DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS

doi:10.3934/dcds.2021068

TRANSFERS OF ENERGY THROUGH FAST DIFFUSION CHANNELS IN SOME RESONANT PDES ON THE CIRCLE

Filippo Giuliani

Departament de Matematiques, Universitat Politecnica de Catalunya
ETSEIB, Avinguda Diagonal 647
08028 Barcelona, Spain

(Communicated by Bernhard Ruf)

Abstract. In this paper we consider two classes of resonant Hamiltonian PDEs on the circle with non-convex (respect to actions) first order resonant Hamiltonian. We show that, for appropriate choices of the nonlinearities we can find time-independent linear potentials that enable the construction of solutions that undergo a prescribed growth in the Sobolev norms. The solutions that we provide follow closely the orbits of a nonlinear resonant model, which is a good approximation of the full equation. The non-convexity of the resonant Hamiltonian allows the existence of *fast diffusion channels* along which the orbits of the resonant model experience a large drift in the actions in the optimal time. This phenomenon induces a transfer of energy among the Fourier modes of the solutions, which in turn is responsible for the growth of higher order Sobolev norms.

Contents

2020 *Mathematics Subject Classification.* Primary: 37K45, 35B34; Secondary: 35B35.

Key words and phrases. Transfers of energy, Hamiltonian PDEs, infinite dimensional dynamical systems, fast diffusion, growth of Sobolev norms.

The author is supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 757802.

1. **Introduction.** We consider the following resonant Hamiltonian PDEs under periodic boundary conditions:

- Nonlinear wave equations with even-power nonlinearity
$$u_{tt} - \Delta u + V_p * u + u^p = 0, \quad u = u(t,x), \quad t \in \mathbb{R}, \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z} \tag{1}$$
with p even and large enough.
- Nonlinear Schrödinger equations with cubic x-dependent nonlinearity
$$\mathrm{i}u_t - \Delta u + V_N * u + \cos(Nx)|u|^2 u = 0, \quad u = u(t,x), \quad t \in \mathbb{R}, \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z} \tag{2}$$
with $N \geq 1$.

We show that, prescribed an arbitrarily large (but finite) growth, for an opportune choice of the nonlinear terms and *time-independent* potentials $V_p(x), V_N(x)$ (supported on few harmonics) we are able to construct arbitrarily small initial data solutions of the equations 1, 2 whose Sobolev norms undergo the prescribed growth (we refer to Section 2 for the precise statements of the results). We point out that this phenomenon is purely nonlinear, since the potentials $V_p(x)$, $V_N(x)$ are chosen such that the origin $u = 0$ is a resonant *elliptic* fixed point - all the linear solutions are time-periodic or quasi-periodic functions that do not exchange energy among their modes.

The construction of solutions whose norms exhibit growth relies on the resonant nature of the equilibrium and the degeneracy of the first order resonant Hamiltonian.

We remark that in the linear setting one can obtain stronger results, such as the existence of unbounded orbits, by adding smooth time-dependent potentials to resonant equations, see for instance [24].

In a neighborhood of a resonant elliptic equilibrium the analysis of the nonlinear dynamics can be performed through Birkhoff normal form methods. We can find a change of coordinates that removes from the Hamiltonian the terms which are not relevant for the dynamics of the equations in a certain range of times. In these coordinates the Hamiltonian is said to be in normal form.

We construct solutions which are close to orbits of a finite-dimensional *resonant model*, which is obtained by restricting the Hamiltonian in normal form on an opportune submainfold of the phase space. The non-convexity of such Hamiltonian allows the existence of certain affine subspaces of the action space, called *diffusion channels*. The orbits that travel on these channels are locked in a resonance and exhibit a relevant drift in (some of) the actions in the optimal time.

A fast instability phenomenon of this kind is illustrated by the following example: consider the two-degrees of freedom Hamiltonian system
$$H(\theta_1, \theta_2, I_1, I_2) = I_1 + \varepsilon \cos(\theta_2), \quad \theta_i \in \mathbb{T}, \ I_i \in (0, r), \ i = 1, 2, \tag{3}$$
for some $r > 0$. When $\varepsilon = 0$ the phase space is foliated by 2-dimensional invariant resonant tori filled by periodic orbits. Hence there is stability in the actions for all time. For $\varepsilon > 0$ the equations of motion are
$$\dot{\theta}_1 = 1, \quad \dot{\theta}_2 = 0, \quad \dot{I}_1 = 0, \quad \dot{I}_2 = \varepsilon \sin(\theta_2).$$
We note that any section $\{\theta_2 = \alpha\}$, $\alpha \in [0, 2\pi)$ is invariant. If we choose as initial conditions $\theta_2(0) = \alpha$ with $\alpha \notin \{0, \pi\}$ it is easy to see that $I_2(t)$ experiences a drift of order $\mathcal{O}(1)$ in a time $T = \mathcal{O}(\varepsilon^{-1})$. Thus all the periodic orbits on a given unperturbed torus, except two, are destroyed and give rise to diffusive solutions.

The first examples of finite-dimensional nearly-integrable Hamiltonian systems exhibiting fast instability have been provided by Nekhoroshev [25]. The analysis of such systems has been carried out by Biasco-Chierchia-Treschev [5], Bounemoura-Kaloshin [8] and Bounemoura [7]. In these works the authors consider two degrees of freedom systems with an unperturbed Hamiltonian that violate the Nekhoroshev's condition for stability (see [25]) and study generic properties of the perturbations that provide fast diffusion. In the present paper we are interested in studying how these fast instability phenomena may be exploited, in the context of Hamiltonian PDEs under periodic boundary conditions, to obtain different ways to transfer energy among Fourier modes of nonlinear wave solutions.

The dynamics of the linearized problem at a resonant elliptic equilibrium is in some way similar to the unperturbed dynamics of the Hamiltonian in 3: there is plenty of resonant invariant tori supporting motions that fill densely lower dimensional submanifolds. As observed by Poincaré, these invariant objects are usually not robust, even under small perturbations. Then we may expect that, under some degeneracy assumptions on the Hamiltonian, some of them may be partially destroyed under the effect of the nonlinear terms and accomodate unstable behaviors. As a counterpart we mention that, under assumptions of integrability and non-degeneracy of the normal form Hamiltonian, the existence of invariant tori very close to resonances, see for instance [1], [12], and results of long-time stability [2], [3] have been provided for completely resonant PDEs on tori.

A drift in the actions in an opportune symplectic reduction of the resonant model corresponds to a transfer of energy between resonant modes. An arbitrarily large growth of Sobolev norms can be obtained if this transition occurs across increasingly high Fourier modes.

In this paper we just consider solutions that display an exchange of energy among few modes. Similar analysis for the dynamics of single resonant clusters have been recently carried out for the search of time-recurrent exchanges of energy, usually called *beatings*, see for instance [16], [22]. In the present paper we consider "one-way" transfers of energy between two sets of modes, say from *low* to *high* modes (see the end of Section 2 for a comparison with periodic beatings). Roughly speaking, the nonlinearities play the role of *external parameters* that modulate the size of the high modes. This allows to obtain the desired growth in the Sobolev norms by choosing appropriately the nonlinear terms. Since the construction of the solutions that we provide is relatively simple we are able to give sharp bounds for the diffusion time.

In the last decades lots of effort has been put to obtain lower bounds of Sobolev norms for solutions of nonlinear Hamiltonian PDEs on compact manifolds. The first works in this direction are due to Bourgain [9], [10] and Kuksin [23]. In 2010 Colliander-Keel-Staffilani-Takaoka-Tao (I-team) proved an outstanding result [11] concerning the H^s-instability of the origin of the cubic nonlinear Schrödinger equation on $\mathbb{T}^2$ with $s > 1$ [1]. The construction of the unstable solutions is inspired by Arnold diffusion techniques: it relies on the presence of orbits of a finite dimensional good approximation of the equation (called *toy model*) that shadow a chain of invariant hyperbolic manifolds (periodic orbits in a suitable symplectic reduction). After this work many papers have been devoted to the analysis and extensions of that scheme, exclusively for NLS models (concerning results on other models, using a different approach, we cite the seminal works by Gérard-Grellier [13], [14] on the Szegö equation on the circle). In [20] Guardia-Kaloshin provide estimates for the diffusion time of solutions obtained by refining the analysis of the dynamics of the toy model, Guardia-Haus-Procesi [19] extended the result of the I-team to Schrödinger

[1] We remark that the energy provides a complete control of the H^1-norm.

equations with all types of analytic nonlinearities. The study of the H^s-instability of different invariant objects rather than elliptic fixed points have been carried out just recently. We mention the paper [21] by Hani for a proof of H^s-instability of plane waves and Guardia-Hani-Haus-Maspero-Procesi [18] for the case of finite gap solutions for $s \in (0,1)$. We point out that the I-team mechanism strongly relies on the fact that the Schrödinger equation possess several symmetries. In particular the dynamics of the resonant models considered in the aforementioned papers is rather special and far from being generic [2]. For this reason it seems hard to implement a similar strategy for different Hamiltonian PDEs. Then it is natural to look for alternative mechanisms and we believe that a first step in this direction should be to find out different ways of transferring energy between modes, even restricting the study to single resonant clusters. About this, we mention the very recent result [15] by Guardia-Martin-Pasquali and the author of the present paper concerning chaotic-like transfers of energy for the wave, beam and Hartree equations on $\mathbb{T}^2$.

2. **Main results.**

Let $s>0$, we introduce the Sobolev spaces

$$H^s := \left\{ u \in L^2(\mathbb{T}) : u = \sum_{j\in\mathbb{Z}} u_j\, e^{\mathrm{i}jx}\,,\ ||u||_s^2 := \sum_{j\in\mathbb{Z}} |u_j|^2 \langle j\rangle^{2s} \right\}$$

where $\langle j\rangle := \max\{1,|j|\}$. We define the following norm

$$||u||_{E^s} := ||u||_{s+1/2} + ||u_t||_{s-1/2}.$$

Notations. When we say that a parameter p is large (small) enough, or we write $p \gg 1$ ($p \ll 1$), we mean that there exists a universal constant p_0 large (small) enough such that $p \geq p_0$ ($p \leq p_0$).

The notation $p \lesssim q$ denotes that there exists a pure constant $C>0$ such that $p \leq Cq$. The notation $p = \mathcal{O}(q)$ denotes that there are two pure constants $0 < C_1 < C_2$ such that $C_1 q \leq p \leq C_2 q$.

The first result regards a class of nonlinear wave equations with even-power nonlinearities.

Theorem 2.1. *Let $s>2$. Given $\delta \ll 1$ and $\mathcal{C} \gg 1$ there exists $p_0 = p_0(s,\delta,\mathcal{C})$ such that for all **even** numbers $p \geq p_0$ the equation*

$$u_{tt} - \Delta u + V_p * u + u^p = 0, \tag{4}$$

where

$$V_p(x) := 1 + \cos(x) + p^2\cos(px), \tag{5}$$

possesses a solution $u(t,x)$ such that $||u(0)||_{E^s} \leq \delta$ and

$$\frac{||u(T)||_{E^s}}{||u(0)||_{E^s}} \geq \mathcal{C} \tag{6}$$

with

$$T = \mathcal{O}(2^p\, p^{-1}\ ||u(0)||_{E^s}^{1-p}). \tag{7}$$

[2] One can find arbitrarily long sequences of invariant manifolds on the same energy level. Moreover the heteroclinic connections between these manifolds are not transversal, in any possible sense.

The above theorem ensures the existence of a solution with arbitrarily small initial datum and arbitrarily large growth $\mathcal{C}$, but it does not guarantee that the norm at time T is arbitrarily large. However this can be obtained for sufficiently higher order Sobolev norms.

Theorem 2.2. *Given $\delta \ll 1$ and $K \gg 1$ there exists $p = p(\delta)$ such that the following holds.*

There exists $s_0 = s_0(p, K)$ such that the equation 4 where V_p as in 5, possesses a solution $u(t, x)$ such that

$$||u(0)||_{E^s} \leq \delta, \qquad ||u(T)||_{E^s} \geq K \qquad \forall s \geq s_0, \tag{8}$$

with T as in 7. Moreover if $K = \delta^{-\alpha}$ with $\alpha > 1$ we can provide the following polynomial time estimate with respect to the growth

$$T \lesssim \mathcal{C}^{p-1}, \qquad \mathcal{C} := \frac{K}{\delta}. \tag{9}$$

Some comments are in order:

- The nonlinearity in equation 4 can be replaced by any analytic function $f(u)$ with a zero of order p at the origin $u = 0$. Moreover the sign of the nonlinearity does not play any role.

  The degree p is used as a parameter to deal with increasingly high order resonances in the first step of a Birkhoff normal form procedure. More precisely, it turns out that the modes ± 1 (low modes) and $\pm p$ (high modes) are in resonance.
- The degeneracy of the resonant Hamiltonian of the nonlinear term is due to the evenness of the degree p. An evidence of the unstable behavior of wave equations with even-power nonlinearities on $\mathbb{T}$ is given, for instance, by the result of non-existence of periodic solutions given in [4].
- Although the results are rather different, it is interesting to notice that the diffusion time for arbitrarily small initial data solutions obtained through the I-team mechanism is super-exponential with respect to the growth, $\exp(\mathcal{C}^k)$ for some $k > 1$ (see for instance [19]), while 9 is a polynomial bound.
- We remark that the growth of H^s-Sobolev norms with $s \geq s_0$ for small data solutions obtained in Theorem 2.2 is not trivial, especially because the diffusion time does not increase with the index s.

  A result of this kind can be achieved if there is a transfer of energy and the norm of the initial datum remains small when s increases. For the case taken into account this happens because: (i) the Fourier support of the initial datum includes the modes ± 1, whose Sobolev weights are the same for all s, and (ii) the time of diffusion has lower and upper bounds that do not depend on the initial amount of energy of the high modes $\pm p$. The latter fact is due to the mechanism we are considering, which relies on the existence of diffusion channels (see Remark 7).

The next theorem concerns cubic NLS equations which are not x-translation invariant. A similar model has been considered in [17] for the search of time-periodic beating solutions.

Theorem 2.3. *Let $s > 0$. Given $\delta \ll 1$ and $K \gg 1$ there exists $N_0 = N_0(s, \delta, K)$ such that for all $N \geq N_0$ there exists a trigonometric polynomial $V_N : \mathbb{T} \to \mathbb{R}$ with real Fourier coefficients supported on 4 modes associated to wavenumbers k_1, k_2, k_3 and k_4, where $|k_1|, |k_2|, |k_3| \lesssim \sqrt{N}$ and $|k_4| = \mathcal{O}(N)$, such that the following holds:*

The equation

$$\mathrm{i}u_t - \Delta u + V_N * u + \cos(Nx)|u|^2 u = 0 \tag{10}$$

possesses a solution $u(t,x)$ such that

$$||u(0)||_s \leq \delta, \quad ||u(T)||_s \geq K$$

with

$$T = \mathcal{O}(N^s \; ||u(0)||_s^{-2}). \tag{11}$$

Moreover if $K = \delta^{-\alpha}$ for some $\alpha > 1$ we can provide the following polynomial time estimate with respect to the growth

$$T \lesssim \mathcal{C}^6, \qquad \mathcal{C} := \frac{K}{\delta}. \tag{12}$$

Some comments are in order:

- We remark that in dimension one the cubic NLS is completely integrable and, by the presence of infinitely many constants of motion, the Sobolev norms are controlled for all time. In equation 10 the Hamiltonian structure and the mass (or the L^2-norm) are still preserved, but the nonlinear term breaks the momentum conservation. The frequency of the cosine x-function is used as a parameter to involve modes of very different size scale in the 4-resonant interactions. Broadly speaking, the ratio between the size of the low and high modes turns out to be a power of N.
- We remark that when $V_N = 0$ the H^1-norm of solutions with small L^2-norm is controlled for all time. This can be seen by using that $\cos(Nx)$ is uniformly bounded in N and by applying the Gagliardo-Nirenberg inequality. When $V_N \neq 0$ the H^1-norm has still an upper bound for all time, but it is not uniform in N.

The convolution potentials V_p in 4 and V_N in 10 have the role to decouple the dynamics of the normalized Hamiltonian on a finite dimensional submanifold from the dynamics of the normal modes, but they still preserve the resonant nature of the equations. The solutions that we construct bifurcate from a periodic or quasi-periodic in time function $w(t,x)$ (see 20, 107) that is obtained as a solution of the linearized problem at the origin by exciting a finite number of modes. The orbit $w(\cdot,x)$ fills densely a lower dimensional submanifold of an embedded resonant torus. The solutions $u(t,x)$ provided by the above theorems remain close to $w(t,x)$ in a weak norm for $t \in [0,T]$, but clearly not in the H^s-topology.

The diffusion channels that we exploit are contained in the level sets of quadratic constants of motion, respectively the momentum and the mass for the wave equation 4 and the Schrödinger equation 10 (see 49 and 113). The unstable solutions that we obtained come in one-parameter families, parametrized by the values of the aforementioned first integrals.

Comparison with beating solutions. To optimize the ratio between the Sobolev norms at time $t = 0$ and at some other time $t = T$ we want to set the initial energy of the high modes almost at zero, say ε-small. Thanks to the use of diffusion channels the exchanging time turns out to have an upper bound independent of ε (see for example Lemma 3.3 and Remarks 7, 14). This is not the case if the same amount of energy is transferred among the modes of a periodic beating solution. These solutions are usually obtained following periodic orbits of a resonant model which are very close to heteroclinic or homoclinic loops. Let us suppose for simplicity that the transfer of energy involves just two modes. Setting the initial energy of one of the modes almost at zero corresponds to consider an orbit with very large period that visits a small neighborhood $\mathcal{U}$ of a saddle point (or a hyperbolic manifold). If the size of $\mathcal{U}$ is $\mathcal{O}(\varepsilon)$ then the time spent to escape from it is $\mathcal{O}(|\log(\varepsilon)|)$.

If one wants to prove that this kind of transfers of energy produces a growth of H^s-Sobolev norms for $s \gg 1$ then ε has to be chosen such that the norm at initial time does not increase with s. Then $\varepsilon \sim C^{-s}$ for some constant $C > 0$ (see for instance 95 and 114) and the factor $|\log(\varepsilon)|$ makes the diffusion time explodes as $s \to \infty$.

2.1. Scheme of the proofs.

The proofs of Theorems 2.1, 2.3 follow the same steps. Since the result on the nonlinear wave equations is more complicated we give full details for the proof of Theorem 2.1 and provide an outline of the proof for Theorem 2.3.

Let us briefly describe the general strategy. First we introduce the Hamiltonian structure of equations 4 and 10. Then we perform a Birkhoff normal form algorithm (see Propositions 1, 3), namely we provide a change of coordinates that remove some terms from the Hamiltonian that do not affect the dynamics in a neighborhood of the origin for a certain range of times. The Hamiltonian in normal form turns out to possess finite-dimensional invariant subspaces. The restriction of the normalized system to such spaces defines our *resonant model*. We analyze the dynamics of the resonant model by using action-angle variables [3]. We construct an orbit that exhibit a certain drift in the actions in the optimal diffusion time (see Lemmata 3.3, 4.1). Thanks to a rescaling argument and Gronwall lemma we prove that there exists a solution of the full PDE that remains close (in a weak norm) to the unstable orbit of the resonant model for sufficiently long time (see Propositions 2, 4). The last step consists in the proof of the bounds for the Sobolev norms of the solution at time $t = 0$ and $t = T$, where T is the rescaled diffusion time.

Acknowledgments. The idea of applying the theory of fast instability in the context of Hamiltonian PDEs has been suggested by V. Kaloshin to L. Biasco and M. Procesi some time ago. Recently L. B. and M. P. told me about that discussion and suggested me to read the papers [5], [8] about diffusion channels. This has inspired the present work. So I would like to thank them all.

I am grateful also to M. Guardia, R. Scandone, E. Haus and R. Feola for many useful comments and discussions.

This paper has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 757802.

3. Proof of Theorem 2.1.

3.1. Hamiltonian structure.

In this section we introduce a useful set of coordinates and we discuss the Hamiltonian structure of the equation 4. To simplify the notation we drop the subindex p from the potential, namely we write $V_p = V$. Let us denote by $\Lambda := -\Delta + V*$. The wave equation 4 can be written as a first order system

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -\Lambda u - u^p, \end{cases}$$

[3] We refer to [6] for the analysis of analogous finite-dimensional models.

that, in the following complex coordinates

$$z^+ := \frac{\Lambda^{1/4} u - \mathrm{i}\Lambda^{-1/4} v}{\sqrt{2}}, \quad z^- := \frac{\Lambda^{1/4} u + \mathrm{i}\Lambda^{-1/4} v}{\sqrt{2}}, \tag{13}$$

reads as

$$\begin{cases} \mathrm{i}\dot{z}^+ = -\Lambda^{1/2} z^+ - \mathsf{g}(z^+, z^-), \\ \mathrm{i}\dot{z}^- = \Lambda^{1/2} z^- + \mathsf{g}(z^+, z^-), \end{cases} \tag{14}$$

with

$$\mathsf{g}(z^+, z^-) := \frac{1}{\sqrt{2}} \Lambda^{-1/4} \left(\Lambda^{-1/4} \left(\frac{z^+ + z^-}{\sqrt{2}} \right) \right)^p .$$

Remark 1. We provide a solution of 14 that undergoes growth in the high Sobolev norms $||\cdot||_s$. Then it is easy to recover the same result for the norms $||\cdot||_{E^s}$ by undoing the change of coordinates 13.

We introduce the infinitely many coordinates

$$z^+ = \sum_{j\in\mathbb{Z}} z_j^+ e^{\mathrm{i}jx}, \quad z^- = \sum_{j\in\mathbb{Z}} z_j^- e^{-\mathrm{i}jx}$$

that transform the system 14 in an infinite dimensional system of ODEs in the unknowns (z_j^+, z_j^-), $j \in \mathbb{Z}$. Such system is equipped with a Hamiltonian structure given by the symplectic form $-\mathrm{i}\sum_{j\in\mathbb{Z}} dz_j^+ \wedge dz_j^-$, which in turn induces the Poisson structure

$$\{F, G\} := -\mathrm{i} \sum_{j\in\mathbb{Z}} \big(\partial_{z_j^+} F\, \partial_{z_j^-} G - \partial_{z_j^-} F\, \partial_{z_j^+} G\big), \tag{15}$$

where F and G are two real-valued functions defined on the phase space. The Hamiltonian of 14 is given by

$$\begin{aligned} H(z_j^+, z_j^-) = & \sum_{j\in\mathbb{Z}} \omega(j) z_j^+ z_j^- \\ & + \frac{1}{(p+1)\sqrt{2}^{(p+1)}} \sum_{j_1+\dots+j_{p+1}=0} \frac{(z_{j_1}^+ + z_{-j_1}^-)\dots(z_{j_{p+1}}^+ + z_{j_{-(p+1)}}^-)}{\sqrt{\omega(j_1)\dots\omega(j_{p+1})}}, \end{aligned} \tag{16}$$

where

$$\omega(j) := \sqrt{j^2 + V_j}, \quad j \in \mathbb{Z}$$

are the linear frequencies of oscillation. We shall look for a solution mainly Fourier supported on the symmetric *tangential* set

$$S := S^+ \cup S^-, \quad S^{\pm} := \{\pm 1, \pm p\}.$$

By the choice of the convolution potential as in 5 the linear frequencies of oscillation are given by

$$\omega(j) := \begin{cases} 1 & j = 0, \\ \sqrt{2}|j| & j \in S, \\ |j| & j \notin S \cup \{0\}. \end{cases} \tag{17}$$

Remark 2. We could choose $V_0 = n$ with an integer $n \geq 1$. Moreover $\sqrt{2}$ can be replaced by any badly approximable number. Indeed all we need is to use the fact that

$$|\sqrt{2}n + m| \geq \frac{\gamma}{n} \quad \forall n, m \in \mathbb{Z}, \quad n \neq 0 \tag{18}$$

for some $\gamma \in (0, 1)$.

Remark 3. The frequencies $\omega(j)$ with $j \in S$ (the *tangential frequencies*) are irrational, while the normal frequencies are integer numbers. This is the key property that allows to decouple the resonant dynamics of the tangential and normal modes.

We point out that the real subspace

$$\mathtt{R} := \left\{ \overline{z_j^+} = z_j^- \right\} \tag{19}$$

is invariant under the flow of H. Since we shall work on $\mathtt{R}$ it is convenient to adopt the following notation

$$z_j := z_j^+, \quad \overline{z_j} := z_j^- .$$

We observe that by exciting the tangential modes $\{\pm 1, \pm p\}$ we obtain a solution $w(t,x)$ of the linearized problem at the origin

$$\mathrm{i}\dot{z}_j = \omega(j)\, z_j, \quad j \in \mathbb{Z},$$

of the form

$$w(t,x) = \sum_{k=1,N} a_{\pm k}\, e^{\pm \mathrm{i}\sqrt{2}kt} \cos(kx), \quad a_{\pm k} \in \mathbb{C}. \tag{20}$$

These linear solutions can be seen as periodic motions supported on invariant embedded tori of dimension 4. We expect that even a small perturbation, which here is provided by the nonlinearity, may destroy these resonant manifolds and give rise to diffusive orbits.

We write the Hamiltonian 16 as $H = H^{(2)} + H^{(p+1)}$ where

$$\begin{aligned} H^{(2)}(z,\overline{z}) :=& \sum_{j\in\mathbb{Z}} \omega(j) z_j \overline{z_j}, \\ H^{(p+1)}(z,\overline{z}) :=& \frac{1}{(p+1)\sqrt{2}^{(p+1)}} \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^{\mathbb{Z}}\times\mathbb{N}^{\mathbb{Z}}, \\ |\alpha|+|\beta|=p+1, \\ \pi(\alpha,\beta)=0}} C_{\alpha,\beta}\, z^\alpha\, \overline{z}^\beta \end{aligned} \tag{21}$$

with $|\alpha| = \sum_{j\in\mathbb{Z}} \alpha_j$, $\pi(\alpha,\beta) := \sum_{j\in\mathbb{Z}} j\,(\alpha_j - \beta_j)$,

$$z^\alpha := \prod_{j\in\mathbb{Z}} z_j^{\alpha_j}, \quad \overline{z}^\beta := \prod_{j\in\mathbb{Z}} \overline{z_j}^{\beta_j}$$

and the coefficients

$$C_{\alpha,\beta} := \frac{(p+1)!}{\alpha!\beta!} \prod_{j\in\mathbb{Z}} \omega(j)^{-\frac{\alpha_j+\beta_j}{2}} \in \mathbb{R}. \tag{22}$$

Remark 4. We observe that a monomial $z^\alpha\,\overline{z}^\beta$ commutes with the momentum Hamiltonian

$$M(z,\overline{z}) := -\mathrm{i} \sum_{j\in\mathbb{Z}} j |z_j|^2 \tag{23}$$

if and only if $\pi(\alpha,\beta) = 0$.

The vector field of H is defined by components as

$$X_H := \left(X_H^{(z_j)}, X_H^{(\overline{z}_j)} \right), \quad X_H^{(z_j)} := \mathrm{i}\partial_{\overline{z_j}} H, \quad X_H^{(\overline{z_j})} := -\mathrm{i}\partial_{z_j} H, \quad j \in \mathbb{Z}.$$

3.2. **Birkhoff normal form.** In this section we perform a Birkhoff normal form procedure in order to highlight the terms of the Hamiltonian 21 which give the effective dynamics of equation 4 for a certain range of time. We shall work on the space of sequences

$$\ell^1 := \left\{ z\colon \mathbb{Z} \to \mathbb{C} \,|\, ||z||_{\ell^1} := \sum_{j\in\mathbb{Z}} |z_j| < \infty \right\}. \tag{24}$$

We point out that ℓ^1 is an algebra with respect to the convolution product. We denote by B_η the ball centered at the origin of ℓ^1 with radius $\eta > 0$, namely

$$B_\eta := \{z \in \ell^1 : ||z||_{\ell^1} \leq \eta\}.$$

Definition 3.1. Let

$$F = F(z,\overline{z}) := \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^{\mathbb{Z}}\times\mathbb{N}^{\mathbb{Z}},\\ |\alpha|+|\beta|=q,\\ \pi(\alpha,\beta)=0}} F_{\alpha,\beta}\, z^\alpha\, \overline{z}^\beta$$

be a homogeneous Hamiltonian of degree $q \geq 2$ preserving momentum. We give the following definitions:
(i) Let $0 \leq k \leq q$, we denote by $F^{(q,k)}$ the projection of F on the monomials supported on

$$\mathcal{A}_{q,k} := \left\{(\alpha,\beta) \in \mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}} : \pi(\alpha,\beta) = 0,\ |\alpha| + |\beta| = q,\ \#(\alpha,\beta,S^c) = k\right\},$$

where $\#(\alpha,\beta,S^c) := \sum_{j\notin S}(|\alpha_j| + |\beta_j|)$.

Similarly we define $F^{(q,\leq k)}$, $F^{(q,\geq k)}$ as the projection of F on the monomials supported respectively on

$$\mathcal{A}_{q,\leq k} = \bigcup_{i=1,\dots,k} \mathcal{A}_{q,i}, \quad \mathcal{A}_{q,\geq k} = \bigcup_{i=k,\dots,q} \mathcal{A}_{q,i}.$$

(ii) We define the following norms

$$[\![F]\!] := \sup_{(\alpha,\beta)} |F_{\alpha,\beta}|, \qquad ||X_F||_\eta := \sum_{j\in\mathbb{Z}} \sup_{B_\eta} \left| X_F^{(z_j)} \right|.$$

Lemma 3.2. *Let F, G be two homogeneous Hamiltonians preserving momentum of degree q and $\tilde{q}$ respectively. The Poisson bracket $\{F, G\}$ defined in 15 is a homogeneous Hamiltonian preserving momentum of degree $q + \tilde{q} - 2$.*

Moreover we have the following estimates

$$|F(z,\overline{z})| \leq [\![F]\!] ||z||_{\ell^1}^q, \tag{25}$$

$$||X_F(z,\overline{z})||_{\ell^1} \leq q\, [\![F]\!] ||z||_{\ell^1}^{q-1}, \tag{26}$$

$$[\![\{F,G\}]\!] \leq q\, \tilde{q}\, [\![F]\!] [\![G]\!]. \tag{27}$$

Proof. The proof follows the same lines of the proof of Lemma 3.2 in [16]. □

We denote by Π_{Ker} and Π_{Rg} the projection respectively on the kernel and the range of the adjoint action of $H^{(2)}$

$$\mathrm{ad}_{H^{(2)}}[F] := \{F, H^{(2)}\}.$$

The adjoint action of $H^{(2)}$ is diagonal on the monomials $z^\alpha \overline{z}^\beta$ with eigenvalues $-\mathrm{i}\Omega(\alpha,\beta)$, where

$$\Omega(\alpha,\beta) := \sum_{j\in\mathbb{Z}} \omega(j)(\alpha_j - \beta_j).$$

Proposition 1. ***(Birkhoff normal form)*** *Recall 21. There exists $\eta > 0$ small enough such that there exists a symplectic change of coordinates $\Gamma\colon B_\eta \to B_{2\eta}$ which takes the Hamiltonian H into its (partial) Birkhoff normal form up to order $p+1$, namely*

$$H \circ \Gamma = H^{(2)} + H_{\mathrm{res}} + H^{(p+1,\geq 2)} + R \tag{28}$$

where:

(i) The resonant Hamiltonian is given by

$$H_{\mathrm{res}} := \Pi_{\mathrm{Ker}} H^{(p+1,0)} = \frac{1}{\sqrt{2}^{p+1}}\big(2\Re(z_1^p\,\overline{z_p}\,) + 2\Re(z_{-1}^p\,\overline{z_{-p}}\,)\big). \tag{29}$$

(ii) The remainder R is such that

$$||X_R||_\eta \leq C_1\gamma^{-1}\eta^{2p-1} + \widetilde{C_1}\gamma^{-2}\eta^{3p-2} \tag{30}$$

with

$$C_1 = 2^{p-1}p^3((p+1)!)^2, \quad \widetilde{C_1} = (3p-1)\,p^5 2^{\frac{3}{2}p-\frac{5}{2}}\,((p+1)!)^3. \tag{31}$$

Moreover the map Γ is invertible and close to the identity

$$||\Gamma^{\pm 1} - \mathrm{Id}||_\eta \leq C_0\,\gamma^{-1}\,\eta^p \tag{32}$$

with

$$C_0 = \sqrt{2}^{p-1}p^2(p+1)!$$

and it preserves the real subspace $\mathtt{R}$ *in 19.*

Remark 5. With a slight abuse of notation we have renamed the Birkhoff coordinates $(z_j)_j$ as the original ones.

Proof. We consider the following homogeneous Hamiltonian

$$F = \sum_{\mathcal{A}_{p+1,\leq 1}} F_{\alpha,\beta} z^\alpha\,\overline{z}^\beta$$

with

$$F_{\alpha,\beta} := \begin{cases} \dfrac{\mathrm{i}\,C_{\alpha,\beta}}{\Omega(\alpha,\beta)\sqrt{2}^{p+1}(p+1)} & \text{if } \Omega(\alpha,\beta)\neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

The coefficients $F_{\alpha,\beta}$ are uniformly bounded because $\mathcal{A}_{p+1,\leq 1}$ is a finite set. Thus the Hamiltonian F is well defined and by using Young's inequality is easy to prove that the associated equation is locally well posed in ℓ^1. By a standard bootstrap argument one can prove that, provided η is small enough, the flow Φ_F^t maps B_η into $B_{2\eta}$ for $t\in[0,1]$. We define $\Gamma := \Phi_F^1$.

By the definition of $F_{\alpha,\beta}$ and the fact that $\omega(j)\in\mathbb{R}$ for all $j\in\mathbb{Z}$ (see 17) the vector field X_F preserves the real subspace $\mathtt{R}$ in 19.

By definition F satisfies the following homological equation

$$\{F, H^{(2)}\} + H^{(p+1,\leq 1)} = \Pi_{\mathrm{Ker}} H^{(p+1,\leq 1)}. \tag{33}$$

We claim that

$$\Pi_{\mathrm{Ker}} H^{(p+1,1)} = 0. \tag{34}$$

If $(\alpha,\beta)\in\mathcal{A}_{p+1,1}$ then there exists $j\notin S$ such that (note that $\omega(j)=\omega(-j)$ for all $j\in\mathbb{Z}$)

$$\Omega(\alpha,\beta)=\omega(1)(\alpha_1+\alpha_{-1}-\beta_1-\beta_{-1})+\omega(p)(\alpha_p+\alpha_{-p}-\beta_p-\beta_{-p})+|j|(\alpha_j-\beta_j)$$

with $(\alpha_j,\beta_j)=(1,0)$ or $(\alpha_j,\beta_j)=(0,1)$ and $\sum_{i\in S}\alpha_i+\beta_i=p$. Then by 17 we have that $\Omega(\alpha,\beta)=\sqrt{2}n+m$ for some $n,m\in\mathbb{Z}$ with $0<|n|\le p^2$, hence (see 18)

$$|\Omega(\alpha,\beta)|\ge\frac{\gamma}{p^2}>0. \tag{35}$$

This proves the claim 34. Now we prove the estimate 32 on the map Γ. By 22, 17 we have

$$|C_{\alpha,\beta}|\le(p+1)!. \tag{36}$$

If $(\alpha,\beta)\in\mathcal{A}_{p+1,0}$ and $\Omega(\alpha,\beta)\neq0$ then by the definition of the tangential frequencies in 17

$$|\Omega(\alpha,\beta)|\ge\sqrt{2}.$$

Hence by Lemma 3.2-26 we have

$$[\![F]\!]\le\gamma^{-1}\sqrt{2}^{-(p+1)}p^2p!,\qquad ||X_F||_\eta\le\gamma^{-1}\sqrt{2}^{-(p+1)}p^2(p+1)!\eta^p. \tag{37}$$

By the mean value theorem and using that $\Phi_F^t\colon B_\eta\to B_{2\eta}$ for $t\in[0,1]$ we have

$$||\Gamma(z)-z||_\eta\le\sup_{t\in[0,1]}||X_F(\Phi_F^t(z))||_\eta\le\gamma^{-1}\sqrt{2}^{p-1}p^2(p+1)!\eta^p.$$

This gives the bound 32 for Γ. If η is small enough we can invert Γ by Neumann series and get a similar bound for the inverse.

Now we prove formula 29. By 34 we focus on monomials of the following form

$$\prod_{i\in S}z_i^{\alpha_i}\,\overline{z_i}^{\beta_i}$$

where

$$\sum_{i\in S}\alpha_i+\sum_{i\in S}\beta_i=p+1,\qquad \alpha_i,\beta_i\ge0. \tag{38}$$

These monomials are resonant if (recall 17)

$$\big(\alpha_1+\alpha_{-1}-\beta_1-\beta_{-1}\big)+p(\alpha_p+\alpha_{-p}-\beta_p-\beta_{-p})=0. \tag{39}$$

While the momentum conservation reads as

$$\big(\alpha_1-\alpha_{-1}-\beta_1+\beta_{-1}\big)+p(\alpha_p-\alpha_{-p}-\beta_p+\beta_{-p})=0. \tag{40}$$

First we observe that 39 implies

$$p|\alpha_p+\alpha_{-p}-\beta_p-\beta_{-p}|=|\alpha_1+\alpha_{-1}-\beta_1-\beta_{-1}|. \tag{41}$$

If $|\alpha_p+\alpha_{-p}-\beta_p-\beta_{-p}|\neq0$ then by 38 we have $p+1\ge|\alpha_1+\alpha_{-1}-\beta_1-\beta_{-1}|\ge p$.

The case $|\alpha_1+\alpha_{-1}-\beta_1-\beta_{-1}|=p+1$ is clearly impossible since the left hand side of 41 is even. Thus

$$|\alpha_1+\alpha_{-1}-\beta_1-\beta_{-1}|=p,\quad |\alpha_p+\alpha_{-p}-\beta_p-\beta_{-p}|=1. \tag{42}$$

By the latter equality we deduce that there is exactly one integer number in $\{\alpha_{\pm p},\beta_{\pm p}\}$ which is equal to 1, while all the others are zero. Then 40 implies that

$$|\alpha_1-\alpha_{-1}-\beta_1+\beta_{-1}|=p. \tag{43}$$

It is easy to see that 42 and 43 imply $\alpha_1=\beta_1$ or $\alpha_{-1}=\beta_{-1}$. Without loss of generality suppose that $\alpha_1=\beta_1$, then $|\alpha_{-1}-\beta_{-1}|=p$. This means that $(\alpha_{-1},\beta_{-1})=(p,0)$ or $(\alpha_{-1},\beta_{-1})=(0,p)$ (and so by 38 $\alpha_1=\beta_1=0$).

The resonant monomials corresponding to these cases are

$$z_p\overline{z_1}^p, \quad z_{-p}\overline{z_{-1}}^p$$

and their complex conjugate . We are left with the case $|\alpha_p + \alpha_{-p} - \beta_p - \beta_{-p}| = 0$. By 41 we have

$$|\alpha_1 + \alpha_{-1} - \beta_1 - \beta_{-1}| = 0.$$

By the fact that $\alpha_i, \beta_i \geq 0$ this implies that

$$\alpha_j + \alpha_{-j} = \beta_j + \beta_{-j}, \qquad j = 1, p.$$

Then 38 becomes $2(\alpha_1 + \alpha_{-1} + \alpha_p + \alpha_{-p}) = p + 1$, which is a contradiction since $p + 1$ is odd.

Now we prove 28. The new Hamiltonian is obtained by Taylor expanding $H \circ \Phi_F^t$ at $t = 0$. We have

$$\begin{aligned}
H \circ \Gamma &= H + \{F, H\} + \frac{1}{2}\int_0^1 (1-t)\{F, \{F, H\}\} \circ \Phi_F^t \, dt \\
&= H^{(2)} + \{F, H^{(2)}\} + H^{(p+1,\leq 1)} + H^{(p+1,\geq 2)} + \{F, H^{(p+1)}\} \\
&\quad + \frac{1}{2}\int_0^1 (1-t)\{F, \{F, H^{(2)}\}\} \circ \Phi_F^t \, dt + \frac{1}{2}\int_0^1 (1-t)\{F, \{F, H^{(p+1)}\}\} \circ \Phi_F^t \, dt \\
&\overset{33,34}{=} H^{(2)} + \Pi_{\mathrm{Ker}} H^{(p+1,0)} + H^{(p+1,\geq 2)} + R
\end{aligned}$$

where

$$\begin{aligned}
R :=& \{F, H^{(p+1)}\} - \frac{1}{2}\int_0^1 (1-t)\{F, \Pi_{\mathrm{Rg}} H^{(p+1,\leq 1)}\} \circ \Phi_F^t \, dt \\
&+ \frac{1}{2}\int_0^1 (1-t)\{F, \{F, H^{(p+1)}\}\} \circ \Phi_F^t \, dt.
\end{aligned}$$

By using Lemma 3.2-27 and the bounds 37, 36 we obtain the following estimates

$$\begin{aligned}
[\![H^{(p+1)}]\!] &\leq p! \sqrt{2}^{-(p+1)}, \\
[\![\{F, H^{(p+1)}\}]\!] &\leq \gamma^{-1} 2^{-(p+1)} p^2 ((p+1)!)^2, \\
[\![\{F, \{F, H^{(p+1)}\}\}]\!] &\leq \gamma^{-2} p^5 2^{-\frac{3}{2}p - \frac{1}{2}} ((p+1)!)^3.
\end{aligned}$$

Then using again Lemma 3.2-26 we get the estimate 30. □

3.3. **Dynamics of the resonant model.** We introduce the rotating coordinates

$$z_j = r_j \, e^{\mathrm{i}\omega(j)t}$$

in order to remove the quadratic part of the Hamiltonian $H \circ \Gamma$. If z is a solution of 28 then r satisfies the equation associated to the Hamiltonian

$$\mathcal{H} = H_{\mathrm{res}} + \mathcal{Q}(t) + \mathcal{R}(t) \tag{44}$$

where

$$\begin{aligned}
\mathcal{Q}\big((r_j)_{j\in\mathbb{Z}}, t\big) &:= H^{(p+1,\geq 2)}\big(r_j \, e^{\mathrm{i}\omega(j)t}\big), \\
\mathcal{R}\big((r_j)_{j\in\mathbb{Z}}, t\big) &:= R\big(r_j \, e^{\mathrm{i}\omega(j)t}\big).
\end{aligned} \tag{45}$$

The next step is to study the dynamics of the resonant Hamiltonian H_{res}.

We observe that the finite dimensional submanifold

$$\mathcal{U}_S := \{r \colon \mathbb{Z} \to \mathbb{C} \,|\, r_j = 0 \;\; j \notin S\}$$

is invariant by the flow of $H_{\rm res}$. We introduce the following action-angle variables on $\mathcal{U}_S$

$$r_j = \sqrt{I_j}\, e^{\mathrm{i}\theta_j} \quad j \in S.$$

The reduced Hamiltonian now reads as (recall 29)

$$\mathcal{G} := \mathcal{G}^+ + \mathcal{G}^-, \quad \mathcal{G}^\pm = \sqrt{2}^{1-p}\, I_{\pm 1}^{p/2}\, \sqrt{I_{\pm p}}\, \cos(p\,\theta_{\pm 1} - \theta_{\pm p}). \tag{46}$$

We observe that $\mathcal{G}$ is the sum of two uncoupled integrable Hamiltonians, indeed both $\mathcal{G}^\pm$ have one degree of freedom in an opportune reduction. Then it makes sense to analyze just $\mathcal{G}^+$.

Remark 6. The partial momenta $\mathtt{M}_\pm := \pm I_{\pm 1} \pm p I_{\pm p}$ are constants of motion for $\mathcal{G}_\pm$ respectively.

The following lemma provides the existence of an orbit that exhibit a large drift in one of its actions.

Lemma 3.3. *Let $\varepsilon > 0$ be arbitrarily small and $\mathtt{c} > p\varepsilon$. There exists an orbit of $\mathcal{G}^+$*

$$g^+_{\varepsilon,\mathtt{c}}(t) = (\theta_1(t), \theta_p(t), I_1(t), I_p(t))$$

such that

$$\begin{aligned} I_1(0) &= \mathtt{c} - p\,\varepsilon, & I_p(0) &= \varepsilon, \\ I_1(T_0) &= \mathtt{c}(1-p^{-1}), & I_p(T_0) &= \frac{\mathtt{c}}{p^2}, \end{aligned} \tag{47}$$

with

$$\frac{\sqrt{2}^{p+1}}{\mathtt{c}^{\frac{p-1}{2}}}\, p^{-1} \le T_0 \le \frac{\sqrt{2}^{p+1}}{\mathtt{c}^{\frac{p-1}{2}}\,(1-p^{-1})^{p/2}}\, p^{-1}. \tag{48}$$

Proof. We consider the following linear symplectic change of coordinates

$$\varphi_1 = \theta_1, \quad \varphi_p = -p\,\theta_1 + \theta_p, \quad J_1 = I_1 + p\,I_p, \quad J_p = I_p.$$

The new Hamiltonian reads as

$$\mathcal{G}_* = \lambda\,(J_1 - pJ_p)^{p/2}\sqrt{J_p}\,\cos(\varphi_p).$$

Since $J_1 = \mathtt{M}_+ = I_1 + pI_p$ is a constant of motion for $\mathcal{G}_*$ we can look for solutions traveling along the diffusion channel

$$\{(\mathtt{c} - pI_p, I_p) : I_p \in (0, \mathtt{c}/p)\} = \{J_1 = \mathtt{M}_+ = \mathtt{c}\}. \tag{49}$$

We fix the section $\{\varphi_p = \pi/2\}$, which is invariant by the flow of $\mathcal{G}_*$. The dynamics of $J_p = I_p$ is determined by the equation

$$\dot{J}_p = \lambda\,(\mathtt{c} - pJ_p)^{p/2}\sqrt{J_p} =: f(J_p).$$

The function f is Lipschitz continuous and strictly positive in the interval $[\varepsilon, \mathtt{c}/p)$. Hence we can easily conclude that there exists an orbit with initial condition $J_p(0) = \varepsilon$ which is monotone increasing and reach the value $\mathtt{c}/p^2$ at time

$$T_0 := \sqrt{2}^{p-1}\int_\varepsilon^{\frac{\mathtt{c}}{p^2}} \frac{1}{(\mathtt{c} - p\,J_p)^{p/2}\sqrt{J_p}}\, dJ_p.$$

By using the fact that

$$\mathtt{c} - p\varepsilon \ge \mathtt{c} - p\,J_p \ge \mathtt{c}(1-p^{-1})$$

on the interval of integration, we get the bounds 48. □

Remark 7. The time of diffusion T_0 has lower and upper bounds that do not depend on $\varepsilon = I_p(0)$, see the first line in 47.

Remark 8. By 49 and the fact that $J_p = I_p$ is monotone increasing in the time interval $[0, T_0]$ we have that

$$\sup_{t\in[0,T_0]} I_1(t) = I_1(0) < \mathtt{c}.$$

By the discussion above it is clear that Lemma 3.3 provides also an orbit $g^-_{\varepsilon,\mathtt{c}}(t)$ for $\mathcal{G}^-$ with the same evolution of the actions I_{-1}, I_{-p} as in 47. We consider the family of solutions $g^\pm_{\varepsilon,\mathtt{c}}(t)$ given by the Lemma 3.3 with

$$\mathtt{c} \in [\mathtt{c}_-, \mathtt{c}_+], \qquad \varepsilon > 0, \tag{50}$$

where $\mathtt{c}_\pm$ and ε shall be chosen later and shall depend respectively on p and (p, s).

We define $b(\varepsilon, \mathtt{c}; t, x) = b(t, x) = \sum_{j\in\mathbb{Z}} b_j(t) e^{\mathrm{i}jx}$ with

$$b_j(t) := \begin{cases} \sqrt{I_j(t)}\, e^{\mathrm{i}\theta_j(t)} & j \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then the function $b(t, x)$ is a solution of H_{res} such that

$$\begin{aligned} |b_{\pm1}(0)|^2 &= \mathtt{c} - \varepsilon\, p, & |b_{\pm p}(0)|^2 &= \varepsilon, \\ |b_{\pm1}(T_0)|^2 &= \mathtt{c}(1 - p^{-1}), & |b_{\pm p}(T_0)|^2 &= \frac{\mathtt{c}}{p^2}. \end{aligned} \tag{51}$$

By 48 we have

$$\frac{\sqrt{2}^{p+1}}{p\sqrt{\mathtt{c}_+}^{p-1}} \le T_0 \le \frac{\sqrt{2}^{p+3}}{p\sqrt{\mathtt{c}_-}^{p-1}}. \tag{52}$$

3.4. **Approximation argument.** In this section we show that solutions of the Hamiltonian 44 whose initial conditions are ℓ^1-close enough to the initial datum of (an opportune rescaling of) $b(t, x)$ remain ℓ^1-close to it for all the time that we need to appreciate the drift in the actions 51.

The solutions $u(t, x)$ of H_{res} are invariant under the rescaling

$$u(t, x) \to \mu^{-1} u(\mu^{-p+1} t, x).$$

Given $\mu > 0$ we consider the rescaled solution

$$r^\mu(t, x) := \mu^{-1} b(\mu^{-p+1} t, x). \tag{53}$$

We remark the dependence of $r^\mu(t, x) = r^\mu(\varepsilon, \mathtt{c}; t, x)$ on the parameters ε, the initial value of the energy of the high modes, and $\mathtt{c}$, the height of the partial momenta. The diffusion time is rescaled in the following way

$$T := \mu^{p-1} T_0, \tag{54}$$

hence we need to ensure a good approximation of $r^\mu(t)$ by solutions $r(t)$ of 44 at least in the range of time $[0, T]$. By Remark 8 we have

$$||r^\mu(t)||_{\ell^1} \le 4\sqrt{\mathtt{c}}\mu^{-1} \qquad \text{for } t \in [0, T]. \tag{55}$$

Let us define (recall γ in 18)

$$\mu_0 := p^{\mathtt{a}} \gamma^{-\mathtt{b}}, \quad \mathtt{a} > 0, \quad \mathtt{b} := \frac{1}{\frac{p}{4} - 2}, \quad \mathtt{r} := \frac{\mathtt{c}_+}{\mathtt{c}_-}, \tag{56}$$

where $\mathtt{a}$ is a constant to be chosen large enough.

Proposition 2. *Let $\varepsilon > 0$. There exists a universal constant $p_0 > 0$ such that for all $p \geq p_0$ the following holds. If we consider*

$$\mathtt{a} \geq \, p\, p!\, 4^{p+1}, \qquad \mathtt{c} \in [\mathtt{c}_-, \mathtt{c}_+], \tag{57}$$

$$\mathtt{r}^{\frac{p-1}{2}} = 2, \quad \mathtt{c}_- \geq 1, \quad \mathtt{c}_+ \leq 2^{2-p^{-1}} (p!)^{2p^{-1}} p^{\frac{\mathtt{a}}{2}-14}, \tag{58}$$

then for all $\mu \geq \mu_0 = \mu_0(\mathtt{a}, p)$ in 56 we have that if $r(t)$ is a solution of 44 with initial datum satysfying

$$||r(0) - r^\mu(\varepsilon, \mathtt{c}; 0)||_{\ell^1} \leq \mu^{-\sigma_1}, \quad \sigma_1 := \frac{3p}{4} + 2, \tag{59}$$

then

$$||r(t) - r^\mu(\varepsilon, \mathtt{c}; t)||_{\ell^1} \leq 2\mu^{-\sigma_2}, \quad \sigma_2 := \sigma_1 - \frac{1}{2}, \quad \text{for } t \in [0, T]. \tag{60}$$

Proof. We set $\xi := r - r^\mu$ and we study the evolution of its ℓ^1-norm. We observe that $\Pi_S^\perp \xi = \Pi_S^\perp r$, where $\Pi_S^\perp$ denotes the projection on the normal modes. We have that $\dot{\xi} = Z_0(t) + Z_1(t)\,\xi + Z_2(t, \xi)$ with (recall 45)

$$Z_0 := X_{\mathcal{R}}(r_\mu),$$
$$Z_1 := DX_{H_{\mathrm{res}}}(r^\mu),$$
$$Z_2 := X_{H_{\mathrm{res}}}(r) - X_{H_{\mathrm{res}}}(r^\mu) - DX_{H_{\mathrm{res}}}(r^\mu)\xi + X_{\mathcal{R}}(r) - X_{\mathcal{R}}(r_\mu) + X_{\mathcal{Q}}(r) - X_{\mathcal{Q}}(r_\mu).$$

By the differential form of Minkowsky's inequality we get

$$\frac{d}{dt}||\xi||_{\ell^1} \leq ||Z_0(t)||_{\ell^1} + ||Z_1(t)\,\xi||_{\ell^1} + ||Z_2(t)||_{\ell^1}.$$

Under suitable conditions on the parameters $\mathtt{a}, \mathtt{c}_\pm, p$, we provide bounds on the terms of the right hand side of the above inequality. Later we shall prove that such conditions are satisfied by the assumptions 57, 58. Let us denote

$$\Xi = \Xi(p, \mathtt{c}_-) := \sqrt{2}^{3p-1} p^2 p! \sqrt{\mathtt{c}_-}^{p-1}. \tag{61}$$

We shall impose

- an upper bound for $\mathtt{c}_+$ to obtain the bound of Z_0;
- an upper bound on $\mathtt{r}$ to obtain the bound of Z_1;
- an upper bound on $\mathtt{r}$ and a lower bound for $\mathtt{c}_-$ to obtain the bound of Z_2.

Bound for Z_0: By 30, 55 we have

$$||Z_0||_{\ell^1} \leq C_1 \gamma^{-1} (4\sqrt{\mathtt{c}})^{2p-1} \mu^{-2p+1} + \widetilde{C_1} \gamma^{-2} (4\sqrt{\mathtt{c}})^{3p-2} \mu^{-3p+2}.$$

- We claim that, under the following conditions

$$\mathtt{r}^{\frac{1-p}{2}} \sqrt{2}^{3(p-1)} p!\, p^{\mathtt{a}(\frac{p}{4}-2)-7p} \geq \sqrt{\mathtt{c}}_+^{p}, \tag{62}$$

$$\mathtt{r}^{\frac{1-p}{2}} \sqrt{2}^{3(p-1)} p!\, p^{\mathtt{a}(\frac{5p}{4}-3)-\frac{21\,p}{2}-\frac{3}{2}} \geq \sqrt{\mathtt{c}}_+^{2p-1}, \tag{63}$$

  we have

$$||Z_0||_{\ell^1} \leq \Xi \mu^{-(7/4)p-1}. \tag{64}$$

We want to impose that

$$\begin{aligned} &C_1 (4\sqrt{\mathtt{c}})^{2p-1} \gamma^{-1} \mu^{-2p+1} \leq (\Xi/2) \mu^{-(7/4)p-1}, \\ &\widetilde{C_1} \gamma^{-2} (4\sqrt{\mathtt{c}})^{3p-2} \mu^{-3p+2} \leq (\Xi/2) \mu^{-(7/4)p-1}. \end{aligned} \tag{65}$$

The first inequality in 65 is equivalent to $\mu^{\frac{p}{4}-2}\gamma \geq (2/\Xi) C_1 (4\sqrt{\mathtt{c}})^{2p-1}$.

Thus we ask for

$$\mu_0^{\frac{p}{4}-2}\gamma = p^{\mathtt{a}(p/4-2)}\gamma^{1-\mathtt{b}(\frac{p}{4}-2)} \overset{56}{=} p^{\mathtt{a}(p/4-2)} \geq (2/\Xi)C_1\,(4\sqrt{\mathtt{c}})^{2p-1} \overset{31}{=} (1/\Xi)\,2^p p^3((p+1)!)^2\,(4\sqrt{\mathtt{c}})^{2p-1}\,. \tag{66}$$

If $p \geq 4$ then

$$2^p p^3((p+1)!)^2\,(4\sqrt{\mathtt{c}})^{2p-1} \leq p^{7p+2}\,\sqrt{\mathtt{c}_+}^{2p-1}.$$

Therefore condition 62 implies 66.

The second inequality in 65 is equivalent to $\mu^{\frac{5p}{4}-3}\gamma^2 \geq (2/\Xi)\,\widetilde{C_1}(4\sqrt{\mathtt{c}})^{3p-2}$. Then it is sufficient to ask for

$$\mu_0^{\frac{5p}{4}-3}\gamma^2 = p^{\mathtt{a}(\frac{5p}{4}-3)}\gamma^{2-\mathtt{b}(\frac{5p}{4}-3)} \geq (2/\Xi)\widetilde{C_1}(4\sqrt{\mathtt{c}})^{3p-2} \tag{67}$$

where

$$(2/\Xi)\widetilde{C_1}(4\sqrt{\mathtt{c}})^{3p-2} \overset{31}{=} (2/\Xi)(3p-1)\,p^5 2^{\frac{3}{2}p-\frac{5}{2}}\,((p+1)!)^3(4\sqrt{\mathtt{c}})^{3p-2}.$$

By the definition of $\mathtt{b}$ in 56, we have $2-\mathtt{b}(\frac{5p}{4}-3) < 0$, thus we can disregard the presence of γ in the above inequality. If $p \geq 4$ then

$$(3p-1)\,p^5 2^{\frac{3}{2}(p-1)}\,((p+1)!)^3(4\sqrt{\mathtt{c}})^{3p-2} \leq p^{\frac{7}{2}(3p+1)}\sqrt{\mathtt{c}_+}^{3p-2}.$$

Then condition 63 implies 67. This proved the claim.

Bound for Z_1: Assume that

$$\mathtt{r}^{\frac{p-1}{2}} \leq \frac{2\,p\,p!}{(p-1)}. \tag{68}$$

Taking into account the definition of H_{res} in 29, the bound 55 and Lemma 3.2 we have

$$||Z_1(t)\,\xi||_{\ell^1} \leq p(p-1)\sqrt{2}^{-(p+1)}(4\sqrt{\mathtt{c}})^{p-1}\mu^{-p+1}||\xi||_{\ell^1} \leq \Xi\mu^{-p+1}||\xi||_{\ell^1} \tag{69}$$

provided that

$$p(p-1)\sqrt{2}^{-(p+1)}(4\sqrt{\mathtt{c}})^{p-1} \leq \Xi. \tag{70}$$

This is implied by 68.

Bound for Z_2: We use a bootstrap argument. Let us define T_* as the sup of the times t such that

$$||\xi(t)||_{\ell^1} \leq 2\mu^{-\sigma_2}. \tag{71}$$

We observe that for $t=0$ we have $||\xi(0)||_{\ell^1} \leq \mu^{-\sigma_1}$ and, since $\sigma_1-\sigma_2 = 1/2$ and μ will be taken large enough, we have $T_* > 0$. A posteriori we shall prove that $T_* > T > 0$. We call

$$Z_{2,1} := X_{H_{\rm res}}(r) - X_{H_{\rm res}}(r^\mu) - DX_{H_{\rm res}}(r^\mu)\xi,$$

$$Z_{2,2} := X_{\mathcal{R}}(r) - X_{\mathcal{R}}(r_\mu), \quad Z_{2,3} := X_{\mathcal{Q}}(r) - X_{\mathcal{Q}}(r_\mu).$$

- We claim that, under the following condition

$$\sqrt{\mathtt{c}_-} \geq \mathtt{r}^{\frac{p-1}{2}}\frac{(p^2-1)2^{p-5}}{p!} \tag{72}$$

  we have

$$||Z_{2,1}||_{\ell^1} \leq \Xi\mu^{-p+1}||\xi||_{\ell^1}. \tag{73}$$

By the definition of H_{res} in 29

$$||Z_{2,1}||_{\ell^1} \leq p^2(p+1)\sqrt{2}^{-(p+1)}\sum_{j=2}^{p}||r^\mu||_{\ell^1}^{p-j}||\xi||_{\ell^1}^{j}$$
$$\overset{71,55}{\leq} p^2(p+1)\sqrt{2}^{-(p+1)}2^{p-1}(4\sqrt{\mathtt{c}})^{p-2}\sum_{j=2}^{p}\mu^{(j-p)-\sigma_2(j-1)}||\xi||_{\ell^1}.$$

Since $\sigma_2 \geq 1$ we have $\mu^{(j-p)-\sigma_2(j-1)} \leq \mu^{-p+1}$ for $j \geq 2$ and

$$||Z_{2,1}||_{\ell^1} \leq p^2(p+1)\sqrt{2}^{-(p+1)}2^{p-1}(4\sqrt{\mathtt{c}})^{p-2}\,(p-1)\,\mu^{-p+1}||\xi||_{\ell^1} \leq \Xi\mu^{-p+1}||\xi||_{\ell^1}$$

provided that (recall 61)

$$p^2(p^2-1)\sqrt{2}^{5p-11}\sqrt{\mathtt{c}}^{p-2} \leq \Xi. \tag{74}$$

This is implied by the bound 72. Now recall the bound 30. We have

$$||Z_{2,2}||_{\ell^1} \leq p^2C_1\gamma^{-1}\sum_{j=1}^{2p-1}||r^\mu||_{\ell^1}^{2p-1-j}||\xi||_{\ell^1}^{j} + \widetilde{C_1}p^2\gamma^{-2}\sum_{j=1}^{3p-2}||r^\mu||_{\ell^1}^{3p-2-j}||\xi||_{\ell^1}^{j}. \tag{75}$$

We reason as for the bound on $Z_{2,1}$. We shall use systematically bounds 55, 65, 71.

- We claim that, under the following conditions (recall 61)

$$\sqrt{\mathtt{c}_-} \geq 2^{2p}p^{2-\mathtt{a}(\frac{p}{4}+1)}(2p-1), \tag{76}$$
$$\sqrt{\mathtt{c}_-} \geq 2^{3p-5}p^{2-\mathtt{a}(\frac{p}{4}+1)}(3p-2), \tag{77}$$

  we have

$$||Z_{2,2}||_{\ell^1} \leq \Xi\mu^{-(3/2)p+1}||\xi||_{\ell^1}. \tag{78}$$

We deal with the first term in the right hand side of 75. We have $\mu^{-2p+1+j-\sigma_2(j-1)} \leq \mu^{-2p+2}$, because $\sigma_2 \geq 1$, then

$$\sum_{j=1}^{2p-1}||r^\mu||_{\ell^1}^{2p-1-j}||\xi||_{\ell^1}^{j} \leq 2^{2p-2}(4\sqrt{\mathtt{c}})^{2p-2}(2p-1)\mu^{-2p+2}||\xi||_{\ell^1}.$$

We want to prove that $C_1\gamma^{-1}2^{2p-2}(4\sqrt{\mathtt{c}})^{2p-2}(2p-1)p^2\mu^{-2p+2} \leq (\Xi/2)\mu^{-(3/2)p+1}$. By using 65, it is easy to check that this is implied by condition 76. Regarding the second term in the right hand side of 75, we have $\mu^{-3p+2+j-\sigma_2(j-1)} \leq \mu^{-3p+3}$, because $\sigma_2 \geq 1$, then

$$\sum_{j=1}^{3p-2}||r^\mu||_{\ell^1}^{3p-2-j}||\xi||_{\ell^1}^{j} \leq 2^{3p-3}(4\sqrt{\mathtt{c}})^{3p-3}(3p-2)\mu^{-3p+3}||\xi||_{\ell^1}.$$

We want to prove that $p^2\widetilde{C_1}\gamma^{-2}2^{3p-3}(3p-2)(4\sqrt{\mathtt{c}})^{3p-3}\mu^{-3p+3} \leq (\Xi/2)\mu^{-(3/2)p+1}$. By using 65, one can check that this is implied by 77. This proves the claim.

The most problematic term is $H^{(p+1,\geq 2)}$, because it has the same degree of $H_{\rm res}$. However we recall that the monomials of $H^{(p+1,\geq 2)}$ are Fourier supported on at least two normal modes.

- We claim that, under the conditions

$$\mathtt{r}^{\frac{p-1}{2}} \leq 2, \tag{79}$$
$$\sqrt{\mathtt{c}_-} \geq \mathtt{r}^{\frac{p-1}{2}}\,2^{-(3p+4)}, \tag{80}$$

we have

$$||Z_{2,3}||_{\ell^1} \leq \Xi \mu^{-p+1} ||\xi||_{\ell^1}. \tag{81}$$

By the definitions 21, 45, the bound 36 and by noting that $||\Pi_S^{\perp} r||_{\ell^1} \leq ||\xi||_{\ell^1}$ we have

$$\begin{aligned} ||Z_{2,3}||_{\ell^1} \leq & p^2 p! \sqrt{2}^{-(p+1)} \sum_{j=1}^{p} ||r^{\mu}||_{\ell^1}^{p-j} ||\xi||_{\ell^1}^{j} \\ \overset{71,55}{\leq} & p^2 p! \sqrt{2}^{-(p+1)} 4^{p-1} \sqrt{\mathtt{c}}^{p-1} \mu^{1-p} ||\xi||_{\ell^1} \\ & + p^2 p! \sqrt{2}^{-(p+1)} 4^{p-2} \sqrt{\mathtt{c}}^{p-2} 2^{p-1} ||\xi||_{\ell^1} \sum_{j=2}^{p} \mu^{j-p-\sigma_2(j-1)} \\ \leq & p^2 p! \sqrt{2}^{-(p+1)} 4^{p-1} \sqrt{\mathtt{c}}^{p-1} \mu^{1-p} ||\xi||_{\ell^1} \\ & + p^3 p! \sqrt{2}^{-(p+1)} 4^{p-2} \sqrt{\mathtt{c}}^{p-2} 2^{p-1} \mu^{1-p} ||\xi||_{\ell^1}. \end{aligned}$$

We obtain the bound 81 provided that

$$p^2 p! \sqrt{2}^{-(p+1)} 4^{p-1} \sqrt{\mathtt{c}}^{p-1} \leq \frac{\Xi}{2}, \qquad p^3 p! \sqrt{2}^{-(p+1)} 4^{p-2} \sqrt{\mathtt{c}}^{p-2} 2^{p-1} \leq \frac{\Xi}{2}. \tag{82}$$

Those are implied by 79, 80 respectively. By collecting the previous estimates 64, 69, 73, 78, 81 we obtained

$$\frac{d}{dt} ||\xi||_{\ell^1} \leq \Xi \Big(\mu^{-(7/4)p-1} + \mu^{-p+1} ||\xi||_{\ell^1} \Big).$$

Then by Gronwall lemma

$$||\xi(t)||_{\ell^1} \leq 2\mu^{-\sigma_1} \exp(\Xi \mu^{-p+1} t) \quad \text{for } t \in [0, T_*].$$

For times $t \in [0, c_0 \mu^{p-1} \log(\mu)]$ with

$$c_0 := \frac{1}{2\Xi} \tag{83}$$

we have that $||\xi||_{\ell^1} \leq 2\mu^{\Xi c_0 - \sigma_1} \leq 2\mu^{-\sigma_2}$. Then $T_* \geq c_0 \mu^{p-1} \log(\mu)$. We now prove that $c_0 \mu^{p-1} \log(\mu) > T$. Then $T_* > T$ and we can drop the bootstrap assumption. Recalling 52, 50 we show that

$$c_0 \log(\mu_0) \geq \frac{2\sqrt{2}^{p+1}}{p\sqrt{\mathtt{c}_-}^{p-1}} \geq T_0.$$

By using the definition of μ_0 in 56 and 57 we have

$$c_0 \log(\mu_0) = \mathtt{a} c_0 \log(p \gamma^{-\frac{\mathtt{b}}{\mathtt{a}}}) > \mathtt{a}\, c_0 \overset{83}{=} \frac{\mathtt{a}}{2\Xi} \overset{57}{\geq} \frac{2\sqrt{2}^{p+1}}{p\sqrt{\mathtt{c}_-}^{p-1}}. \tag{84}$$

We conclude the proof by showing that taking $\mathtt{a}, \mathtt{c}_\pm$ as in 57, 58 the conditions 62, 63, 68, 72, 76, 77, 79, 80 are satisfied.

The conditions 79, 80 imply respectively 68, 72 if p is taken large enough .

By the choice of $\mathtt{r}$ in 58 the inequality 79 is satisfied.

If $\mathtt{c}_- \geq 1$ it is easy to see that, for p large enough, the bounds 80, 76 and 77 hold.

We prove that, assuming 57, 58 and taking p large enough, imply 62, 63.

The condition 62 is equivalent to

$$\mathtt{c}_+ \leq 2^{3-p^{-1}} (p!)^{2/p} p^{\mathtt{a}(\frac{1}{2}-\frac{4}{p})-14}. \tag{85}$$

While 63 is equivalent to

$$\mathtt{c}_+ \leq 2^{3-\frac{p}{2p-1}}(p!)^{\frac{2}{2p-1}} p^{\mathtt{a}(\frac{5p-12}{4p-2})-\frac{14p+3}{2p-1}}. \tag{86}$$

For p large enough the right hand side of 86 is greater than the right hand side of 85. To obtain the upper bound of $\mathtt{c}_+$ in 58 from 85 we consider that, if $p \geq 16$ then $p^{4/p} \leq 2$ and this implies

$$p^{\mathtt{a}(\frac{1}{2}-\frac{4}{p})} \geq \frac{p^{\mathtt{a}/2}}{2}.$$

□

Remark 9. The upper bound for $\mathtt{c}_+$ is not optimal and the interval $[\mathtt{c}_-, \mathtt{c}_+]$ can be enlarged. However the approximation argument does not hold for $\mathtt{c}_+ = \mathcal{O}(p^{2\mathtt{a}})$ (see for instance the proof of the bound 64 for Z_0). This fact is fundamental in the estimate 94 for the norm at time T of the unstable solution. See Remark 10.

3.5. **Conclusion of the proof.**

In this section we conclude the proof of Theorem 2.1 by showing that a solution $z(t)$ with initial datum $z(0) = r^\mu(0)$, with an opportune choice of μ, undergoes the prescribed growth of its Sobolev norms.

We fix $\delta \ll 1, \mathcal{C} \gg 1$ and we consider $s > 2$. Recalling the rescaling 53 we consider $\mu = p^{\mathtt{a}}\gamma^{-\mathtt{b}}$ with $\mathtt{a} := p\,p!\,4^p$ (see 57) and $\mathtt{b}$ as in 56. We consider $\mathtt{c} \in [\mathtt{c}_-, \mathtt{c}_+]$ as in 58. To simplify the exposition we fix

$$\mathtt{c}_+ := p^{a/4}. \tag{87}$$

Let us consider $r(t)$ solution of 44 with $r(0) = \Gamma^{-1} r^\mu(0)$. Thanks to the choice of μ as above we can apply the approximation argument in Proposition 2. Let us call

$$z(t) = \Gamma\big((r_j e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big).$$

Now we give a lower bound for $||z(T)||_s$. It turns out that it is sufficient to estimate $|z_{\pm p}(T)|$. We give a lower bound for $z_p(T)$, the one for $z_{-p}(T)$ is obtained in the same way. We have

$$\begin{aligned}|z_p(T)| &\geq |r_p(T)| - |\Gamma_p\big((r_j e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big) - r_p(T)e^{\mathrm{i}\omega(p)T}| \\ &\geq |r_p^\mu(T)| - |r_p(T) - r_p^\mu(T)| - |\Gamma_p\big((r_j e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big) - r_p(T)e^{\mathrm{i}\omega(p)T}|.\end{aligned} \tag{88}$$

First we need a lower bound for $|r_p^\mu(T)|$. By 51 and the rescaling 53

$$|r_p^\mu(T)| \geq \sqrt{\mathtt{c}}p^{-1}\,\mu^{-1}. \tag{89}$$

Now we give an upper bound for $|r_p(T) - r_p^\mu(T)|$. By the estimates 32 and 55 we have that

$$||r(0) - r^\mu(0)||_{\ell^1} \leq C_0\gamma^{-1}(4\sqrt{\mathtt{c}})^p\mu^{-p}.$$

By using that $\mu = p^{\mathtt{a}}\gamma^{-\mathtt{b}}$ it is easy to see that $||r(0) - r^\mu(0)||_{\ell^1} \leq \mu^{-\sigma_1}$ (recall σ_1 in 59), provided that

$$p^{\mathtt{a}(\frac{p}{4}-2)-\frac{11}{4}p-\frac{7}{4}} \geq \sqrt{\mathtt{c}_+}^p. \tag{90}$$

The bound 90 is implied by the choice of $\mathtt{c}_+$ in 87 if p is large enough (recall $\mathtt{a} \sim p!$). Then by Proposition 2 (recall σ_2 in 60)

$$||r(t) - r^\mu(t)||_{\ell^1} \leq 2\mu^{-\sigma_2} \quad \text{for } t \in [0, T]. \tag{91}$$

Hence

$$|r_p(T) - r_p^\mu(T)| \leq 2\mu^{-\sigma_2}. \tag{92}$$

We are left with an upper bound for $|\Gamma_p\big((r_j e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big) - r_p(T)e^{\mathrm{i}\omega(p)T}|$. By 91 and 55 we have

$$|r_p(T)| \leq ||r(T)||_{\ell^1} \leq ||r^\mu(T)||_{\ell^1} + ||r(T) - r^\mu(T)||_{\ell^1} \leq 8\sqrt{\mathtt{c}}\mu^{-1},$$

where the last inequality holds provided that

$$\mu^{1-\sigma_2} \leq 2\sqrt{\mathtt{c}_-},$$

that is equivalent to

$$2p^{\frac{3}{4}\mathtt{a}(p+1)} \geq 2^{\frac{1}{p-1}}\gamma^{\mathtt{b}(\frac{3}{4}p+\frac{1}{2})}.$$

Since $\mathtt{b} > 0$ this holds for p large enough. By using the estimate 32 we have

$$|\Gamma_p\big((r_j e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big) - r_p(T)e^{\mathrm{i}\omega(p)T}| \leq (8\sqrt{\mathtt{c}})^p\, C_0\gamma^{-1}\mu^{-p} \leq \mu^{-\sigma_1} \tag{93}$$

provided that 90 holds and p is large enough. By 88 and collecting the bounds 89, 92, 93 we obtained

$$|z_p(T)| \geq \sqrt{\mathtt{c}}\, p^{-1}\,\mu^{-1} - 2\mu^{-\sigma_2} - \mu^{-\sigma_1} \geq \sqrt{\mathtt{c}}\frac{p^{-1}}{2}\mu^{-1},$$

where the last inequality holds for $p \geq 4$, since $\sigma_1 > 3/2$. This implies that

$$||z(T)||_s^2 \geq \big(|z_p(T)|^2 + |z_{-p}(T)|^2\big)\, p^{2s} \geq (\mathtt{c}/2)\,\mu^{-2}p^{2(s-2)}. \tag{94}$$

Remark 10. Since $\mu = p^{\mathtt{a}}\gamma^{-\mathtt{b}}$, by 94 we have $||z(T)||_s^2 \geq (\mathtt{c}/2)\,p^{2(s-2-\mathtt{a})}\gamma^{2\mathtt{b}}$. If s is kept fixed then we cannot ensure that taking p large enough $||z(T)||_s$ is arbitrarily large.

We observe that $\mathtt{c} \leq \mathtt{c}_+$ and, by Proposition 2, $\mathtt{c}_+$ behaves asymptotically like $p^{k\mathtt{a}}$ with $k \in (0,1)$ (see 58 and recall Remark 9). Thus $\mathtt{c}\,p^{-2\mathtt{a}}$ is not uniformly (in p) bounded from below.

Regarding the Sobolev norm at time zero of $z(t)$, by 51 and choosing $\varepsilon = \varepsilon(p,s)$ in 50 such that

$$\varepsilon = \varepsilon_0 p^{-2s}, \qquad 0 < \varepsilon_0 \leq \frac{\mathtt{c}_-}{1-p^{1-2s}}, \tag{95}$$

we have

$$||z(0)||_s^2 = ||r^{\mu}(0)||_s^2 = 2\mu^{-2}\big((\mathtt{c} - \varepsilon\, p) + \varepsilon p^{2s}\big) \leq 4\mathtt{c}\mu^{-2}. \tag{96}$$

We observe that we can also provide the lower bound

$$||z(0)||_s^2 \geq 2\mathtt{c}\mu^{-2}.$$

Since

$$2\sqrt{\mathtt{c}}\mu^{-1} \leq 2p^{-\mathtt{a}}\sqrt{\mathtt{c}_+}\gamma^{\mathtt{b}} \overset{87}{\leq} 2p^{-\frac{7}{8}\mathtt{a}},$$

to obtain $||z(0)||_s \leq \delta$ we impose that

$$F_\gamma(p) \leq \frac{\delta}{2}, \quad F_\gamma(p) := p^{-\frac{7}{8}\,p\,p!\,4^p}\gamma^{\frac{1}{\frac{p}{2}-4}}. \tag{97}$$

By 94 and 96 the ratio between the Sobolev norms at time $t = T$ and $t = 0$ has the following lower bound

$$\frac{||z(T)||_s}{||z(0)||_s} \geq \frac{p^{s-2}}{2\sqrt{2}}. \tag{98}$$

To obtain 6 we need to impose

$$p^{s-2} \geq 2\sqrt{2}\mathcal{C}. \tag{99}$$

Recalling that $s > 2$, the conditions 97 and 99 can be satisfied by taking $p \geq p_0$ for some $p_0 = p_0(s,\delta,\mathcal{C})$ large enough.

By 54, 52, 58 we have that

$$\begin{aligned} \frac{2^{p-1}}{p}||z(0)||_s^{1-p} \leq T = \mu^{p-1}T_0 &\leq (\sqrt{2\mathtt{c}})^{p-1}T_0\,||z(0)||_s^{1-p} \\ &\leq \frac{2^{p+2}}{p}||z(0)||_s^{1-p}. \end{aligned} \tag{100}$$

This proves 7.

3.6. **Proof of Theorem 2.2.** The proof follows the same steps of the proof of Theorem 2.1. The difference relies in (i) the choice of the index s to obtain the lower bound 8 for the norm of the solution $z(t)$ at time T; (ii) the choice of p in order to get a lower bound on the norm of the initial datum.

Let us fix $\delta \ll 1$, $\mathcal{C} \gg 1$. We choose $\mathtt{a}, \mathtt{b}$ and $\mathtt{c}$ as in previous section. We consider $\mu := p^{\mathtt{a}}\gamma^{-\mathtt{b}}$ with $p = p(\delta, \gamma)$ large enough such that 97 holds and (recall the definition of F_γ in 97)

$$F_\gamma(p) \geq \frac{\delta}{2\sqrt{2}}. \tag{101}$$

Remark 11. The conditions 101 and 97 can be satisfied at the same time for $C_1 \leq p \leq C_2$ with $C_i = C_i(\gamma, \delta)$, $i = 1, 2$. We point out that p can be considered larger than p_0 given in Proposition 2 if δ is taken small enough.

The 101 provides the lower bound for the norm of the initial datum

$$||z(0)||_s \geq \frac{\delta}{2}.$$

We choose s such that 99 holds. This holds for $s \geq s_0$ with $s_0 = s_0(p, \mathcal{C})$ large enough. We observe that, from 96, we can ensure that $||z(0)||_s \leq \delta$ for all s. Indeed it is sufficient to choose $\varepsilon = \varepsilon(s, p)$ small enough. The choice of ε does not affect the other parameters of the problem. This is due to the fact that the transfer of energy occurs via diffusion channels.

Thanks to the lower bound on $||z(0)||_s$ we obtain

$$||z(T)||_s \geq \mathcal{C}||z(0)||_s \geq \frac{\mathcal{C}\delta}{2}.$$

It is enough to choose $\mathcal{C} = 2\,K\,\delta$ to conclude.

Now we show that there exists an unstable solution of 4 with an upper bound on the diffusion time like 9. Assume that $K = \delta^{-\alpha}$ for some $\alpha > 1$. If we choose

$$\mu^{-1} = \frac{\delta}{4\sqrt{2\mathtt{c}}}.$$

then

$$T = \mu^{p-1}T_0 \leq (\delta^{-1})^{p-1}\,8^p\,\mathtt{r}^{\frac{p-1}{2}}\,p^{-1} \leq \mathcal{C}^{p-1}\delta^{\alpha(p-1)}\,\sqrt{2}^{7p-1}\,p^{-1}.$$

We have

$$\lim_{p\to\infty}\left(\sqrt{2}^{7p-1}\,p^{-1}\right)^{\frac{1}{p-1}} = \frac{1}{8\sqrt{2}}.$$

Then there exists $\alpha_0 > 0$ large enough such that for $\alpha \geq \alpha_0$ and δ small enough $T \lesssim \mathcal{C}^{p-1}$.

4. **Proof of Theorem 2.3.** We follow the same steps of the proof of Theorem 2.1 shown in Section 3. First we build the convolution potential V_N. To simplify the notation we write $V_N = V$. We consider the tangential set

$$S := \{k_1, k_2, k_3, k_4\} \subset \mathbb{Z} \tag{102}$$

with

$$k_1, k_3 > 0, \quad k_2 < 0, \quad k_4 := k_1 - k_2 + k_3 + N. \tag{103}$$

Let us also assume that

$$k_3 := \max\{k_1, |k_2|, k_3\} \leq \sqrt{N}. \tag{104}$$

Let us consider $\mathtt{q} := (\mathtt{q}_1, \mathtt{q}_2, \mathtt{q}_3) \in [1,2]^3$ such that

$$|\mathtt{q} \cdot \ell + k| \geq \frac{\gamma}{\langle \ell \rangle^\tau} \quad \forall \ell \in \mathbb{Z}^3, \quad 0 < |\ell| \leq 9, \quad \forall k \in \mathbb{Z}, \quad (\ell, k) \neq (0,0) \tag{105}$$

with $\gamma \in (0,1)$ and $\tau > 0$. It is well known that for τ large enough the set of vectors in $[1,2]^3$ satisfying 105 has positive measure. We set

$$V_j := \begin{cases} \mathtt{q}_i - k_i^2, & j = k_i, \quad i = 1,2,3, \\ \mathtt{q}_1 - \mathtt{q}_2 + \mathtt{q}_3 - k_4^2, & j = k_4, \\ 0 & \text{otherwise.} \end{cases}$$

We consider the Fourier expansion $u = \sum_{j \in \mathbb{Z}} u_j \, e^{\mathrm{i} j x}$ with $u_j := \frac{1}{2\pi} \int_{\mathbb{T}} u \, e^{-\mathrm{i} j x} \, dx$. The equation 10 is Hamiltonian with respect to the symplectic structure

$$-\mathrm{i} \sum_{j \in \mathbb{Z}} du_j \wedge d\overline{u_j}.$$

The Hamiltonian is given by $H = H^{(2)} + H^{(4)}$ where

$$\begin{aligned} H^{(2)}(u_j, \overline{u_j}) &:= \sum_{j \in \mathbb{Z}} \omega(j) u_j \, \overline{u_j}, \\ H^{(4)}(u_j, \overline{u_j}) &:= \sum_{j_1 - j_2 + j_3 - j_4 = \pm N} u_{j_1} \, \overline{u_{j_2}} \, u_{j_3} \, \overline{u_{j_4}} \end{aligned} \tag{106}$$

with

$$\omega(j) := \begin{cases} \mathtt{q}_j & j = k_i, \quad i = 1,2,3, \\ \mathtt{q}_1 - \mathtt{q}_2 + \mathtt{q}_3, & j = k_4, \\ j^2 & \text{otherwise.} \end{cases}$$

Remark 12. As in the case of the wave equation (see Remark 3), the tangential frequencies are irrational real numbers while the normal ones are integers.

The equation 10 can be written as an infinite dimensional system of ODEs for the Fourier coefficients

$$-\mathrm{i} \dot{u}_j = \omega(j) u_j + \sum_{j_1 - j_2 + j_3 - j = \pm N} u_{j_1} \, \overline{u_{j_2}} \, u_{j_3}, \qquad j \in \mathbb{Z}.$$

We consider the solution of the linear problem

$$-\mathrm{i} \dot{u}_j = \omega(j) u_j, \quad j \in \mathbb{Z},$$

obtained by exciting the modes in S, namely

$$w(t,x) = \sum_{k \in S} a_k \, e^{\mathrm{i}(\omega(k) t + k \, x)}. \tag{107}$$

By the definition of the linear frequencies of oscillation and by 105 the orbit $w(\cdot, x)$ is conjugated to a quasi-periodic motion on an embedded 4-d resonant torus that fills densely a lower dimensional submanifold.

As in Section 3.2 we construct a change of coordinates that puts (partially) in normal form the Hamiltonian 106. Again we work with the ℓ^1-topology. Recall 24 and Definition 3.1.

Proposition 3. *Recall 106. There exists $\eta > 0$ small enough such that there exists a symplectic change of coordinates $\Gamma \colon B_\eta \to B_{2\eta}$ which takes the Hamiltonian H into its (partial) Birkhoff normal form up to order 4, namely*

$$H \circ \Gamma = H^{(2)} + H_{\rm res} + H^{(4,\geq 2)} + R \tag{108}$$

where:
(i) the resonant Hamiltonian is given by

$$H_{\rm res} := \Pi_{\rm Ker} H^{(4,0)} = 2\Re\big(u_{k_1}\, \overline{u_{k_2}}\, u_{k_3}\, \overline{u_{k_4}}\big). \tag{109}$$

(ii) The remainder R is such that

$$||X_R||_\eta \lesssim \gamma^{-1}\eta^5 + \gamma^{-2}\eta^7. \tag{110}$$

Moreover the map Γ is invertible and close to the identity

$$||\Gamma^{\pm 1} - {\rm Id}||_\eta \lesssim \gamma^{-1}\eta^3. \tag{111}$$

Proof. The proof follows the same lines of the proof of Proposition 1. Actually the proof is easier since the parameter N, that we need to control, does not enter in the estimates for the map Γ and the remainder R. The only thing that we need to prove is that $\Pi_{\rm Ker} H^{(4,1)} = 0$ and formula 109.
If $(\alpha, \beta) \in \mathcal{A}_{4,1}$ then

$$\Omega(\alpha, \beta) = \mathtt{q} \cdot \ell \pm j^2$$

for some $\ell = \ell(\alpha, \beta) \in \mathbb{Z}^3$ with $3 \leq |\ell| \leq 9$ and $j \notin S$. Since j^2 is an integer, by 105 we have $\Omega(\alpha, \beta) \geq \gamma\, 9^{-\tau} > 0$. This proves that $\Pi_{\rm Ker} H^{(4,1)} = 0$. If $(\alpha, \beta) \in \mathcal{A}_{4,0}$ by 105 and the choice of the potential the only resonant monomials are $u_{k_1}\, \overline{u_{k_2}}\, u_{k_3}\, \overline{u_{k_4}}$ and its complex conjugate. □

We introduce the rotating coordinates

$$u_j = r_j\, e^{{\rm i}\omega(j)t}$$

in order to remove the quadratic part of the Hamiltonian $H \circ \Gamma$. Then r satisfies the equation associated to the Hamiltonian

$$\mathcal{H} = H_{\rm res} + \mathcal{Q}(t) + \mathcal{R}(t) \tag{112}$$

where

$$\mathcal{Q}\big((r_j)_{j\in\mathbb{Z}}, t\big) := H^{(4,\geq 2)}\big((r_j\, e^{{\rm i}\omega(j)t})_{j\in\mathbb{Z}}\big),$$
$$\mathcal{R}\big((r_j)_{j\in\mathbb{Z}}, t\big) := R\big((r_j\, e^{{\rm i}\omega(j)t})_{j\in\mathbb{Z}}\big).$$

We study the dynamics of the resonant Hamiltonian $H_{\rm res}$. We observe that the finite dimensional subspace

$$\mathcal{U}_S := \{r \colon \mathbb{Z} \to \mathbb{C} \,|\, r_j = 0 \ \ j \notin S\}$$

is invariant by the flow of $H_{\rm res}$. We introduce the following action-angle variables on $\mathcal{U}_S$

$$r_{k_j} = \sqrt{I_j}\, e^{{\rm i}\theta_j}, \quad j = 1, 2, 3, 4.$$

The Hamiltonian $H_{\rm res}$ now reads as

$$\mathcal{G} := 2\sqrt{I_1\, I_2\, I_3\, I_4}\,\cos(\theta_1-\theta_2+\theta_3-\theta_4).$$

Remark 13. The Hamiltonian $\mathcal{G}$ commutes with the mass $\mathtt{M} := I_1+I_2+I_3+I_4$.

Lemma 4.1. *Let $\varepsilon>0$ be arbitrarily small and let $\mathtt{c}>\frac{8}{3}\varepsilon$. There exists an orbit of $\mathcal{G}$*

$$g_{\varepsilon,\mathtt{c}}(t)=(\theta_1(t),\dots,\theta_4(t),I_1(t),\dots,I_4(t))$$

such that

$$I_1(0)=I_2(0)=I_3(0)=\frac{\mathtt{c}-\varepsilon}{3},\quad I_4(0)=\varepsilon$$

$$I_1(T_0)=I_3(T_0)=\frac{\mathtt{c}}{6}+\frac{2\varepsilon}{3},\quad I_2(T_0)=\frac{\mathtt{c}}{2}-\frac{4\varepsilon}{3},\quad I_4(T_0)=\frac{\mathtt{c}}{6}$$

with

$$T_0\le\frac{6}{\mathtt{c}}.$$

Proof. We apply the following linear symplectic change of variables

$$\varphi=A\theta,\quad J=A^{-T}\,I,\quad A:=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-1&1&-1&1\end{pmatrix}.$$

We observe that the matrix $A\in SL(4,\mathbb{Z})$, hence this defines a linear automorphism of the torus $\mathbb{T}$. The new Hamiltonian is given by

$$\mathcal{G}_*=2\sqrt{(J_1-J_4)\,(J_2+J_4)\,(J_3-J_4)\,J_4}\,\cos(\varphi_4).$$

We observe that J_1,J_2,J_3 are constants of motion. Then we fix

$$J_i=\alpha:=\frac{\mathtt{c}-\varepsilon}{3}+\varepsilon\quad i=1,3,\qquad J_2=\beta:=\frac{\mathtt{c}-\varepsilon}{3}-\varepsilon$$

and we look for solutions traveling along the following diffusion channel

$$\{(\alpha-I_4,\beta+I_4,\alpha-I_4,I_4):I_4\in(0,\alpha)\},$$

which is contained in the mass level

$$\{J_1+J_2+J_3=\mathtt{M}=\mathtt{c}\}. \tag{113}$$

If we restrict to the invariant section $\{\varphi_4=\pi/2\}$ the equation of motion for $J_4=I_4$ is

$$\dot J_4=2\,(\alpha-J_4)\,\sqrt{(\beta+J_4)\;\,J_4}.$$

Reasoning as in Lemma 3.3 we can conclude that there exists an orbit such that

$$J_4(0)=\varepsilon,\quad J_4(T_0)=\frac{\mathtt{c}}{6}$$

with

$$T_0=\frac{1}{2}\int_\varepsilon^{\mathtt{c}/6}\frac{1}{(\alpha-J_4)\sqrt{(\beta+J_4)\,J_4}}\,dJ_4\le\frac{3}{\mathtt{c}}\int_0^{\mathtt{c}/6}\frac{1}{\sqrt{(\frac{\mathtt{c}}{2}+J_4)\,J_4}}\,dJ_4\le\frac{6}{\mathtt{c}}.$$

$\square$

Remark 14. The diffusion time T_0 has an upper bound that does not depend on $\varepsilon=I_4(0)$.

We set

$$0<\varepsilon\leq\frac{\mathtt{c}}{2^{2s}N^s-1}. \tag{114}$$

We define $b(\varepsilon,\mathtt{c};t,x)=b(t,x)=\sum_{j\in\mathbb{Z}}b_j(t)e^{\mathrm{i}jx}$ with

$$b_j(t):=\begin{cases}\sqrt{I_j(t)}\,e^{\mathrm{i}\theta_j(t)} & j\in S,\\ 0 & \text{otherwise.}\end{cases}$$

Then the function $b(t,x)$ is a solution of H_{res} such that

$$\begin{aligned}|b_{k_1}(0)|^2=|b_{k_2}(0)|^2=|b_{k_3}(0)|^2=\frac{\mathtt{c}-\varepsilon}{3},\quad |b_{k_4}(0)|^2=\varepsilon,\\ |b_{k_1}(T_0)|^2=|b_{k_3}(T_0)|^2=\frac{\mathtt{c}}{6}+\frac{2\varepsilon}{3},\quad |b_{k_2}(T_0)|^2=\frac{\mathtt{c}}{2}-\frac{4\varepsilon}{3},\quad |b_{k_4}(T_0)|^2=\frac{\mathtt{c}}{6}.\end{aligned} \tag{115}$$

In particular from the proof of Lemma 4.1 we deduce that

$$\sup_{t\in[0,T_0]}|b_{k_i}(t)|^2<\mathtt{c},\quad i=1,2,3,4. \tag{116}$$

We follow the same procedure of Section 3.4. The solutions $u(t,x)$ of H_{res} are invariant under the rescaling

$$u(t,x)\to\mu^{-1}u(\mu^{-2}t,x).$$

Then we consider the rescaled solution

$$r^{\mu}(t,x):=\mu^{-1}b(\mu^{-2}t,x). \tag{117}$$

The diffusion time is rescaled in the following way

$$T=\mu^2T_0\leq\frac{6\,\mu^2}{\mathtt{c}}\,. \tag{118}$$

By 116 we have

$$||r^{\mu}||_{\ell^1}\leq 4\sqrt{\mathtt{c}}\mu^{-1}. \tag{119}$$

Proposition 4. *Let $\gamma\in(0,1)$ be the constant in 105 and $\mathtt{c}\geq 1$. There exists $C_0>0$ large enough such that for all $\mu\geq\mu_0:=C_0\gamma^{-2}\mathtt{c}^5$ we have the following. If $r(t)$ is a solution of 44 such that*

$$||r(0)-r^{\mu}(0)||_{\ell^1}\leq\mu^{-5/2},$$

then

$$||r(t)-r^{\mu}(t)||_{\ell^1}\leq 2\mu^{-3/2},\quad \textit{for } t\in[0,T].$$

Proof. We set $\xi:=r-r^{\mu}$ and we study the evolution of its ℓ^1-norm. We observe that $\Pi_S^{\perp}\xi=\Pi_S^{\perp}r$. We have that $\dot{\xi}=Z_0(t)+Z_1(t)\,\xi+Z_2(t,\xi)$ with (recall 4)

$$\begin{aligned}&Z_0:=X_{\mathcal{R}}(r_\mu),\\ &Z_1:=DX_{H_{\mathrm{res}}}(r^{\mu}),\\ &Z_2:=X_{H_{\mathrm{res}}}(r)-X_{H_{\mathrm{res}}}(r^{\mu})-DX_{H_{\mathrm{res}}}(r^{\mu})\xi+X_{\mathcal{R}}(r)-X_{\mathcal{R}}(r_\mu)+X_{\mathcal{Q}}(r)-X_{\mathcal{Q}}(r_\mu).\end{aligned}$$

By the differential form of Minkowsky's inequality we get

$$\frac{d}{dt}||\xi||_{\ell^1}\leq||Z_0(t)||_{\ell^1}+||Z_1(t)\,\xi||_{\ell^1}+||Z_2(t)||_{\ell^1}.$$

By 110 and 119 we have

$$||Z_0(t)||_{\ell^1}\lesssim\gamma^{-1}\sqrt{\mathtt{c}}^5\mu^{-5}+\gamma^{-2}\sqrt{\mathtt{c}}^7\mu^{-7}.$$

We impose that

$$\gamma^{-1}\sqrt{\mathsf{c}}^5\mu^{-5} \leq \mu^{-9/2}, \quad \gamma^{-2}\sqrt{\mathsf{c}}^7\mu^{-7} \leq \mu^{-9/2}.$$

Since $\mathsf{c} \geq 1$ and $\gamma \in (0,1)$, the above inequalities are satisfied if

$$\mu \geq \gamma^{-2}\mathsf{c}^5. \tag{120}$$

We obtained

$$||Z_0(t)||_{\ell^1} \lesssim \mu^{-9/2}. \tag{121}$$

By 109 and 119 we have

$$||Z_1(t)\,\xi||_{\ell^1} \lesssim \mathsf{c}\mu^{-2}||\xi||_{\ell^1}. \tag{122}$$

To obtain a bound for Z_2 we use a bootstrap argument. Let us define T_* as the sup of the times t such that

$$||\xi(t)||_{\ell^1} \leq 2\mu^{-3/2}. \tag{123}$$

We observe that for $t = 0$ we have $||\xi(0)||_{\ell^1} \leq \mu^{-5/2}$, thus $T_* > 0$. A posteriori we shall prove that $T_* > T > 0$. We call

$$Z_{2,1} := X_{H_{\rm res}}(r) - X_{H_{\rm res}}(r^\mu) - DX_{H_{\rm res}}(r^\mu)\xi,$$
$$Z_{2,2} := X_{\mathcal{R}}(r) - X_{\mathcal{R}}(r_\mu), \quad Z_{2,3} := X_{\mathcal{Q}}(r) - X_{\mathcal{Q}}(r_\mu).$$

We have by 123

$$||Z_{2,1}||_{\ell^1} \lesssim \sqrt{\mathsf{c}}\mu^{-1}||\xi||^2_{\ell^1} \lesssim \sqrt{\mathsf{c}}\mu^{-5/2}||\xi||_{\ell^1} \lesssim \mathsf{c}\mu^{-2}||\xi||_{\ell^1}, \tag{124}$$

where the last inequality holds provided that

$$\mu \geq \mathsf{c}^{-1}. \tag{125}$$

This inequality is satisfied if 120 holds, since we assumed that $\mathsf{c} \geq 1$. In a similar way we get (see 110)

$$||Z_{2,2}||_{\ell^1} \lesssim \sum_{j=1}^{5} ||r^\mu||^{5-j}||\xi||^j_{\ell^1} \lesssim \mathsf{c}^2||\xi||_{\ell^1}(\mu^{-4} + \sum_{j=2}^{5}\mu^{j-5}\mu^{-\frac{3}{2}(j-1)}).$$

Since $(1-\frac{3}{2})j - 5 + \frac{3}{2} \leq -4$, because $j \geq 1$, then

$$||Z_{2,2}||_{\ell^1} \lesssim \mathsf{c}^2\mu^{-4}\,||\xi||_{\ell^1} \lesssim \mathsf{c}\mu^{-2}||\xi||_{\ell^1}, \tag{126}$$

where the last inequality holds provided that

$$\mu \geq \sqrt{\mathsf{c}}.$$

This inequality is satisfied if 120 holds. Regarding the bound for $Z_{2,3}$, we recall that the vector field $X_{H^{(4,\geq 2)}}(r^\mu) = 0$ because $H^{(4,\geq 2)}$ is supported on at least two normal sites. We have

$$\begin{aligned} ||Z_{2,3}||_{\ell^1} = ||X_{H^{(4,\geq 2)}}(r)||_{\ell^1} &\lesssim \mathsf{c}\mu^{-2}||\xi||_{\ell^1} + \sqrt{c}\mu^{-1}||\xi||^2_{\ell^1} + ||\xi||^3_{\ell^1} \\ &\lesssim \big(\mathsf{c}\mu^{-2} + \sqrt{\mathsf{c}}\mu^{-5/2} + \mu^{-9/2}\big)||\xi||_{\ell^1} \\ &\lesssim \mathsf{c}\mu^{-2}||\xi||_{\ell^1}, \end{aligned} \tag{127}$$

where the last inequality holds provided that 120 holds.

By collecting the previous estimates 121, 122, 124, 126, 127 we obtained

$$\frac{d}{dt}||\xi||_{\ell^1} \leq C\,\mathsf{c}\Big(\mu^{-9/2} + \mu^{-2}||\xi||_{\ell^1}\Big)$$

for some pure constant $C > 0$. By Gronwall Lemma we have

$$||\xi(t)||_{\ell^1} \leq 2\mu^{-5/2}\,e^{C\,\mathsf{c}\,\mu^{-2}t} \quad \text{for } t \in [0, T_*].$$

For times $t \in [0, c_0\mu^2 \log(\mu)]$ with

$$c_0 := \frac{1}{4C\,\mathtt{c}} \tag{128}$$

we have that $||\xi||_{\ell^1} \leq 2\mu^{-5/2}\mu^{1/4} \leq 2\mu^{-1/4}\mu^{-2} \leq \mu^{-2}$, for μ large enough. Then $T_* > c_0\mu^2\log(\mu)$. We now prove that $c_0\mu^2\log(\mu) > T$. Then $T_* > T$ and we can drop the bootstrap assumption. We have

$$c_0\log(\mu) = \frac{1}{4C\,\mathtt{c}}\log(\mu) \geq \frac{6}{\mathtt{c}}$$

if μ is large enough. Then by 118 we conclude. □

We fix $\delta \ll 1$, $K \gg 1$, $s > 0$, $\mathtt{c} \geq 1$ and we consider $\mu = N^{\frac{3}{4}s}$. Recalling 104 we take $N = N(\mathtt{c}, \gamma, s, \delta, K)$ large enough such that

$$N^{\frac{3}{4}s} \geq C_0\gamma^{-2}\mathtt{c}^5, \tag{129}$$

$$\frac{1}{\sqrt{2\mathtt{c}}}\,N^{\frac{s}{4}} \geq \delta^{-1}, \tag{130}$$

$$\sqrt{\frac{\mathtt{c}}{6}}\,N^{\frac{s}{4}} \geq K, \tag{131}$$

where C_0 is the constant introduced in Proposition 4. Let us consider $r(t)$ solution of 112 with $r(0) = \Gamma^{-1}r^\mu(0)$. By 111, 119 we have

$$||r(0) - r^\mu(0)||_{\ell^1} \leq 64\gamma^{-1}\mathtt{c}^{3/2}\mu^{-3}.$$

Then by 129 and the definition of μ, $||r(0) - r^\mu(0)||_{\ell^1} \leq \mu^{-5/2}$ if C_0 is large enough. Therefore we are in position to apply Proposition 4. Let us call

$$z(t) = \Gamma\big((r_j e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big).$$

Reasoning as in Section 3.5, we can obtain the following lower bound for the Sobolev norms of z at time $t = T$ by using 103, 115

$$||z(T)||_s^2 \geq \big(|z_{k_4}(T)|^2\big)\,k_4^{2s} \geq \mu^{-2}\frac{\mathtt{c}}{6}\,N^{2s} = \frac{\mathtt{c}}{6}\,N^{\frac{s}{2}} \overset{131}{\geq} K^2.$$

Regarding the Sobolev norm at time zero, we have by 115

$$\begin{aligned}
||z(0)||_s^2 = ||r^\mu(0)||_s^2 &= \big(\frac{\mathtt{c}-\varepsilon}{3}(\langle k_1\rangle^{2s} + \langle k_2\rangle^{2s} + \langle k_3\rangle^{2s}) + \varepsilon\langle k_4\rangle^{2s}\big)\mu^{-2} \\
&\leq \big((\mathtt{c}-\varepsilon)\langle k_3\rangle^{2s} + \varepsilon\langle k_4\rangle^{2s}\big)\mu^{-2} \\
&\overset{104}{\leq} \big((\mathtt{c}-\varepsilon)N^s + \varepsilon(2N)^{2s}\big)\mu^{-2} \\
&\overset{114}{\leq} 2\mathtt{c}\,N^s\,\mu^{-2} \overset{130}{\leq} \delta^2.
\end{aligned}$$

We conclude by giving the estimate 11 on the diffusion time. By 118 and 130 we get

$$T \leq \mu^2 T_0 \leq 12N^s||z(0)||_s^{-2}.$$

Now we prove the bound 12 on the diffusion time respect to the growth. Let us fix $s > 0$, $\delta \ll 1$, $\mathtt{c} = 1$ and assume that $K = \delta^{-\alpha}$ with $\alpha > 1$. Then $\mathcal{C} := K/\delta = \delta^{-(1+\alpha)}$. If

$$\delta^{-1} \geq C_0^{\frac{1}{3\alpha}}6^{-\frac{1}{2\alpha}}\gamma^{-\frac{2}{3\alpha}}$$

then the condition 131 implies 129, 130. We can choose $N^{s/4} = \sqrt{6}\,\delta^{-\alpha}$. Therefore by 118

$$T \leq 6\,\mu^2 T_0 = \mathcal{O}(\mathcal{C}^{\frac{6\alpha}{1+\alpha}}) \lesssim \mathcal{C}^6.$$

REFERENCES

[1] P. Baldi, M. Berti and R. Montalto, KAM for autonomous quasi-linear perturbations of KdV, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, **33** (2016), 1589–1638.

[2] D. Bambusi, Nekhoroshev theorem for small amplitude solutions in nonlinear Schrödinger equations, *Math. Z.*, **230** (1999), 345–387.

[3] D. Bambusi and N. N. Nekhoroshev, Long time stability in perturbations of completely resonant PDE's, *Acta Applicandae Mathematica*, **70** (2002), 1–22.

[4] M. Berti and M. Procesi, Quasi-periodic solutions of completely resonant forced wave equations, *Comm. Partial Differential Equations*, **31** (2006), 959–985.

[5] L. Biasco, L. Chierchia and D. Treschev, Stability of nearly integrable, degenerate Hamiltonian systems with two degrees of freedom, *J. Nonlinear Sci.*, **16** (2006), 79–107.

[6] L. Biasco, J. E. Massetti and M. Procesi, An abstract birkhoff normal form theorem and exponential type stability of the 1d NLS, *Communications in Mathematical Physics*, **375** (2020), 2089–2153.

[7] A. Bounemoura, Generic perturbations of linear integrable hamiltonian systems, *Regul. Chaotic Dyn.*, **21** (2016), 665–681.

[8] A. Bounemoura and V. Kaloshin, Generic fast diffusion for a class of non-convex hamiltonians with two degrees of freedom, *Mosc. Math. J.*, **14** (2014), 181–203.

[9] J. Bourgain, Aspects of long time behaviour of solutions of nonlinear hamiltonian evolution equations, *Geom. Funct. Anal.*, **5** (1995), 105–140.

[10] J. Bourgain, On the growth in time of higher sobolev norms of smooth solutions of hamiltonian PDE, *Internat. Math. Res. Notices*, (1996), 277–304.

[11] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation, *Invent. Math.*, **181** (2010), 39–113.

[12] R. Feola, F. Giuliani and M. Procesi, Reducible kam tori for the Degasperis-Procesi equation, *Communications in Mathematical Physics*, **377** (2020), 1681–1759.

[13] P. Gérard and S. Grellier, The cubic Szegö equation, *Ann. Sci. Éc. Norm. Supér*, **43** (2010), 761–810.

[14] P. Gérard and S. Grellier, Effective integrable dynamics for a certain nonlinear wave equation, *Anal. PDE*, **5** (2012), 1139–1155.

[15] F. Giuliani, M. Guardia, P. Martin and S. Pasquali, Chaotic-like transfers of energy in hamiltonian pdes, *Communications in Mathematical Physics*, (2021).

[16] B. Grébert and L. Thomann, Resonant dynamics for the quintic nonlinear Schrödinger equation, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, **29** (2012), 455–477.

[17] B. Grébert and C. Villegas-Blas, On the energy exchange between resonant modes in nonlinear Schrödinger equations, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, **28** (2011), 127–134.

[18] M. Guardia, E. Haus, Z. Hani, A. Maspero and M. Procesi, Strong nonlinear instability and growth of Sobolev norms near quasiperiodic finite-gap tori for the 2d cubic nls equation, *J. Eur. Math. Soc.*, (2020).

[19] M. Guardia, E. Haus and M. Procesi, Growth of Sobolev norms for the analytic nls on $\mathbb{T}^2$, *Adv. Math.*, **301** (2016), 615–692.

[20] M. Guardia and V. Kaloshin, Growth of sobolev norms in the cubic defocusing nonlinear Schrödinger equation, *J. Eur. Math. Soc.*, **17** (2015), 71–149.

[21] Z. Hani, Long-time instability and unbounded sobolev orbits for some periodic nonlinear Schrödinger equations, *Arch. Ration. Mech. Anal.*, **211** (2014), 929–964.

[22] E. Haus and M. Procesi, Kam for beating solutions of the quintic NLS, *Communications in Mathematical Physics*, **354** (2017), 1101–1132.

[23] S. B. Kuksin, Oscillations in space-periodic nonlinear Schrödinger equations, *Geom. Funct. Anal.*, **7** (1997), 338–363.

[24] A. Maspero, Lower bounds on the growth of sobolev norms in some linear time dependent Schrödinger equations, *Math. Res. Lett.*, **26** (2019), 1197–1215.

[25] N. N. Nekhorošev, An exponential estimate of the time of stability of nearly integrable Hamiltonian systems, *Uspehi Mat. Nauk*, **32** (1977), 5–66, 287.

Received September 2020; 1st revision January 2021; 2nd revision March 2021.

E-mail address: filippo.giuliani@upc.edu