Beyond all orders breakdown of the homoclinic connection to L_3 in the Restricted Planar Circular 3-Body Problem

I. Baldomá, M. Giralt and M. Guardia

Universitat Politècnica de Catalunya (UPC)

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Table of contents

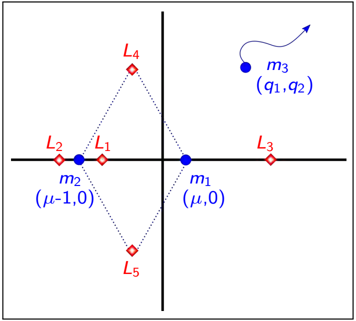
Restricted Planar Circular 3-Body Problem

- Restricted: one body is massless, $m_3 = 0$.
- Planar: the bodies move on the same plane.
- Circular: the primaries (m_1 and m_2) perform a circular motion.

With:

- $m_1 = 1 - \mu$ and $m_2 = \mu$ with $\mu \in \left(0, \frac{1}{2}\right]$.
- Rotating framework with primaries fixed at $(\mu, 0)$ and $(\mu - 1, 0)$.

We study the third body: $(q(t), p(t)) \in \mathbb{R}^4$.



Approach: Perturbative study for $0 < \mu \ll 1$.

Hamiltonian: $h = h_0 + \mu h_1$

$$h_0(q,p) = \frac{\|p\|^2}{2} - q^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{1}{\|q\|},$$

$$\mu h_1(q;\mu) = \frac{1}{\|q\|} - \frac{(1-\mu)}{\|q-(\mu,0)\|} - \frac{\mu}{\|q-(\mu-1,0)\|}.$$

- The system is autonomous.
- h_0 is integrable (2-BP).
- μh_1 is a perturbation when "far" from the primaries.
- Five critical points, (Lagrange points).

The Lagrange point L_3

L_3 is a critical point of saddle-center type. For $\mu > 0$ small it has eigenvalues:

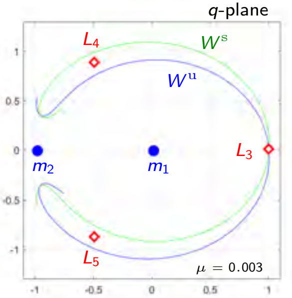
$$\pm\sqrt{\mu\frac{21}{8}}(1+\mathcal{O}(\mu)), \qquad \pm i(1+\mathcal{O}(\mu)).$$

Eigenvalues at different time-scales: **singular perturbation** problem.

- L_3 has one dimensional stable and unstable manifolds W^{s} and W^{u}.

  The manifolds either coincide or do not intersect.

- We focus on the upper branches. The lower branches are symmetric.



Goal:

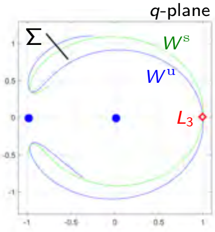
To **measure the distance** between the stable and unstable manifolds of L_3 for $0 < \mu \ll 1$. Due to the **rapidly rotating dynamics**, the invariant manifolds are **exponentially close** to each other with respect to $\sqrt{\mu}$.

Main Result

Main Theorem:

Take the section Σ (see Figure) and let $(q^{\mathrm{u,s}}, p^{\mathrm{u,s}})$ be the first intersection of $W^{\mathrm{u,s}}$ with Σ. For μ small enough

$$\|q^{\mathrm{u}} - q^{\mathrm{s}}\| + \|p^{\mathrm{u}} - p^{\mathrm{s}}\| = \mu^{-\frac{2}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left(M + \mathcal{O}\left(\frac{1}{|\log(\mu)|} \right) \right)$$



- The **constant A** is given by

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}\, dx \approx 0.177744.$$

Computed by J. Font (1984), C. Simó, P. Sousa-Silva and M. Terra (2013).

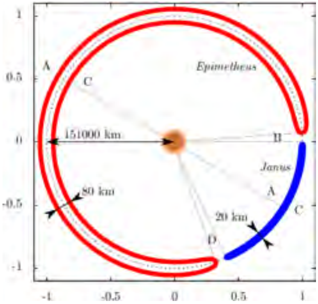
- The **constant M** corresponds to a Stokes constant. It does not have a closed formula, but it can be computed numerically by means of the so-called inner equation.

Dynamics on a neighborhood of L_3 and its invariant manifolds

- Horseshoe-shaped orbits: quasi-periodic orbits encompassing L_3, L_4 and L_5.

- The interest of these orbits arises when modeling the motion of co-orbital satellites (Janus and Epimetheus, for example).

L. Niederman, A. Pousse and P. Robutel. 2020.
On the co-orbital motion in the 3-BP: Existence of quasi-periodic horseshoe-shaped orbits.



- The center stable and center unstable manifolds of L_3 act as boundaries of regions of effective stability around L_4 and L_5.

C. Simó, P. Sousa-Silva and M. Terra. 2013.
Practical stability domains near $L_{4,5}$ in the restricted three-body problem: some preliminary facts.

Next steps

- Prove the existence of chaotic dynamics around L_3:

  Look for Smale's horseshoes associated to the invariant manifolds of "small" Lyapunov orbits.

- Prove the existence of a sequence $\mu_k \to 0$ such that, for these parameters, there exist secondary homoclinic connections.

  E. Barrabés, J. M. Mondelo and M. Ollé. (2008). Dynamical aspects of multi-round horseshoe-shaped homoclinic orbits in the RTBP.

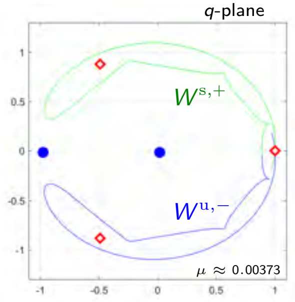
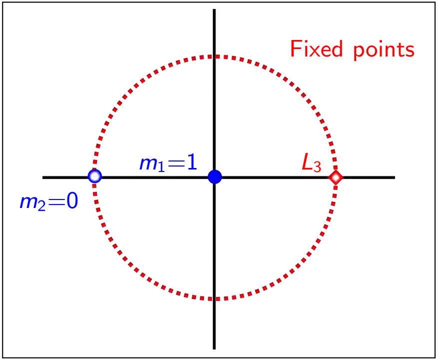


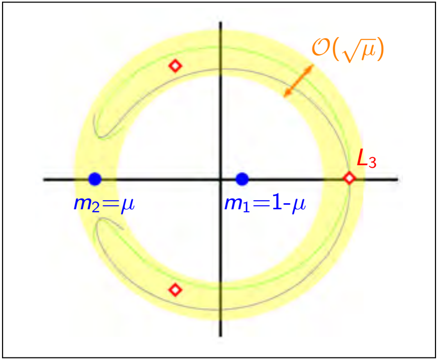
Table of contents

Unperturbed system

$\mu = 0$



$\mu > 0$



- The eigenvalues of L_3 are $\pm\sqrt{\frac{21}{8}\mu}(1+\mathcal{O}(\mu))$ and $\pm i(1+\mathcal{O}(\mu))$.

Goal:

Apply a singular change of coordinates to obtain a **new first order** of the system with a saddle and separatrices.

New "unperturbed system"

Hamiltonian in "good" coordinates and scaled time:

$$H = H_{\mathrm{p}} + H_{\mathrm{osc}} + \mathcal{O}(\mu^{\frac{1}{4}}),$$

$$H_{\mathrm{p}}(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}}, \qquad H_{\mathrm{osc}}(x, y; \mu) = \frac{xy}{\mu^{\frac{1}{2}}},$$

with symplectic form $d\lambda \wedge d\Lambda + i\, dx \wedge dy$.

- The linearization of the vector field at L_3 is

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ -\frac{7}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\mu^{\frac{1}{2}}} & 0 \\ 0 & 0 & 0 & -\frac{i}{\mu^{\frac{1}{2}}} \end{pmatrix} + \mathcal{O}(\mu^{\frac{1}{4}}).$$

New "unperturbed system":

A pendulum-like Hamiltonian H_{p} with two homoclinic connections plus a fast oscillator H_{osc}.

Problem:

Terms $\mathcal{O}(\mu^{\frac{1}{4}})$ are not explicit.

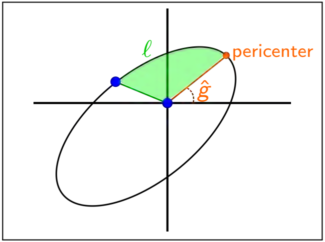
How do we obtain these coordinates?

- Step 1: Performing an action-angle change of coordinates, to decouple on a first order the saddle and center behaviour.

- Step 2: Scaling the coordinates to capture the slow-fast dynamics.

Step 1: Delaunay elements

The Delaunay elements $(\ell, L, \hat{g}, G)$ are action-angle coordinates for the 2-body problem (for negative energy).



- Mean anomaly, ℓ.
- Argument of the pericenter, $\hat{g}$.
- Square root of the semi major axis, L.
- Angular momentum, G.

Equations:

$$\dot{\ell} = \frac{1}{L^3}, \quad \dot{\hat{g}} = 0, \quad \dot{L} = 0, \quad \dot{G} = 0,$$

The Delaunay elements on a rotating framework are (ℓ, L, g, G) with $g = \hat{g} - t$.

Problems with Delaunay elements:

- They are not well defined for circular orbits: $L - G = 0$.
- They are not explicit.

Step 1: Poincaré elements

The Poincaré elements, (λ, L, η, ξ), are defined as:

$$\lambda = \ell + g, \qquad L, \qquad \eta = \sqrt{L-G}e^{ig}, \qquad \xi = i\sqrt{L-G}e^{-ig}.$$

with symplectic form $d\lambda \wedge dL + i d\eta \wedge d\xi$.

- The Lagrange point L_3 is $(\lambda, L, \eta, \xi) = (0, 1, 0, 0) + \mathcal{O}(\mu)$.
- The linearized part of the vector field associated to L_3 is

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} + \mathcal{O}(\mu).$$

We have uncoupled up to a first order the center and the degenerate behavior.

Step 2: Scaled coordinates

We consider the scaling:

$$L = 1 + \mu^{\frac{1}{2}} \Lambda, \qquad \eta = \mu^{\frac{1}{4}} x, \qquad \xi = \mu^{\frac{1}{4}} y.$$

- The Lagrange point L_3 is $(\lambda, \Lambda, x, y) = (0, 0, 0, 0) + \mathcal{O}(\sqrt{\mu})$.
- The linearization of the vector field at L_3 is

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ -\frac{7}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\mu^{\frac{1}{2}}} & 0 \\ 0 & 0 & 0 & -\frac{i}{\mu^{\frac{1}{2}}} \end{pmatrix} + \mathcal{O}(\mu^{\frac{1}{4}}).$$

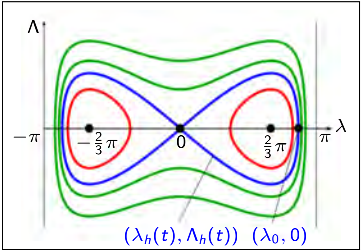
Hamiltonian in scaled coordinates

$$H = H_{\mathrm{p}} + H_{\mathrm{osc}} + \mathcal{O}(\mu^{\frac{1}{4}}),$$

$$H_{\mathrm{p}}(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 - \cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}}, \qquad H_{\mathrm{osc}}(x, y; \mu) = \frac{xy}{\mu^{\frac{1}{2}}}.$$

Table of contents

Hamiltonian $H_{\mathrm{p}}(\lambda, \Lambda)$



- The Hamiltonian

$$H_{\mathrm{p}} = -\frac{3}{2}\Lambda^2 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}},$$

has two homoclinic connections or separatrices.

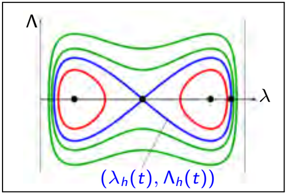
- Parametrization of the right separatrix:

$$(\lambda_h(t), \Lambda_h(t)), \quad t \in \mathbb{R},$$

initial condition $(\lambda_0, 0)$ at $t = 0$.

- There is no explicit expression for (λ_h, Λ_h).
- We expect the perturbed invariant manifolds to be close to $(\lambda_h, \Lambda_h, 0, 0)$.

Analytical properties of the separatrix



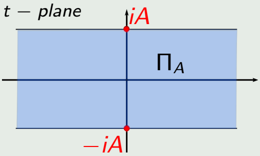
- Parametrization of the right separatrix:

$$(\lambda_h(t), \Lambda_h(t)), \quad t \in \mathbb{R},$$

initial condition $(\lambda_0, 0)$ at $t = 0$.

- Goal: analytically extend (λ_h, Λ_h) to the complex plane.

Theorem A:



$(A \approx 0.177744)$

Let:

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}} dx.$$

- (λ_h, Λ_h) analytically extends to the complex strip

$$\Pi_A = \{t \in \mathbb{C} \; : \; |\mathrm{Im}\, t| < A\},$$

- (λ_h, Λ_h) has two singularities $t = \pm iA$ at $\partial\Pi_A$.

Importance of the singularities of the separatrix

Recall that A appears on the formula for the distance between the manifolds of L_3:

$$\mathrm{dist} \sim M\mu^{-\frac{2}{3}} e^{-\frac{A}{\sqrt{\mu}}}.$$

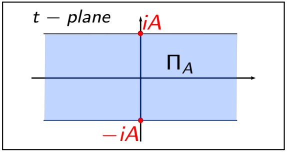
Recall the full Hamiltonian:

$$H = -\frac{3}{2}\Lambda^2 - \cos\lambda - \frac{1}{\sqrt{2+2\cos\lambda}} + \frac{xy}{\sqrt{\mu}} + \mathcal{O}(\mu^{\frac{1}{4}}).$$

Let $(\Delta x, \Delta y) = (x^{\mathrm{u}} - x^{\mathrm{s}}, y^{\mathrm{u}} - y^{\mathrm{s}})$ the difference between the stable and the unstable manifolds. Then

$$\dot{\Delta x} \approx \frac{i}{\sqrt{\mu}}\Delta x \quad \Longrightarrow \quad \Delta x(t) \approx \Delta x(0) e^{\frac{i}{\sqrt{\mu}}t},$$

$$\dot{\Delta y} \approx -\frac{i}{\sqrt{\mu}}\Delta y \quad \Longrightarrow \quad \Delta y(t) \approx \Delta y(0) e^{-\frac{i}{\sqrt{\mu}}t}.$$



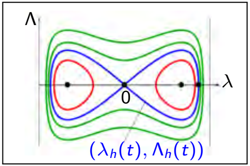
Proving that $(\Delta x, \Delta y)$ is analytic in Π_A implies

$$|\Delta x(0)| \lesssim e^{-\frac{A}{\sqrt{\mu}}} |\Delta x(-iA)|,$$

$$|\Delta y(0)| \lesssim e^{-\frac{A}{\sqrt{\mu}}} |\Delta y(iA)|.$$

$(\Delta x, \Delta y)$ bounded in a complex domain $\Longrightarrow$ exponentially small bound in $\mathbb{R}$.

Parametrization of the separatrix



- There is no explicit expression for the parametrization of the separatrix, (λ_h, Λ_h).

- We apply the change of coordinates $\ell = \cos\left(\frac{\lambda_h}{2}\right)$:

$$(\dot{\ell})^2 = \frac{3}{\ell}(\ell - 1)^2(\ell + 1)(\ell - a_+)(\ell - a_-), \qquad a_\pm = -\frac{1}{2} + \frac{\sqrt{2}}{2}.$$

- Then,

$$t = \mathcal{F}(\ell(t)) = \int_{a_+}^{\ell(t)} \frac{1}{s-1}\sqrt{\frac{s}{3(s+1)(s-a_+)(s-a_-)}}ds,$$

with $\mathcal{F}$ defined on a Riemann surface and an appropriate path.

Where the singularities of $\ell(t)$ are?

If the **inverse function theorem** cannot be applied at $\ell = \ell^*$ then $\ell(t)$ may have a singularity at a neighborhood of $t^* = \mathcal{F}(\ell^*)$.

Classification of the singularities

$$t^* = \mathcal{F}(\ell^*) = \int_{a_+}^{\ell^*} \frac{1}{s-1}\sqrt{\frac{s}{3(s+1)(s-a_+)(s-a_-)}}\,ds, \qquad a_\pm = -\frac{1}{2} + \frac{\sqrt{2}}{2}.$$

The inverse function theorem can not be applied at $t^* = \mathcal{F}(\ell^*)$ when

$$\ell^* = 0, 1, -1, a_+, a_- \qquad \text{and} \qquad |\ell^*| \to \infty.$$

Lemma:

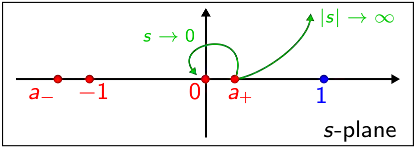
- $t^* = \mathcal{F}(0)$ is a branching point of $\ell(t)$ of order $+\frac{2}{3}$. (Singularity)
- $t^* = \mathcal{F}(1) = \pm\infty$ corresponds with the saddle. (No singularity)
- $t^* = \mathcal{F}(-1)$ is an analytic point of $\ell(t)$. (No singularity)
- $t^* = \mathcal{F}(a_+)$ is an analytic point of $\ell(t)$. (No singularity)
- $t^* = \mathcal{F}(a_-)$ is an analytic point of $\ell(t)$. (No singularity)
- $t^* = \mathcal{F}(\infty)$ is a pole of $\ell(t)$ of order 1. (Singularity)

Computation of the singularities

We have to compute

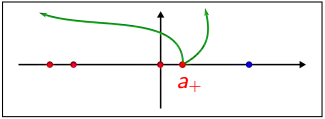
$$t^* = \int_{a_+}^{0} f(s)ds, \qquad t^* = \int_{a_+}^{\infty} f(s)ds,$$

with $f(s) = \frac{1}{s-1}\sqrt{\frac{s}{3(s+1)(s-a_+)(s-a_-)}}ds.$

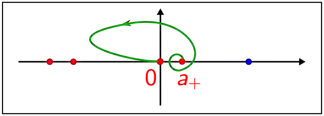


- The values of t^* **depend on the integration path** considered.
- We need to **locate all the singularities** t^* for all possible paths and select the ones with smallest imaginary module.

Examples:



- $\mathrm{Im}\, t^* = -\pi\sqrt{\frac{2}{21}} \approx 0.969516.$



- $t^* = i\int_0^{a_+} \frac{1}{1-x}\sqrt{\frac{x}{3(x+1)(a_+-x)(x-a_-)}}dx$

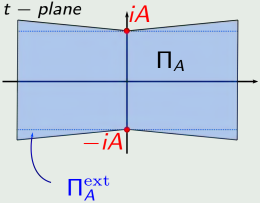
  $= iA \approx i0.177744.$

Singularities of the separatrix

Theorem A:

Let:

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}} dx.$$



- (λ_h, Λ_h) analytically to the strip

$$\Pi_A = \{t \in \mathbb{C} \; : \; |\mathrm{Im}\, t| < A\}.$$

and has only two singularities $\pm iA$ at $\partial \Pi_A$.

- (λ_h, Λ_h) extends analytically to Π_A^{ext} (see figure).
- For $|t - iA| \ll 1$,

$$\lambda_h(t) = \pi + c(t - iA)^{\frac{2}{3}} + \mathcal{O}(t - iA)^{\frac{4}{3}},$$

$$\Lambda_h(t) = -\frac{\dot{\lambda}_h(t)}{3}.$$

Table of contents

Invariant manifolds as a perturbation of the separatrix

- We expect the perturbed invariant manifolds to be close to $(\lambda_h, \Lambda_h, 0, 0)$.
- We perform the symplectic change of variables:

$$(\lambda, \Lambda, x, y) \mapsto (u, w, x, y),$$

where

$$\lambda = \lambda_h(u), \qquad \Lambda = \Lambda_h(u) - \frac{w}{3\Lambda_h(u)}.$$

This change give us control over the domain of the variable u.

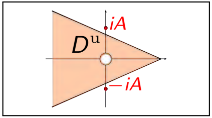
- The homoclinic connection is $(u, 0, 0, 0)$.

Lemma: $\Lambda_h(u)$ has only one zero in Π_A^{ext} at $u = 0$.

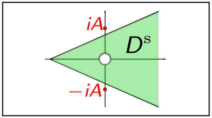
Invariant manifolds as a graph

We look for the perturbed manifolds as **graphs with respect to u**:

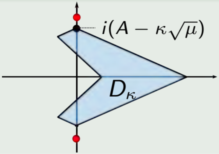
$$z^{\mathrm{u}}(u) = (w^{\mathrm{u}}(u), x^{\mathrm{u}}(u), y^{\mathrm{u}}(u)), \quad u \in D^{\mathrm{u}}$$



$$z^{\mathrm{s}}(u) = (w^{\mathrm{s}}(u), x^{\mathrm{s}}(u), y^{\mathrm{s}}(u)), \quad u \in D^{\mathrm{s}}$$



Theorem B:



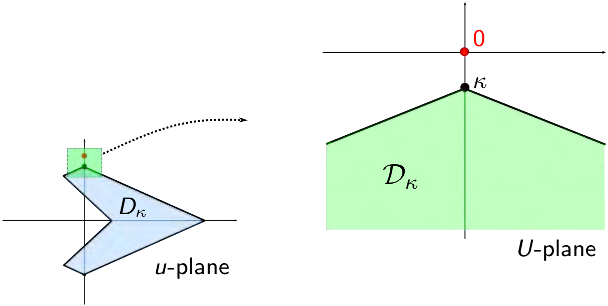
For $\mu > 0$ small enough, the difference $\Delta z = z^{\mathrm{u}} - z^{\mathrm{s}}$ is well defined and analytic in the domain D_κ.

D_κ contains a segment of the real line and has points close to $u = \pm iA$.

Table of contents

Inner change of coordinates

To capture the asymptotic first order of Δz, we need to give the dominant terms of the difference close to the singularities.



We perform the symplectic scaling:

$$U = \mu^{-\frac{1}{2}}(u - iA), \qquad W = \mu^{\frac{1}{3}} w, \qquad X = \mu^{-\frac{1}{12}} x, \qquad Y = \mu^{-\frac{1}{12}} y.$$

We can perform an analogous scaling around the singularity $u = -iA$.

Inner equation

Theorem C:

The Hamiltonian H expressed in inner coordinates can be written as

$$H^{\mathrm{in}}(U,W,X,Y;\mu) = \mathcal{H}(U,W,X,Y) + \mathcal{O}(\mu^{\frac{1}{3}}).$$

where the Hamiltonian $\mathcal{H}$ gives the **inner equation** of the system:

$$\mathcal{H} = W + XY - \frac{3}{4}U^{\frac{2}{3}}W^2 - \frac{1}{3U^{\frac{2}{3}}}\left(\frac{1}{\sqrt{1+\mathcal{J}}} - 1\right),$$

with

$$\mathcal{J} = \frac{4W^2}{9U^{\frac{2}{3}}} - \frac{16W}{27U^{\frac{4}{3}}} + \frac{16}{81U^2} + \frac{4(X+Y)}{9U}\left(W - \frac{2}{3U^{\frac{2}{3}}}\right) - \frac{4i(X-Y)}{3U^{\frac{2}{3}}} - \frac{X^2+Y^2}{3U^{\frac{4}{3}}} + \frac{10XY}{9U^{\frac{4}{3}}}.$$

Even though the original Hamiltonian H has not a closed formula, $\mathcal{H}$ is explicit.

Table of contents

Difference Inner equation

Consider two suitable invariant graphs (with respect to U) of the inner equation,

$$Z^{\mathrm{u}} = (W^{\mathrm{u}}, X^{\mathrm{u}}, Y^{\mathrm{u}}), \qquad Z^{\mathrm{s}} = (W^{\mathrm{s}}, X^{\mathrm{s}}, Y^{\mathrm{s}}).$$

Theorem D:

There exists $M^{\mathrm{in}} \in \mathbb{C}$ such that

$$\Delta Y(U) = Y^{\mathrm{u}}(U) - Y^{\mathrm{s}}(U) = M^{\mathrm{in}} e^{-iU} \left(1 + \mathcal{O}\left(\frac{1}{|U|}\right)\right),$$

as $\mathrm{Im}\, U \to -\infty$.

Corollary:

Analogously, studying $u = -iA$, we obtain an expression for ΔX.

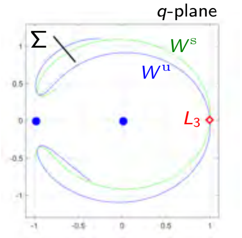
M^{in} corresponds to a Stokes constant. It can be numerically approximated using,

$$|M^{\mathrm{in}}| = \lim_{\mathrm{Im}\, U \to -\infty} \left|\Delta Y(U) e^{iU}\right| \approx 1.63.$$

A more accurate approximation for M^{in} is not straightforward, since ΔY is exponentially small.

Difference

- It only remains to translate the results for the inner equation to the original coordinates.
- By using matching complex techniques we relate the invariant graphs of the inner equation $Z^{\mathrm{u,s}}$ with the perturbed invariant graphs, $z^{\mathrm{u,s}}$.



Main Theorem:

Take the section Σ (see figure) and let $(q^{\mathrm{u,s}}, p^{\mathrm{u,s}})$ be the first intersection of $W^{\mathrm{u,s}}$ with Σ. For μ small enough

$$\|q^{\mathrm{u}} - q^{\mathrm{s}}\| + \|p^{\mathrm{u}} - p^{\mathrm{s}}\| = \mu^{-\frac{2}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left(M + \mathcal{O}\left(\frac{1}{|\log(\mu)|} \right) \right)$$

- $A \approx 1.77744$ given in Theorem A.
- The constant M satisfies $M = \alpha M^{\mathrm{in}}$ for some explicit $\alpha \neq 0$.

Final comments

- The Hamiltonian we deal with has not an explicit formula. We use series expansions to obtain the relevant terms of the Hamiltonian and to obtain the estimates for the remainders.

- The inner equation is explicit.

- The Stokes constant $\left|M^{\mathrm{in}}\right| \approx 1.63$. One could implement a computer assisted proof to obtain rigorous estimates for M^{in}.

Thanks for your attention!