell \rangle^\tau} \qquad \gamma = O(\varepsilon^2).$$

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2. Thanks to the Hamiltonian structure and the momentum conservation we can find a large class of solutions: we do not need to use the *reversibility*, we do not need to assume *parity conditions*.

$$\{(\eta(t,x), \psi(t,x)) = \rho(\eta(-t,x), \psi(-t,x)) = (\eta(-t,-x), -\psi(-t,-x))\}$$

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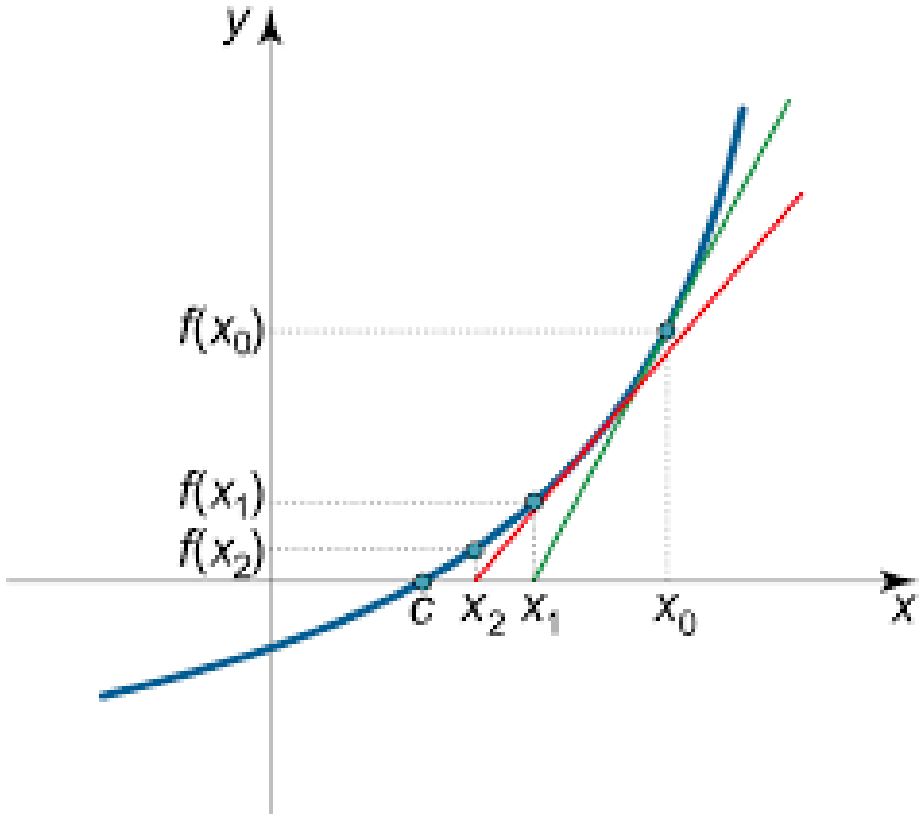
3. Since we cannot bifurcate from linear solutions we need to perform a nonlinear analysis that requires a deep knowledge of the normal form of the WW system.

4. **Linear stability**: We are able to construct a KAM (reducible) normal form around the invariant torus that supports the quasi-periodic solution. The (normal) eigenvalues of the linearized problem at the torus are purely imaginary.

General strategy of the proof



Invariance equation

We look for a smooth embedding $U = (\eta, \psi)\colon \mathbb{T}^d \to H^1_0(\mathbb{T};\mathbb{R}) \times H^1_0(\mathbb{T};\mathbb{R})$ such that

$$U \circ \Psi^t_\omega = \Phi^t_H \circ U$$

where $\Psi^t_\omega(\varphi) = \varphi + \omega t$. Note that the dynamics on the torus is prescribed, but ω is an unknown of the problem.

Nonlinear functional equation

$$F(\omega; U) = \omega \cdot \partial_\varphi U - X_H(U) = 0.$$

$\omega \cdot \partial_\varphi := \sum_{i=1}^d \omega(i)\partial_{\varphi_i}$

Linearized problem

$$\mathcal{L}_\omega := dF(\omega; 0) = \omega \cdot \partial_\varphi U - dX_H(0) = \omega \cdot \partial_\varphi - \begin{pmatrix} 0 & |D| \\ -1 & 0 \end{pmatrix}$$

The eigenvalues are: $\mu_{\ell,j}(\omega) := \mathrm{i}\big(\omega \cdot \ell \pm \sqrt{|j|}\big)$

Assume that $\mathcal{L}_\omega = \mathrm{diag}(\mu_{\ell j})$ then

$$\mathcal{L}_\omega h = g \quad \Leftrightarrow \quad h_{\ell j} = \sum \frac{g_{\ell j}}{\mathrm{i}\mu_{\ell j}} e^{\mathrm{i}(\ell\cdot\varphi + jx)}$$

For many (in the sense of Lebesgue measure) ω we have

$$|\mu_{\ell,j}(\omega)| \geq \frac{\gamma}{\langle \ell \rangle^\tau} \qquad \tau \gg d.$$

$$H^s := H^s(\mathbb{T}^{d+1}) = \{u \in L^2(\mathbb{T}^{d+1}) : u = \sum_{\ell,j} u_{\ell j}\, e^{\mathrm{i}(\ell\cdot\varphi + jx)},\ ||u||_s^2 := \sum_{\ell,j} |u_{\ell j}|^2 \langle \ell, j\rangle^{2s}\}$$

Loss of derivatives

$$\mathcal{L}_\omega^{-1} \colon H^s \to H^{s-\tau}$$

Then the Implicit Function Theorem cannot be applied.

We shall work on a scale of Sobolev spaces.

Nash-Moser Scheme

We apply a Nash-Moser scheme: $\rightsquigarrow$ construct a sequence of approximate solutions $(U_n)_{n\geq 0}$ converging to a solution:

$$U_{n+1} = U_n - \Pi_n(\mathcal{L}_\omega(U_n))^{-1}\Pi_n F(U_n)$$

$$\Pi_n u = \sum_{|(\ell,j)|<n} u_{\ell j}\, e^{\mathrm{i}(\ell\cdot\varphi + jx)}$$

The operators Π_n are *smoothing operators*: $\rightsquigarrow$ U_n is C^∞ for any n!

Main issues

(1) Finding U_0 that initializes the iteration $|F(U_0)|\gamma^{-n} \ll 1$;

(2) invert the linearized operator at any step of the scheme (in a whole neighborhood of the origin), providing tame estimates such as

$$\|(\mathcal{L}_\omega(U_n))^{-1}h\|_s \leq C(s)\Big(\|h\|_{s+\mu} + \|U_n\|_{s+\mu}\|h\|_{s_0+\mu}\Big)$$

for $s \in [s_0, \mathbf{S}]$, with $\mathbf{S}$ large. The above estimates can be obtained for "many" ω and under smallness assumption on the low norms $||U_n||_{s_0+\mu}$.

Analysis of the linearized operator

Linearized operator in normal directions

In a suitable set of coordinates the linear problem is "triangular": the tangential and normal dynamics can be decoupled.

Linearized operator in normal directions

$$\mathcal{L}_\omega := \omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x V + G(\eta)B & -G(\eta) \\ (1 + BV_x) + BG(\eta)B & V\partial_x - BG(\eta) \end{pmatrix}$$

$$V = V(\eta, \psi) := (\partial_x \Phi)(x, \eta(x)) = \psi_x - \eta_x B\,,$$

$$B = B(\eta, \psi) := (\partial_y \Phi)(x, \eta(x)) = \frac{G(\eta)\psi + \eta_x \psi_x}{1 + \eta_x^2}$$

After a symmetrization procedure and a block-diagonalization we reduce to the study of

$$\mathcal{L}_\omega = \mathcal{L}_\omega(U_n) := \omega \cdot \partial_\varphi + V(\varphi, x)\,\partial_x + (1 + a(\varphi, x))|D|^{1/2} + R(\varphi, x)$$

$$V(\varphi, x), a(\varphi, x), R(\varphi, x) = O(\varepsilon) \qquad (\textbf{ in low norm})$$

The idea is to diagonalize it (in Fourier, make constant coefficients) in order to invert it and get suitable tame estimates on the inverse.

Reducibility

We look for a tame, symplectic, quasi-periodically dependent on time change of coordinates

$$\Phi(\omega t) = \Phi \colon H^s \mapsto H^s \qquad \forall s$$

such that

$$\Phi\, \mathcal{L}_\omega\, \Phi^{-1} = \mathcal{L}_\infty = \omega \cdot \partial_\varphi + \mathrm{diag}_j(\mathrm{i} d_j).$$

Due to fully nonlinear nature we need to construct Φ in two main steps $\Phi = \Upsilon \circ \Psi$.

(1) **Regularization procedure**: reduction in decreasing orders (decreasing symbols).

$$\Psi_1 \mathcal{L}_\omega \Psi_1^{-1} = \omega \cdot \partial_\varphi + m_1\, \partial_x + (1 + \tilde{a}(\varphi, x))\, \mathrm{i}\, |D|^{1/2} + \ldots, \quad m_1 \in \mathbb{R}.$$

$$\Psi_2 \Psi_1 \mathcal{L}_\omega \Psi_1^{-1} \Psi_2^{-1} = \omega \cdot \partial_\varphi + m_1\, \partial_x + (1 + m_{1/2})\, \mathrm{i}\, |D|^{1/2} + \ldots, \quad m_1, m_{1/2} \in \mathbb{R}.$$

$$\ldots$$

To prove that m_i are constants we have to deal with resonances.

$$\mathcal{L}_\omega^+ := \Psi \mathcal{L}_\omega \Psi^{-1} = \omega \cdot \partial_\varphi + D(\omega) + \mathfrak{R},$$

$\mathfrak{R}$ is regularizing (in space) and "small" in size.

Symbols

$a(x, j) \in C^\infty(\mathbb{T} \times \mathbb{Z}; \mathbb{C})$ is a symbol of order m if $\forall \alpha, \beta \in \mathbb{N}$ there exists $C_{\alpha,\beta} > 0$ such that

$$|\Delta_j^\beta \partial_x^\alpha a| \leq C_{\alpha,\beta} \langle j \rangle^{m-\beta}.$$

Pseudo-differential operator

$$Au = \mathrm{Op}(a)\, u = \sum_{j \in \mathbb{Z}} a(x, j)\, u_j\, e^{\mathrm{i} j x}, \quad s_A := \text{order of A} = \text{order of } a(x, j)$$

Example : $a(x, j) = f(x)\, \mathrm{i} j$, $\mathrm{A} = f(x)\, \partial_x$.

Most important property

$$\mathrm{Op}(A) \circ \mathrm{Op}(B) \text{ is pseudo-diff. with order } s_A + s_B.$$

The commutator between two pseudo-diff. operators gains one derivative!

$$[\mathrm{Op}(A), \mathrm{Op}(B)] \text{ has order } s_A + s_B - 1.$$

Reduction of the highest order

We want to find Ψ_1 such that

$$\Psi_1 \mathcal{L}_\omega \Psi_1^{-1} = \omega \cdot \partial_\varphi + m_1 \, \partial_x + (1 + \tilde{a}(\varphi, x)) \, \mathrm{i} \, |D|^{1/2} + \ldots, \quad m_1 \in \mathbb{R}.$$

Flow of a first order pseudo-differential PDE

$$\partial_\tau \Phi^\tau = b(\tau, \varphi, x) \partial_x \Phi^\tau$$

$$b(\tau, \varphi, x) := \frac{\beta(\varphi, x)}{1 + \tau \beta_x(\varphi, x)}, \qquad \Psi_1 = \Psi_1(\beta) = \Phi^\tau_{|_{\tau=1}}$$

β has to be chosen opportunely as the solution of a *transport equation*.

The problem is equivalent to find a diffeo $(\varphi, x) \mapsto (\varphi, x + \beta(\varphi, x))$ straightening the following weakly perturbed constant vector field on the torus $\mathbb{T}^{d+1}$

$$\omega \cdot \frac{\partial}{\partial \varphi} + V(\varphi, x) \frac{\partial}{\partial x} \quad \rightsquigarrow \quad \omega \cdot \frac{\partial}{\partial \varphi} + m_1 \frac{\partial}{\partial x}$$

by finding a C^∞ change of coordinates assuming the smallness of $V(\varphi, x)$ only on a low norm. Since $V(\varphi, x) = O(\varepsilon)$ the vector field is degenerate.

Standing wave case

The spatial average of $V(\varphi, x)$ is zero. In our case it is not.

Recall $(U_n)_{n\geq 0}$ is the sequence of approximate solutions provided by the Nash-Moser iteration.

Momentum reduction

We need to prove that U_n is a traveling quasi-periodic function for all $n \geq 0$. Then

$$V(U_n; \varphi, x) = V(\varphi, x) = \widetilde{V}(\varphi - \mathtt{v}x)$$

and the vector field

$$\omega \cdot \frac{\partial}{\partial \varphi} + V(\varphi, x)\frac{\partial}{\partial x}$$

is non-degenerate on a lower-dimensional torus $\mathbb{T}^d$.

Constructing the straightening diffeo in this case is a small divisors problem which can be solved by performing a Nash-Moser scheme (Feola-G.-Montalto-Procesi, Journal of Functional Analysis, 2019).

(2) **KAM reducibility scheme**: reduction in size of the remainder $\mathfrak{R}$.

$$\mathcal{L}_\omega^+ := \omega \cdot \partial_\varphi + D(\omega) + \mathfrak{R}$$

$$\Upsilon \, \mathcal{L}_\omega^+ \, \Upsilon^{-1} = \omega \cdot \partial_\varphi + D_\infty(\omega), \quad D_\infty(\omega) = \mathrm{diag}_{j \in \mathbb{Z}\setminus\{0\}}(\mathrm{i}\, d_j(\omega))$$

Melnikov conditions

We have a nice expansion in orders:

$$d_j = m_1(\zeta)\, j + (1 + m_{1/2}(\zeta))\sqrt{|j|} + m_0(\zeta)\,\mathrm{sign}(j) + r_j, \quad \sup_j |j|^{1/2} r_j < \infty$$

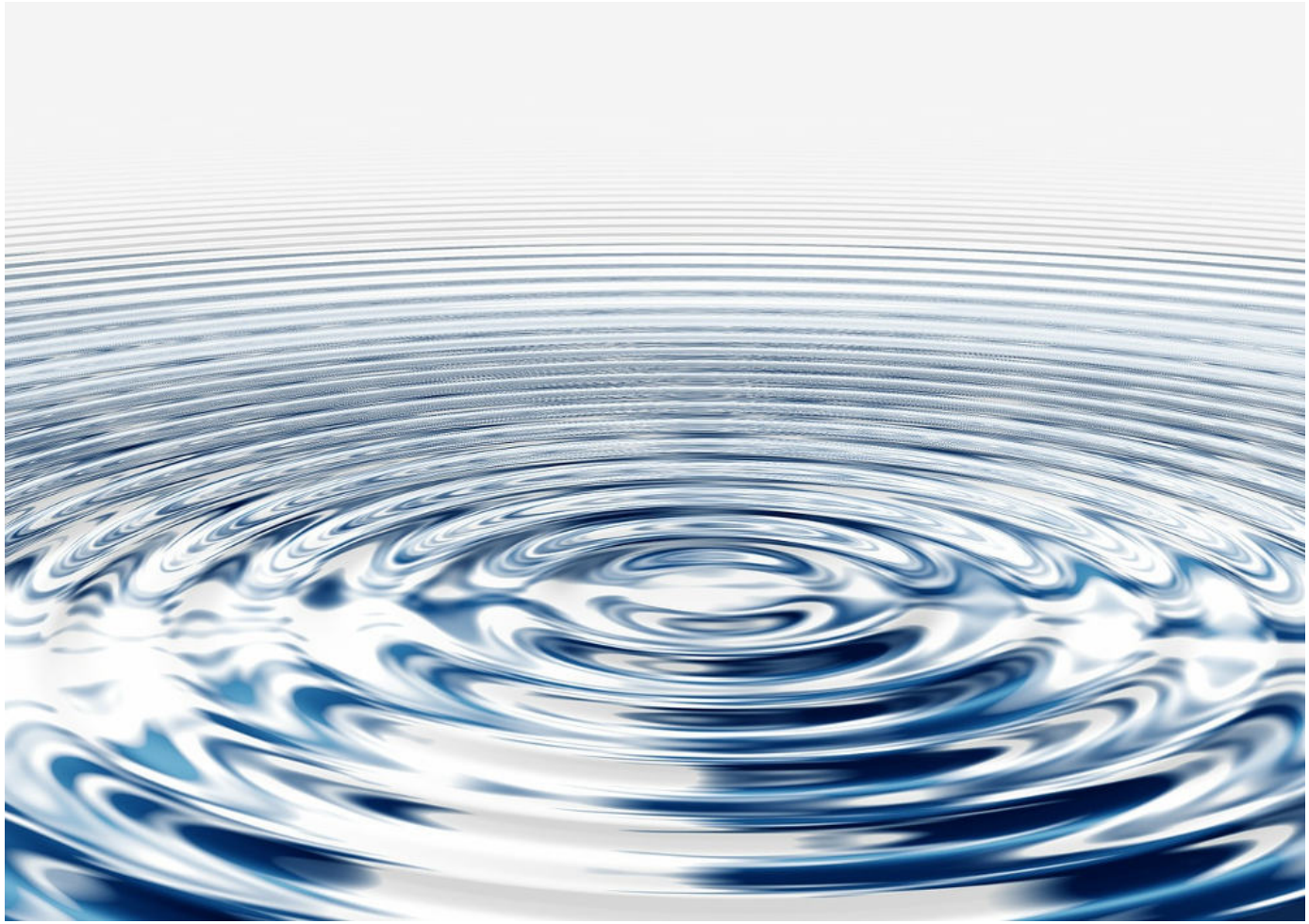
Melnikov conditions: $|\omega \cdot \ell + d_j - d_k| \geq \dfrac{\gamma}{\langle \ell \rangle^\tau} \quad j, k \in \mathbb{Z} \setminus \{0\},\ \ell \in \mathbb{Z}^d$

The momentum condition $\mathtt{v} \cdot \ell + j - k = 0$ is fundamental to impose them.
We have to construct always changes of coordinates that preserve momentum!

Linear stability

Due to the Hamiltonian structure, the eigenvalues $d_j \in \mathbb{R}$ $\Rightarrow$ **linear stability**.

Normal form and resonances



ε^2-corrections

$$d_j = m_1(\zeta)\, j + (1 + m_{1/2}(\zeta))\sqrt{|j|} + m_0(\zeta)\,\mathrm{sign}(j) + r_j, \quad \sup_j |j|^{1/2} r_j < \infty$$

$$\omega \cdot \ell + d_j - d_k = \sum_{j_i \in S} \sqrt{|j_i|}\,\ell_i + \sqrt{|j|} - \sqrt{|k|} + O(\varepsilon^2 |\zeta|)$$

The blue vanishes on resonances. We need an expansion of the eigenvalues also in size! **At order ε^2 we need exact expression for ω and m_1.**

$$\omega(j) = \sqrt{|j|} + \varepsilon^2 \mathbb{A}\zeta + O(\varepsilon^4), \qquad \mathbb{A} \text{ is called } \textbf{twist} \text{ matrix},$$

$$m_1 = \varepsilon^2 \mathbf{m}_1 + O(\varepsilon^4) \qquad \mathbf{m}_1 := \sum_{j \in S} j\,|j|\,\zeta_j$$

Non-degeneracy conditions

Proving that for a generic choice of the tangential sites

$$\begin{cases} \det \mathbb{A} \neq 0 \\ \det(\mathbb{A} - \mathbf{m}_1\, \mathtt{v}) \neq 0, \end{cases} \qquad \mathtt{v} := (j_1, \dots, j_d)$$

we are able to impose the Melnikov conditions for a positive measure set of ω's.

The goals

- Find nonlinear quasi-periodic approximate solutions $U_0 = (\eta_0, \psi_0)$ with frequency ω such that
$$|F(U_0)|\gamma^{-n} \ll 1.$$
- Extract parameters from the equation that modulate the frequency ω.

We construct a symplectic, bounded change of coordinates Φ_B such that

$$H \circ \Phi_B = \underbrace{\mathcal{N}}_{\text{normal form}} + P$$

where

- $\mathcal{N}$ is integrable restricted to a finite-dimensional invariant subspace.
- X_P is a small remainder.

The function (η_0, ψ_0) will be a solution of $\mathcal{N}$. Solution of $\mathcal{N}$ are approximate solutions for WW.

Hamiltonian

$$H = H^{(2)} + H^{(3)} + H^{(4)} + ..., \qquad H^{(2)} := \sum_{j \in \mathbb{Z}\setminus\{0\}} \sqrt{|j|}\, |u_j|^2$$

$$H^{(n)} = \sum_{\sigma_1 j_1 + \cdots + \sigma_n j_n = 0} C^{\sigma_1 \dots \sigma_n}_{j_1 \cdots j_n}\, u^{\sigma_1}_{j_1} \dots u^{\sigma_n}_{j_n}, \quad u^+_j := u_j,\ u^-_j = \overline{u_j}$$

Let $u = v + z$, v tangential variable, z normal variable

$$v = \Pi_S u = \sum_{j \in S} u_j e^{\mathrm{i}jx}, \quad z = \Pi^\perp_S u = \sum_{j \notin S} u_j e^{\mathrm{i}jx}$$

$$H^{(n,k)} = \sum_{\substack{\sigma_1 j_1 + \cdots + \sigma_n j_n = 0, \\ k = \text{number of } j_i\text{'s not in S}}} C^{\sigma_1 \dots \sigma_n}_{j_1 \cdots j_n}\, u^{\sigma_1}_{j_1} \dots u^{\sigma_n}_{j_n}$$

By the momentum conservation

If $k \leq 1$ then $H^{(n,k)}$ has finite Fourier support and its vector field has finite rank.

Weak Birkhoff normal form procedure

We normalize just the terms $H^{(n,k)}$ with $k \leq 1$.

Why use this procedure?

- The Birkhoff maps are time-one flow maps of unbounded PDEs, possibly with no pseudo-differential structure. So they could be ill-posed. With the weak procedure they are flows of finite-dimensional ODEs.
- We cannot control the resonances. If you want to perform several steps then you deal with many non-trivial resonances.
- The linearized operator $\mathcal{L}_\omega(U_n)$ in these Birkhoff coordinates is the one in original coordinates up to a finite rank operator. Hence the original pseudo-differential structure is preserved.

Normal form

We construct changes of coordinates that are time-one flow maps of homogenous Hamiltonians

$$F^{(n)} = \sum_{\sigma_1 j_1 + \cdots + \sigma_n j_n = 0} F^{\sigma_1 \dots \sigma_n}_{j_1 \dots j_n} \, u^{\sigma_1}_{j_1} \dots u^{\sigma_n}_{j_n} .$$

Homological equation

$$\{H^{(2)}, F^{(n)}\} + K^{(n)} = \Pi_{ker} K^{(n)}$$

Π_{ker} is the projection on the kernel of the adjoint action $\{H^{(2)}, \cdot\}$.
$\Pi_{ker} K^{(n)}$ is supported on resonances

$$(\vec{\sigma}, \vec{j}) \quad : \quad \sum_i \sigma_i j_i = 0, \qquad \sum_i \sigma_i \sqrt{|j_i|} = 0.$$

Trivial resonances

$\vec{j} := (j, \pm j, k, \pm k, ...)$ and $\vec{\sigma} := (+, -, +, -, ...)$. The monomials supported on trivial resonances are of the form

$$|u_j|^2 |u_k|^2 .... \qquad \text{or} \qquad u_j \, \overline{u_{-j}} |u_k|^2 ...$$

The $|u_j|^2$ are called actions. The $|u_j|^2 + |u_{-j}|^2$ are called super-actions.

4-waves resonances

When $\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 + \sigma_4 j_4 = 0$ we have two types of resonances:

- *Trivial resonances*: $(j, j, k, k) \leftrightarrow |u_j|^2 |u_k|^2$ *integrable* monomials
- *Benjamin-Feir*: 4-tuple belonging to

$$\bigcup_{\lambda \in \mathbb{Z}\setminus\{0\}, b \in \mathbb{N}} \left\{ j_1 = -\lambda b^2,\ j_2 = \lambda (b+1)^2,\ j_3 = \lambda (b^2+b+1)^2,\ j_4 = \lambda (b+1)^2 b^2 \right\},$$

with $\sigma_1 = \sigma_3 = -\sigma_2 = -\sigma_4$.

Zakharov-Dyachenko proved that the coefficients $C^{\sigma_1\sigma_2\sigma_3\sigma_4}_{j_1 j_2 j_3 j_4} = 0$ if $(\vec{\sigma}, \vec{j})$ is a Benjamin-Feir resonance.

Zakharov-Dyachenko (ZD normal form)

$$\frac{1}{4\pi} \sum_{k \in \mathbb{Z}} |k|^3 |u_k|^4 + \frac{1}{\pi} \sum_{\substack{k_1, k_2 \in \mathbb{Z},\ \mathrm{sign}(k_1) = \mathrm{sign}(k_2) \\ |k_2| < |k_1|}} |k_1| |k_2|^2 |u_{k_1}|^2 |u_{k_2}|^2$$

Weak Birkhoff normal form

We construct a change of coordinates Φ_B such that

$$\mathcal{H} := H \circ \Phi_B = H^{(2)} + \mathcal{H}^{(4,0)}_{res} + \sum_{k=3}^{7} \mathcal{H}^{(2k,0)}_{res} + \mathcal{H}^{(\geq 3, \geq 2)} + \mathcal{H}^{(\geq 16, \leq 1)}$$

- The Hamiltonians $\mathcal{H}^{(n,0)}_{res}$ Poisson commutes with $H^{(2)}$.
- $\mathcal{H}^{(\geq 16, \leq 1)}$ is small enough in a neighborhood of the origin.
- The vector field $\mathcal{H}^{(\geq 3, \geq 2)}$ vanishes on the subspace $\mathcal{U}_S := \{u_j = 0 \; j \notin S\}$.

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By an **Identification argument** we show that

$$\mathcal{H}^{(4,0)}_{res} := \frac{1}{4\pi} \sum_{k \in S} |k|^3 |u_k|^4 + \frac{1}{\pi} \sum_{\substack{k_1, k_2 \in S, \, \mathrm{sign}(k_1) = \mathrm{sign}(k_2) \\ |k_2| < |k_1|}} |k_1| |k_2|^2 |u_{k_1}|^2 |u_{k_2}|^2$$

By using a genericity argument we have that $\mathcal{H}^{(2k,0)}_{res}$ with $3 \leq k \leq 7$ are *integrable*.

The normal form

$$\mathcal{N} := H^{(2)} + \mathcal{H}^{(4,0)}_{res} + \sum_{k=3}^{7} \mathcal{H}^{(2k,0)}_{res}$$

depends only on the actions $I_j := |u_j|^2$, $j \in S$. Moreover $\mathcal{U}_S$ is invariant.

We put action angle variables on $\mathcal{U}_S$

$$v_j = \sqrt{I_j}\, e^{\mathrm{i}\theta_j} \qquad j \in S.$$

The equations of motion are

$$\dot\theta_j = \sqrt{|j|} + \big(\mathbb{A} I\big)_j + O_2(|I|), \qquad \dot I_j = 0$$

Then the torus $\mathbb{T}_\zeta := \{I = \zeta\}$ has frequency

$$\omega(\zeta) = \lambda + \mathbb{A}\zeta + O_2(|\zeta|), \qquad \lambda(j) := \sqrt{|j|}.$$

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Twist condition

If $\det \mathbb{A} \neq 0$ then

Frequency-amplitude map : $\quad \zeta \mapsto \omega(\zeta) \qquad$ is a local diffeomorphism.

We use ζ as a parameter to modulate the frequency ω.

Normal form around the torus $\mathbb{T}_\zeta$

We work in a neighborhood of $\mathbb{T}_\zeta$

$$I_j = \zeta_j + y_j, \quad j \in S.$$

The normal form around it is

$$\omega(\zeta) \cdot y + \frac{1}{2}\langle M(\theta)z, z\rangle + P(\theta, y, z)$$

Benjamin-Feir resonances

To find the ε^2-correction to the eigenvalues m_1 we need to partially reduce $M(\theta)$.
These corrections will be given by the resonant terms of the form $O(v^2 z^2)$.
Degree $4 \Rightarrow$ **Benjamin-Feir resonances.**

Convergent procedure (Weak BNF+Reg.+Linear BNF) vs **Formal procedure**

Identification of normal forms

We need an identification argument of normal forms to prove that the coefficients of the Hamiltonian of the linearized operator $\mathcal{L}_\omega(U_n)$ related to the Benjamin-Feir resonances vanish.

Thanks for the attention!