A proof of the KAM theorem by a Nash-Moser approach

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- Nearly-integrable Hamiltonian systems, Kronecker tori
- The "classical" KAM theorem (in finite differentiability class)
- KAM theorem for perturbations of Isochronous Hamiltonian systems
- Functional setting, Nash-Moser theorem
- Proof of the Nash-Moser theorem:
  - Check of assumptions $(P1)$-$(P2)$-$(P3)$
  - Approximate right inverse of the linearized operator

Nearly integrable Hamiltonian systems

Let us consider a Hamiltonian $H\colon \mathbb{T}^n \times \mathcal{B}_{\mathbb{R}^n}(0,R) \to \mathbb{R}$ of class C^k (with k large) and of the form

$$H(\varepsilon;\theta,I) = H_0(I) + \varepsilon H_1(\theta,I), \quad \theta \in \mathbb{T}^n,\ I \in \mathcal{B}_{\mathbb{R}^n}(0,R)$$

with the symplectic form $\Omega := dI \wedge d\theta$.

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Unperturbed dynamics $\varepsilon = 0$

$$(\dot\theta,\dot I) = X_{H_0}(\theta,I) = J\nabla H_0(\theta,I) \quad \begin{cases} \dot\theta = \partial_I H_0(I) =: \omega_0(I), \\ \dot I = -\partial_\theta H_0(I) = 0. \end{cases}$$

The phase space is foliated by invariant tori

$$\mathbb{T}^n \times \mathcal{B}_{\mathbb{R}^n}(0,R) := \bigcup_{\xi \in \mathcal{B}_{\mathbb{R}^n}(0,R)} \mathcal{T}_\xi, \qquad \mathcal{T}_\xi := \{(\theta,I) : I = \xi\} \cong \mathbb{T}^n \times \{\xi\}$$

which support orbits winding around the torus

$$\theta(t) = \omega_0(\xi)t + \theta(0), \quad I(t) \equiv \xi.$$

Kronecker tori

Linear flow on $\mathbb{T}^n$ with frequency ω

$$\Psi^t_\omega(\varphi) = \omega t + \varphi, \quad \varphi \in \mathbb{T}^n, \quad \omega \in \mathbb{R}^n.$$

A torus supporting such motion is called a **Kronecker** torus.

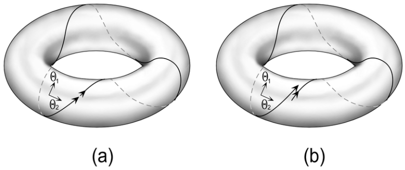
Dynamics on a Kronecker torus

If ω is **non-resonant** (or irrational), namely

$$\omega \cdot \ell \neq 0 \qquad \forall \ell \in \mathbb{Z}^n \backslash \{0\},$$

the orbit $\{\omega t\}_{t\in\mathbb{R}}$ fills densely the torus.

If ω is **resonant** the closure of the orbit is diffeomorphic to a lower-dimensional torus.



Embedded tori

General definition of a Kronecker torus

We say that $\mathcal{T} \cong \mathbb{T}^n$, which is invariant by the flow Φ_H^t of a Hamiltonian H, is a **Kronecker** torus with frequency ω if its dynamics is *conjugated* with the linear flow $\Psi_\omega^t(\varphi) = \omega t + \varphi$ on $\mathbb{T}^n$.
More precisely, there exists a smooth embedding $i\colon \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{R}^n$, $\varphi \mapsto i(\varphi) := (\theta(\varphi), I(\varphi))$ such that $\mathcal{T} = i(\mathbb{T}^n)$ and

$$i \circ \Psi_\omega^t = \Phi_H^t \circ i. \qquad (1)$$

The curves $t \mapsto i(\omega t + \varphi), \varphi \in \mathbb{T}^n$ are said **quasi-periodic** solutions of X_H.

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Example: $\mathcal{T}_\xi$ are Kronecker tori

$$\mathcal{T}_\xi = i(\mathbb{T}^n), \quad i(\varphi) = (\varphi, \xi), \quad \omega = \omega_0(\xi),$$

$$\Phi_{H_0}^t(i(\varphi)) = \Phi_{H_0}^t(\varphi, \xi) = \big(\omega_0(\xi)t + \varphi, \xi\big) = i(\omega_0(\xi)t + \varphi) = i(\Psi_{\omega_0(\xi)}^t(\varphi))$$

Moreover $\mathcal{T}_\xi$ is **Lagrangian**, namely $i_*\Omega$ vanishes on it (and it has the maximal dimension n).

When $\varepsilon > 0$, what happens to the tori $\mathcal{T}_\xi$ and their quasi-periodic motions?

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Example: Isochronous system

$$H(I) = I_1 + \sqrt{2} I_2 + \varepsilon I_2$$

When $\varepsilon = 0$ we have only 2-dimensional tori supporting quasi-periodic motions with frequency vector $(1, \sqrt{2})$.
If $\varepsilon \neq 0$, $\sqrt{2} + \varepsilon \in \mathbb{Q}$ then the solutions are all periodic.

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$$\text{Frequency map :} \quad I \mapsto \omega_0(I).$$

Non-degeneracy condition

We say that the frequency map is **non-degenerate** if

$$\det \partial_I \omega_0 \neq 0.$$

Then this map is a **local diffeomorphism**, and the frequencies are *modulated* by the actions I.

Diophantine vectors

Small divisors problem

$$f(\varphi) = \sum_{\ell \in \mathbb{Z}^n} f_\ell \, e^{\mathrm{i}\ell\cdot\varphi}, \quad \omega \cdot \partial_\varphi f(\varphi) = g(\varphi), \quad f_\ell = \frac{g_\ell}{\mathrm{i}\,\omega\cdot\ell} \quad \ell \neq 0$$

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Diophantine vectors

We require the frequencies to be strongly irrational. Let $\gamma \in (0,1), \tau > 0$

$$DC(\gamma,\tau) := \left\{ \omega \in \mathbb{R}^n : |\omega \cdot \ell| > \frac{\gamma}{|\ell|^\tau} \quad \forall \ell \in \mathbb{Z}^n \backslash \{0\} \right\}.$$

Measure

Let $\tau > n-1$ and $K \subset \mathbb{R}^n$ compact. Then $|K \backslash DC(\gamma,\tau)| \lesssim \gamma$.

KAM theorem in finite differentiability class

Let $n \geqslant 1$, $\gamma \in (0,1)$, $\tau > n-1$. Assume that $H = H_0 + \varepsilon H_1$ is of class $C^k(\mathbb{T}^n \times \mathcal{B}_{\mathbb{R}^n}(0,R); \mathbb{R})$ for some $k = k(n)$ large enough and the map

$$\xi \mapsto \omega_0(\xi) := \partial_I H_0(\xi)$$

is a local diffeomorphism. If $\sqrt{\varepsilon}\gamma^{-1} \ll 1$ then for each $\xi \in \mathcal{B}_{\mathbb{R}^n}(0,R)$ such that

$$\omega_0(\xi) \in DC(\gamma, \tau)$$

the Kronecker torus $\mathcal{T}_\xi$ persists, slightly deformed, and it supports quasi-periodic solutions for X_H with same frequency $\omega_0(\xi)$.

Positive measure set

If $\tau > n-1$ then the set of KAM tori has positive Lebesgue measure.

Replacing $\sqrt{\varepsilon}$ by ε we are led to the ξ-parameter family of isochronous Hamiltonian systems

$$H(\xi, \varepsilon; \theta, y) = \omega_0(\xi) \cdot y + \varepsilon P(\xi, \varepsilon; \theta, y).$$

Thanks to the twist condition we can consider $\alpha = \omega_0(\xi)$ as a parameter instead of the unperturbed actions ξ, hence

$$H_{\alpha,\varepsilon}(\theta, y) := H(\alpha, \varepsilon; \theta, y) = \alpha \cdot y + \varepsilon P(\alpha, \varepsilon; \theta, y).$$

When $\varepsilon = 0$, $\{y = 0\}$ is an invariant Kronecker torus with frequency α.

KAM theorem for perturbations of isochronous systems

If $\varepsilon\gamma^{-1} \ll 1$ then for any $\omega \in DC(\gamma, \tau)$ there exists a $\alpha = \alpha_\infty(\omega, \varepsilon) = \omega + O(\varepsilon)$ such that there exists a Kronecker torus with frequency ω close to $\{y = 0\}$ which is invariant for $X_{H(\alpha_\infty(\omega,\varepsilon),\varepsilon;\cdot)}$.

Functional setting

We look for an embedding $i\colon \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{R}^n$, $\varphi \mapsto i(\varphi) := (\theta(\varphi), y(\varphi))$ such that

$$i \circ \Psi_\omega^t = \Phi_{H_{\alpha,\varepsilon}}^t \circ i.$$

Differentiating in t and using that $\{\omega t\}$ is dense on $\mathbb{T}^n$ we obtain the following functional equation

$$\omega \cdot \partial_\varphi i - X_{H_{\alpha,\varepsilon}}(i) = 0.$$

Technical trick: we introduce the counter term $\zeta \in \mathbb{R}^n$ considering the Hamiltonian

$$H_{\alpha,\varepsilon,\zeta}(\theta, y) = H_{\alpha,\varepsilon}(\theta, y) + \zeta \cdot \theta.$$

Given $\omega \in DC(\gamma, \tau)$, we will solve the equation

$$\mathcal{F}_\omega(\varepsilon; \alpha, \zeta, i(\varphi)) := \omega \cdot \partial_\varphi i(\varphi) - X_{H_{\alpha,\varepsilon,\zeta}}(i(\varphi)) = 0$$

in the unknowns (α, ζ, i). We know that

$$\mathcal{F}_\omega(\varepsilon = 0; \omega, 0, (\varphi, 0)) = 0$$

and we define $\mathfrak{I}(\varphi) := i(\varphi) - (\varphi, 0)$.

Given an embedding $i(\varphi)$, we call $Z(\varphi) = (Z_1(\varphi), Z_2(\varphi)) \colon \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{R}^n$

$$Z(\varphi) := \mathcal{F}_\omega(\varepsilon; \alpha, \zeta, i(\varphi)) = \omega \cdot \partial_\varphi i(\varphi) - X_{H_{\alpha,\varepsilon,\zeta}}(i(\varphi))$$

the **error function**. It measures how much $i(\varphi)$ is a good approximate solution of $\mathcal{F}_\omega = 0$.

Lemma

$$\zeta := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \Big(-[Dy(\varphi)]^T Z_1(\varphi) + [D\theta(\varphi)]^T Z_2(\varphi) \Big)\, d\varphi$$

Hence if $i(\varphi)$ is a solution of $H_{\alpha,\varepsilon,\zeta}$ then $i(\varphi)$ is a solution of $H_{\alpha,\varepsilon}$ ($\zeta = 0$).

proof: Let $\psi \in \mathbb{T}^n$. We denote by $i_\psi(\varphi) := (\theta_\psi(\varphi), y_\psi(\varphi)) := (\theta(\varphi+\psi), y(\varphi+\psi))$. Since the system is autonomous the following Hamiltonian action functional is constant

$$\mathcal{I}(\psi) := \int_{\mathbb{T}^n} \Big(y_\psi(\varphi) \cdot (\omega \cdot \partial_\varphi \theta_\psi(\varphi)) - H(\alpha, \zeta, i_\psi(\varphi)) \Big) \, d\varphi.$$

Differentiating at $\psi = 0$ and integrating by parts we get, for all $\eta \in \mathbb{R}^n$,

$$\begin{aligned} 0 = d\mathcal{I}(0)[\eta] &= - \int_{\mathbb{T}^n} \Omega\big(\omega \cdot \partial_\varphi i - X_{H_{\alpha,\varepsilon,\zeta}}(i), Di(\varphi)[\eta]\big) \, d\varphi \\ &= - \int_{\mathbb{T}^n} \Omega\big(Z(\varphi) - (0, \zeta), Di(\varphi)[\eta]\big) \, d\varphi \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{T}^n} \Omega\big((0,\zeta), Di(\varphi)[\eta]\big) \, d\varphi &\overset{\Omega = dy \wedge d\theta}{=} \int_{\mathbb{T}^n} \zeta \cdot D\theta(\varphi)[\eta] \, d\varphi \\ &= \int_{\mathbb{T}^n} \zeta \cdot \eta \, d\varphi + \zeta \cdot \int_{\mathbb{T}^n} D\Theta(\varphi)[\eta] \, d\varphi = (2\pi)^n \zeta \cdot \eta. \end{aligned}$$

Then for all $\eta \in \mathbb{R}^n$

$$\zeta \cdot \eta = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \Omega(Z(\varphi), Di(\varphi)[\eta]) d\varphi.$$

We consider

$$\mathcal{F}_\omega \colon (0,1) \times Y^s \to Z^s, \quad (\varepsilon, \mathcal{U}) \mapsto \mathcal{F}_\omega(\varepsilon, \mathcal{U}), \quad \mathcal{U} := (\alpha, \zeta, i),$$

$$Y^s := \mathbb{R}^n \times \mathbb{R}^n \times H^s(\mathbb{T}^n; \mathbb{T}^n \times \mathbb{R}^n), \quad \|\!|\mathcal{U}|\!\|_s := |\alpha|_{\mathbb{R}^n} + |\zeta|_{\mathbb{R}^n} + \|i\|_s,$$

$$\|i\|_s := \|i(\varphi)\|_{H^s(\mathbb{T}^n;\mathbb{T}^n\times\mathbb{R}^n)} = \|\theta(\varphi)\|_{H^s(\mathbb{T}^n;\mathbb{R}^n)} + \|y(\varphi)\|_{H^s(\mathbb{T}^n;\mathbb{R}^n)}$$

$$Z^s := H^{s-1}(\mathbb{T}^n; \mathbb{T}^n \times \mathbb{R}^n), \quad |\cdot|_s := \|\cdot\|_{s-1}.$$

Let $s_0 > n/2$, $\varepsilon > 0$. Define $\mathtt{B} := \mathtt{B}_{\varepsilon, s_0} := \{(\alpha, \zeta, i) \in Y^{s_0} : \|\mathfrak{I}\|_{s_0} \leqslant \varepsilon\}$.

Nash-Moser Theorem

Under the assumptions $(P1)$-$(P2)$-$(P3)$-(INV) there exists $\varepsilon_0 \in (0,1)$ and $s_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$ there exists

$$\mathcal{U}_\infty = \mathcal{U}_\infty(\omega, \varepsilon) := \big(\alpha_\infty(\omega, \varepsilon), \zeta_\infty(\omega, \varepsilon), i_\infty(\omega, \varepsilon; \varphi)\big) \in Y^\infty \cap \mathtt{B}$$

such that $\mathcal{F}_\omega(\varepsilon, \mathcal{U}_\infty) = 0$ and

$$\|\!|\mathcal{U}_\infty - \mathcal{U}_*|\!\|_{s_0} \leqslant C(s_0)\varepsilon, \quad \mathcal{U}_* := (\omega, 0, (\varphi, 0)).$$

Assumptions $(P1)$-$(P2)$

$(P1)$

$\mathcal{F}_\omega : (0,1) \times \big(Y^s \cap \mathtt{B}\big) \to Z^s$ for all $s \geqslant s_0$, it is continuos and it satisfies the tame estimates

$$|\mathcal{F}_\omega(\varepsilon, \mathcal{U})|_s \leqslant C(s)(1 + |||\mathcal{U}|||_s) \quad \forall \varepsilon \in (0,1),\ \forall \mathcal{U} \in Y^s \cap \mathtt{B}. \tag{2}$$

proof:

$$\mathcal{F}_\omega(\varepsilon, \mathcal{U}) = \begin{pmatrix} \omega \cdot \partial_\varphi \theta(\varphi) - \alpha - \varepsilon \partial_y P(\alpha, \varepsilon; \theta(\varphi), y(\varphi)) \\ \omega \cdot \partial_\varphi y(\varphi) + \zeta + \varepsilon \partial_\theta P(\alpha, \varepsilon; \theta(\varphi), y(\varphi)) \end{pmatrix}$$

$\omega \cdot \partial_\varphi$ is a linear operator and P is a smooth function, hence the continuity assumption holds.
To prove (2) we have to apply the Composition Lemma to the functions $\partial_{y,\theta} P(\varepsilon; \cdot) \circ \mathcal{U}$, recalling that $\mathcal{U} \in \mathtt{B}$ (smallness assumption on the **low norm**).

$(P2)$

$\mathcal{F}_\omega(0, \mathcal{U}_*) = 0$ where $\mathcal{U}_* := (\omega, 0, (\varphi, 0))$.

$(P3)$

For all $\varepsilon \in (0,1)$ $\mathcal{F}_\omega(\varepsilon,\cdot)\colon Y^s \cap \mathtt{B} \to Z^s$ there exists a linear **tame** operator $\mathcal{L}_\omega(\varepsilon,\cdot) := d_{\alpha,\zeta,i}\mathcal{F}_\omega(\varepsilon,\cdot)$ such that for all $\mathcal{U} \in Y^s \cap \mathtt{B}$, $h \in Y^s$

$$|\mathcal{F}_\omega(\varepsilon,\mathcal{U}+h) - \mathcal{F}_\omega(\varepsilon,\mathcal{U}) - \mathcal{L}_\omega(\varepsilon,\mathcal{U})[h]|_s = o(|\!|\!| h |\!|\!|_s) \tag{3}$$

$$|\mathcal{L}_\omega(\varepsilon,\mathcal{U})[h]|_s \leqslant C(s)\Big(|\!|\!| h |\!|\!|_s + |\!|\!| \mathcal{U} |\!|\!|_s |\!|\!| h |\!|\!|_{s_0}\Big) \tag{4}$$

$$|\mathcal{L}_\omega(\varepsilon,\mathcal{U}_1)[h] - \mathcal{L}_\omega(\varepsilon,\mathcal{U}_2)[h]|_{s_0} \leqslant C(s_0) |\!|\!| \mathcal{U}_1 - \mathcal{U}_2 |\!|\!|_{s_0} |\!|\!| h |\!|\!|_{s_0} \tag{5}$$

proof: We have to use the Composition Lemma, the fact that $\mathcal{U} \in \mathtt{B}$ and the following expressions

$$\mathcal{L}_\omega(\varepsilon,\mathcal{U})[h] = \begin{pmatrix} -\hat{\alpha} - \varepsilon\big[\partial^2_{y\theta}P(\varepsilon;\alpha,\zeta,i)\hat{\theta} + \partial^2_{yy}P(\varepsilon;\alpha,\zeta,i)\hat{y}\big] \\ \hat{\zeta} + \varepsilon\big[\partial^2_{\theta\theta}P(\varepsilon;\alpha,\zeta,i)\hat{\theta} + \partial^2_{\theta y}P(\varepsilon;\alpha,\zeta,i)\hat{y}\big] \end{pmatrix}, \quad h := (\hat{\alpha},\hat{\zeta},\hat{\imath})$$

$$\text{Taylor remainder:} \quad \mathcal{R}_2(\mathcal{U},h) = \int_0^1 (1-\tau)\, D^2\mathcal{F}_\omega(\varepsilon;\mathcal{U}+\tau\, h)[h,h]\, d\tau$$

$$|\mathcal{R}_2(\mathcal{U},h)|_s \leqslant C(s)\Big(|\!|\!| h |\!|\!|_s |\!|\!| h |\!|\!|_{s_0} + |\!|\!| \mathcal{U} |\!|\!|_s |\!|\!| h |\!|\!|^2_{s_0}\Big)$$

(INV) assumption

(INV)

There exists $\rho > 0$ such that for any $\varepsilon \in (0,1)$, $\mathcal{U} \in \mathtt{B} \cap Y^{\infty}$ there exists an approximate right inverse of $\mathcal{L}_\omega(\varepsilon, \mathcal{U})$, namely an operator $\mathtt{T}_\omega(\varepsilon, \mathcal{U})$ such that for all $h \in Y^{s+\rho}$

$$|\mathtt{T}_\omega(\varepsilon, \mathcal{U})[h]|_s \leqslant C(s,\varepsilon)\Big(\| h \|_{s+\rho} + \| \mathcal{U} \|_{s+\rho} \| h \|_{s_0} \Big)$$

$$\begin{aligned} |(\mathcal{L}_\omega(\varepsilon, \mathcal{U})\, \mathtt{T}_\omega(\varepsilon, \mathcal{U}) - \mathrm{I})[h]|_s \leqslant C(s,\varepsilon)\Big(& |\mathcal{F}_\omega(\varepsilon, \mathcal{U})|_{s_0} \| h \|_{s+\rho} + |\mathcal{F}_\omega(\varepsilon, \mathcal{U})|_{s+\rho} \| h \|_{s_0} \\ & + \| \mathcal{U} \|_{s+\rho} |\mathcal{F}_\omega(\varepsilon, \mathcal{U})|_{s_0} \| h \|_{s_0} \Big) \end{aligned}$$

Strategy

We consider an approximate solution $\mathcal{U}_0(\varphi) = (\alpha, \zeta, i_0(\varphi))$ such that

$$\gamma^{-1} \|\mathfrak{I}_0\|_{s_0+\mu} \ll 1, \quad \|Z\|_{s_0+\mu} \ll 1$$

$$\mathfrak{I}_0 := i_0(\varphi) - (\varphi, 0) = (\Theta_0(\varphi), y_0(\varphi))$$

The symplectic form $(i_0)_* \Omega$ on $\mathbb{T}^n$ is "approximately" zero ($i_0(\mathbb{T}^n)$ is not Lagrangian).

- We show that there exists a Lagrangian embedding $i_\delta(\varphi)$ close to $i_0(\varphi)$ such that $\mathcal{U}_\delta := (\alpha, \zeta, i_\delta(\varphi))$ is an approximate solution (good as well as $\mathcal{U}_0$).
- Thanks to the isotropicity of i_δ, there exists a a **symplectic** change of coordinates defined in a neighborhood of $i_\delta(\mathbb{T}^n)$ that *triangularizes* the linearized problem. Then, in these coordinates, it is easy to find an approximate right inverse $\mathtt{T}$ for $d\mathcal{F}_\omega(\varepsilon, \mathcal{U}_\delta)$.
- We show that $\mathtt{T}$ is an approximate right inverse also for $d\mathcal{F}_\omega(\varepsilon, \mathcal{U}_0)$.

Differential forms

The symplectic form is exact $\Omega = dy \wedge d\theta$ and

$$\Omega = d\lambda, \qquad \lambda(\theta, y)[\hat{\theta}, \hat{y}] = y \cdot \hat{\theta}.$$

We consider the pull-back 1-form on $\mathbb{T}^n$ is

$$(i_0)_* \lambda(\varphi)[\hat{\varphi}] = \lambda(i_0(\varphi))[D\, i_0(\varphi)[\hat{\varphi}]] = \big(D\theta_0(\varphi)\big)^T y_0(\varphi) \cdot \hat{\varphi}$$

We can express $(i_0)_* \lambda$ as a vector field on $\mathbb{T}^n$

$$(i_0)_* \lambda(\varphi) = \sum_{j=1}^{n} a_j(\varphi)\, d\varphi_j \qquad a_j(\varphi) := \big(D\theta_0(\varphi)\big)^T y_0(\varphi) \cdot \mathtt{e}_j.$$

Then $(i_0)_* \Omega(\varphi) = \sum_{k<j} A_{kj}(\varphi)\, d\varphi_k \wedge d\varphi_j$ with $A_{kj} := \partial_{\varphi_k} a_j - \partial_{\varphi_j} a_k$.

The coefficients $A_{kj}(\varphi)$ measure the **lack of isotropy** of i_0.

Lemma

Recall that $(i_0)_*\Omega(\varphi) = \sum_{k<j} A_{kj}(\varphi)\, d\varphi_k \wedge d\varphi_j$. Then

$$A_{kj}(\varphi) = (\omega \cdot \partial_\varphi)^{-1}\Big(\Omega(DZ(\varphi)\mathsf{e}_k, Di_0(\varphi)\mathsf{e}_j) + \Omega(Di_0(\varphi)\mathsf{e}_k, DZ(\varphi)\mathsf{e}_j)\Big).$$

$$\|A_{kj}\|_s \lesssim_s \gamma^{-1}\big(\|\mathfrak{I}_0\|_{s_0+1}\|Z\|_{s+\tau+1} + \|\mathfrak{I}_0\|_{s+\tau+1}\|Z\|_{s_0+1}\big)$$

Then $A_{kj} = O(\gamma^{-1}Z)$.

proof: we want to compute the Lie derivative $L_\omega\big((i_0)_*\Omega\big) := \frac{d}{dt}_{|t=0}(\Psi^t_\omega)_*\,(i_0)_*\Omega$. Since Ψ^t_ω is just a translation we have

$$(\Psi^t_\omega)_*\,(i_0)_*\Omega = \sum_{k<j} A_{kj}(\varphi+\omega t)\, d\varphi_k \wedge d\varphi_j \Rightarrow L_\omega\big((i_0)_*\Omega\big) = \sum_{k<j}\big(\omega\cdot\partial_\varphi A_{kj}\big)\, d\varphi_k \wedge d\varphi_j.$$

We use the Cartan's formula $L_\omega(i_0)_*\Omega = d\big((i_0)_*\Omega(\omega,\cdot)\big) + (d(i_0)_*\Omega)(\omega,\cdot)$. We have $d(i_0)_*\Omega = d(i_0)_*d\lambda = (i_0)_*d^2\lambda = 0$. By using that $\Omega(X_H(u),\cdot) = -dH(u)[\cdot]$ we have

$$(i_0)_*\Omega(\omega,\cdot) = \Omega(X_H(i_0) + (0,\zeta) + Z, D\,i_0[\cdot]) = -dH(i_0)[Di_0] + \zeta\cdot D\theta_0 + \Omega(Z, D\,i_0)$$

If we call $(i_0)_*\Omega(\omega, \cdot) = \sum_{j=1}^n b_j(\varphi)\, d\varphi_j$ is easy to see that

$$\partial_{\varphi_k} b_j = -\frac{\partial^2(H \circ i_0)}{\partial_{\varphi_k}\partial_{\varphi_j}} + \Omega(DZ\,[\mathsf{e}_j], Di_0\,[\mathsf{e}_k])$$

Then

$$L_\omega(i_0)_*\Omega = d\big((i_0)_*\Omega(\omega,\cdot)\big) = \sum_{k>j}\big(\partial_{\varphi_k} b_j - \partial_{\varphi_j} b_k\big)\, d\varphi_k \wedge d\varphi_j = \sum_{k<j}\big(\omega\cdot\partial_\varphi A_{kj}\big)\, d\varphi_k \wedge d\varphi_j .$$

If g is zero average and $(\omega \cdot \partial_\varphi)^{-1} g$ is the only zero-average solution of $\omega \cdot \partial_\varphi h = g$ then

$$\|(\omega \cdot \partial_\varphi)^{-1} g\|_s^2 = \sum_{\ell \in \mathbb{Z}^n} \frac{|g_\ell|^2}{|\omega \cdot \ell|^2} \langle \ell \rangle^{2s} \underbrace{\lesssim}_{\omega \in DC(\gamma,\tau)} \gamma^{-2} \sum_{\ell \in \mathbb{Z}^n} |g_\ell|^2 \langle \ell \rangle^{2(s+\tau)} \lesssim \gamma^{-2} \|g\|_{s+\tau}^2$$

Recalling that $\pi_0 A_{kj} = 0$ for all $j, k = 1, \dots, n$ we conclude.

Now we look for a torus embedding $i_\delta(\varphi)$, which is close to $i_0(\varphi)$, and such that $(i_\delta)_*\Omega = (i_\delta)_* d\lambda = d(i_\delta)_*\lambda = 0$. So this is equivalent to look for $i_\delta(\varphi)$ such that $(i_\delta)_*\lambda$ is closed. We use the following results:

Helmotz decomposition

A smooth vector field $\mathsf{a}(\varphi) = \sum_{j=1}^n a_j(\varphi)\, d\varphi_j$ on $\mathbb{T}^n$ may be decomposed as the sum of a conservative and a divergence-free vector field

$$\mathsf{a} = \nabla U + \mathsf{c} + \rho, \quad U\colon \mathbb{T}^n \to \mathbb{R}, \quad \mathsf{c} \in \mathbb{R}^n, \quad \operatorname{div}\rho = 0, \quad \pi_0\rho = 0.$$

We have

- $U = \Delta^{-1}(\operatorname{div}\mathsf{a})$;
- $\rho(\varphi) = \sum_{j=1}^n \rho_j(\varphi)\, d\varphi_j$ with $\rho_j(\varphi) = \Delta^{-1}\sum_{k=1}^n \partial_{\varphi_k} A_{kj}$, $A_{kj} := \partial_{\varphi_k} a_j - \partial_{\varphi_j} a_k$;
- $c_j := \pi_0 a_j$.

Corollary

Let $\mathsf{a}(\varphi) = \sum_{j=1}^n a_j(\varphi)\, d\varphi_j$ be a 1-form on $\mathbb{T}^n$.

- $\mathsf{a} - \rho$ is closed.

Lemma

The torus embedding $i_\delta(\varphi) := (\theta_0(\varphi), y_\delta(\varphi))$ defined by

$$y_\delta(\varphi) = y_0(\varphi) - [D\theta_0(\varphi)]^T \rho(\varphi), \quad \rho_j(\varphi) := \Delta^{-1} \sum_{k=1}^{n} \partial_{\varphi_k} A_{kj}(\varphi)$$

is Lagrangian. Furthermore $y_0 - y_\delta = O(\gamma^{-1} Z)$, more precisely

$$\|y_\delta - y_0\|_{H^s(\mathbb{T}^n;\mathbb{R}^n)} \lesssim_s \gamma^{-1}\big(\|\mathfrak{I}_0\|_{s_0+1}\|Z\|_{s+\tau+2} + \|\mathfrak{I}_0\|_{s+\tau+2}\|Z\|_{s_0+1}\big). \tag{6}$$

proof: $(i_\delta)_*\lambda = (i_0)_*\lambda - \rho$, which is closed by the Corollary above. Recall that $A_{kj} = O(\gamma^{-1}Z)$.
The (6) follows by the fact that $\|D\theta_0^{\pm 1}\|_s \lesssim_s 1 + \|\mathfrak{I}_0\|_{s+1}$ and the bound on A_{kj}.

Error function for i_δ

Set $\mathcal{U}_\delta := (\alpha, \zeta, i_\delta(\varphi))$. We define the error function

$$Z_\delta(\varphi) := \mathcal{F}_\omega(\varepsilon, \mathcal{U}_\delta(\varphi)) = \omega \cdot \partial_\varphi i_\delta(\varphi) - X_{H_{\alpha,\varepsilon,\zeta}}(i_\delta(\varphi)).$$

Lemma

There exists $\sigma > 0$ such that

$$\|Z - Z_\delta\|_s \lesssim_s \|Z\|_{s+\sigma}\|\mathfrak{I}_0\|_{s_0+\sigma} + \|Z\|_{s_0+\sigma}\|\mathfrak{I}_0\|_{s+\sigma} + \|Z\|_{s+\sigma}\|Z\|_{s_0+\sigma}$$

Thus $O(Z) = O(Z_\delta)$.

proof:

$$Z - Z_\delta = \underbrace{\omega \cdot \partial_\varphi(i_0 - i_\delta)(\varphi)}_{(1)} - \underbrace{\left(X_{H_{\alpha,\varepsilon,\zeta}}(i_0(\varphi)) - X_{H_{\alpha,\varepsilon,\zeta}}(i_\delta(\varphi))\right)}_{(2)}$$

By the fundamental theorem of calculus we have

$$(2) = \int_0^1 \partial_y X_{H_{\alpha,\varepsilon,\zeta}}(\tau\, i_\delta + (1-\tau)\, i_0)\,[y_0 - y_\delta]\,d\tau$$

Since the unperturbed vector field is y-independent and the Hamiltonian P is of class C^k with k large we have that

$$\|(2)\|_s \lesssim_s \varepsilon\gamma^{-1}\big(\|\mathfrak{I}_0\|_{s_0+\sigma}\|Z\|_{s+\sigma} + \|\mathfrak{I}_0\|_{s+\sigma}\|Z\|_{s_0+\sigma}\big).$$

$$(1) = \omega \cdot \partial_\varphi (0, [D\theta_0(\varphi)]^{-T} \rho(\varphi)),$$

$$\omega \cdot \partial_\varphi [D\theta_0(\varphi)]^{-T} \rho(\varphi) = \underbrace{\big(\omega \cdot \partial_\varphi [D\theta_0(\varphi)]^{-T}\big)}_{(i)} \underbrace{\rho}_{O(\gamma^{-1} Z)} + \underbrace{[D\theta_0(\varphi)]^{-T}}_{O(1)} \underbrace{(\omega \cdot \partial_\varphi \rho)}_{O(Z)}$$

ρ and $\omega \cdot \partial_\varphi \rho$ have estimates similar to the A_{kj}'s. We have $\|D\theta_0^{\pm T}\|_s \lesssim_s 1 + \|\mathfrak{I}_0\|_{s+1}$. Regarding (i) we have

$$\omega \cdot \partial_\varphi [D\theta_0(\varphi)]^{-T} = -[D\theta_0(\varphi)]^{-T} (\omega \cdot \partial_\varphi [D\theta_0(\varphi)]^{T}) [D\theta_0(\varphi)]^{-T}$$

We know that

$$\omega \cdot \partial_\varphi \theta_0 = \alpha + \varepsilon \partial_y P + Z_1$$

then differentiating we get

$$\omega \cdot \partial_\varphi \; D\theta_0(\varphi) = \varepsilon\big(\partial_{\theta y} P(i_0(\varphi)) D\theta_0 + \partial_{yy} P(i_0(\varphi)) D y_0\big) + \partial_\varphi Z_1(\varphi).$$

$$\|\omega \cdot \partial_\varphi [D\theta_0(\varphi)]^{-T} \rho(\varphi)\|_s \lesssim_s \|Z\|_{s+\sigma} \|\mathfrak{I}_0\|_{s_0+\sigma} + \|Z\|_{s_0+\sigma} \|\mathfrak{I}_0\|_{s+\sigma} + \|Z\|_{s+\sigma} \|Z\|_{s_0+\sigma}$$

The map G_δ

The following map

$$G_\delta(\psi,\eta) := \begin{pmatrix} \theta_0(\psi) \\ y_\delta(\psi) + [D\theta_0(\psi)]^{-T}\eta \end{pmatrix}$$

is symplectic and in the new coordinates (ψ,η) the torus $i_\delta(\mathbb{T}^n)$ corresponds to $\{\eta = 0\}$.

We set $K_{\alpha,\varepsilon,\zeta} := H_{\alpha,\varepsilon,\zeta} \circ G_\delta = K + \zeta \cdot \theta_0(\psi)$, $K := H \circ G_\delta$.

$$X_{K_{\alpha,\varepsilon,\zeta}}(\psi,\eta) = [DG_\delta(\psi,\eta)]^{-1} X_{H_{\alpha,\varepsilon,\zeta}}(G_\delta(\psi,\eta)).$$

Proposition

If we call $\mathtt{L} := \omega\cdot\partial_\varphi - d_{\alpha,\zeta,i}X_K(\alpha,\varepsilon,i_\delta)$ the linearized operator at $\{\eta = 0\}$ then we can write $\mathtt{L} = \mathtt{D} + \mathcal{R}$, where $\mathtt{D}$ is invertible and $\mathcal{R} = O(Z)$

$$\|\mathtt{D}^{-1}g\|_s \lesssim_s \gamma^{-1}\big(\|g\|_{s+\sigma} + \|\mathfrak{I}_0\|_{s+\sigma}\|g\|_{s_0+\sigma}\big)$$

$$\|\mathcal{R}[\hat{\imath}]\|_s \lesssim \|\hat{\imath}\|_{s+\sigma}\|Z\|_{s_0+\sigma} + \|\hat{\imath}\|_{s_0+\sigma}\|Z\|_{s+\sigma} + \|\mathfrak{I}_0\|_{s+\sigma}\|\hat{\imath}\|_{s_0+\sigma}\|Z\|_{s_0+\sigma}$$

The linearized operator at $(\psi, 0)$ reads as

$$\begin{pmatrix} \omega \cdot \partial_\varphi \hat{\psi} - \partial_\psi K_1(\psi)[\hat{\psi}] - K_2(\psi)[\hat{\eta}] - \partial_\alpha K_1(\psi)[\hat{\alpha}] \\ \omega \cdot \partial_\varphi \hat{\eta} + [D\theta_0(\psi)]^T \hat{\zeta} + \partial_\psi (D_\alpha K_0(\alpha, \varphi)[\hat{\alpha}]) + D_{\psi\psi} K_0(\alpha, \varphi)[\hat{\psi}] + D_\psi K_1(\alpha, \varphi)^T \hat{\eta} \end{pmatrix}$$

At a solution $K_1 = \omega$, $\partial_\psi K_0 = 0$. Then, if we neglect the terms which are zero at a true solution we get

$$\mathtt{L} := \begin{pmatrix} \omega \cdot \partial_\varphi \hat{\psi} - K_2(\psi)[\hat{\eta}] - \partial_\alpha K_1(\psi)[\hat{\alpha}] \\ \omega \cdot \partial_\varphi \hat{\eta} + [D\theta_0(\psi)]^T \hat{\zeta} + \partial_\psi (D_\alpha K_0(\alpha, \varphi)[\hat{\alpha}]) \end{pmatrix}$$

We want to solve $\mathtt{L}(\hat{\zeta}, \hat{\alpha}, \hat{\psi}, \hat{\eta}) = (g_1(\varphi), g_2(\varphi))$.

If we set $\widetilde{G}_\delta(\alpha, \zeta, \psi, \eta) = (\alpha, \zeta, G_\delta(\psi, \eta))$ then

$$\mathtt{T}_\omega := D\widetilde{G}_\delta(\varphi, 0)\, \mathtt{D}^{-1}\, [DG_\delta(\varphi, 0)]^{-1}$$

is an approximate right inverse of $d_{\alpha,i,\zeta}\mathcal{F}_\omega(\varepsilon, \mathcal{U}_\delta)$.

Last Proposition

The operator $\mathtt{T}_\omega$ is an approximate right inverse of $d_{\alpha,i,\zeta}\mathcal{F}_\omega(\varepsilon, \mathcal{U}_0)$.

$$\|\mathtt{T}_\omega g\|_s \lesssim_s \gamma^{-1}\big(\|g\|_{s+\sigma} + \|\mathfrak{I}_0\|_{s+\sigma}\|g\|_{s_0+\sigma}\big)$$

$$\begin{aligned}\|\big(d_{\alpha,i,\zeta}\mathcal{F}_\omega(\varepsilon, \mathcal{U}) \circ \mathtt{T}_\omega - \mathrm{I}\big)g\|_s \lesssim_s\ & \|g\|_{s+\sigma}\|Z\|_{s_0+\sigma} + \|g\|_{s_0+\sigma}\|Z\|_{s+\sigma} \\ & + \|\mathfrak{I}_0\|_{s+\sigma}\|g\|_{s_0+\sigma}\|Z\|_{s_0+\sigma}\end{aligned}$$

Lemma

$$\|DG_\delta(\varphi,0)^{\pm 1}[\hat{\imath}]\|_s \lesssim_s \|\hat{\imath}\|_s + \|\mathfrak{I}_0\|_{s+\sigma}\,\|\hat{\imath}\|_{s_0}$$

$$\|D^2 G_\delta(\varphi,0)[\hat{\imath}_1,\hat{\imath}_2]\|_s \lesssim_s \|\hat{\imath}_1\|_s\|\hat{\imath}_2\|_{s_0} + \|\hat{\imath}_1\|_{s_0}\|\hat{\imath}_2\|_s + \|\mathfrak{I}_0\|_{s+\sigma}\|\hat{\imath}_1\|_{s_0}\|\hat{\imath}_2\|_{s_0}$$

proof:

$$DG_\delta(\varphi,0) = \begin{pmatrix} D\theta_0 & 0 \\ Dy_\delta & [D\theta_0]^{-T} \end{pmatrix}, \quad DG_\delta(\varphi,0)^{-1} = \begin{pmatrix} [D\theta_0]^{-1} & 0 \\ -[D\theta_0]^T\, Dy_\delta\,[D\theta_0]^{-1} & [D\theta_0]^T \end{pmatrix}$$

We use the notation $\hat{\imath}_k := (\hat{\psi}_k, \hat{\eta}_k)$

$$D^2 G_\delta(\varphi,0)[\hat{\imath}_1,\hat{\imath}_2] = \begin{pmatrix} D^2\theta_0(\varphi)[\hat{\psi}_1,\hat{\psi}_2] \\ D^2 y_\delta(\varphi)[\hat{\psi}_1,\hat{\psi}_2] + \partial_\psi[D\theta_0]^{-T}[\hat{\psi}_1,\hat{\eta}_2] + \partial_\psi[D\theta_0]^{-T}[\hat{\psi}_2,\hat{\eta}_1] \end{pmatrix}$$

proof:

$$d_{\alpha,i,\zeta}\mathcal{F}_\omega(\varepsilon,\mathcal{U}_0)-d_{\alpha,i,\zeta}\mathcal{F}_\omega(\varepsilon,\mathcal{U}_\delta)=\varepsilon\int_0^1 d_y d_{\alpha,i,\zeta}X_P(\tau\, i_\delta+(1-\tau)i,\alpha)[y_0-y_\delta,\hat{\imath}]\,d\tau=:\mathcal{E}_0$$

$$d_{\alpha,i,\zeta}\mathcal{F}_\omega(\varepsilon,\mathcal{U}_\delta)=DG_\delta\,\mathtt{L}\,[D\widetilde{G_\delta}]^{-1}+\mathcal{E}_1=DG_\delta\,\mathtt{D}\,[D\widetilde{G_\delta}]^{-1}+\mathcal{E}_1+\mathcal{E}_2$$

$$\mathcal{E}_1:=D^2G_\delta(\varphi,0)[DG_\delta^{-1}(\varphi,0)\mathcal{F}_\omega(\varepsilon,\mathcal{U}_\delta),\ DG_\delta^{-1}(\varphi,0)[\cdot]],$$

$$\mathcal{E}_2:=DG_\delta\,\mathcal{R}\,[D\widetilde{G_\delta}]^{-1}$$

Then

$$d_{\alpha,i,\zeta}\mathcal{F}_\omega(\varepsilon,\mathcal{U})=d_{\alpha,i,\zeta}\mathcal{F}_\omega(\varepsilon,\mathcal{U}_\delta)+\mathcal{E}_0+\mathcal{E}_1+\mathcal{E}_2$$

$$d_{\alpha,i,\zeta}\mathcal{F}_\omega(\varepsilon,\mathcal{U})\circ\mathtt{T}_\omega-\mathrm{I}=\big(\mathcal{E}_0+\mathcal{E}_1+\mathcal{E}_2\big)\circ\mathtt{T}$$

Bound for $\mathcal{E}_0$: By the Taylor-estimates on the vector field

$$\|\mathcal{E}_0[\hat{\zeta}, \hat{\imath}]\|_s \lesssim_s \varepsilon\gamma^{-1}\big(\|Z\|_{s+\sigma} + \|\mathfrak{I}_0\|_{s+\sigma}\|Z\|_{s_0+\sigma}\big)\|\hat{\imath}\|_{s_0} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{s_0+\sigma}\|Z\|_{s_0+\sigma}\|\hat{\imath}\|_s$$
$$\underbrace{\lesssim_s}_{\varepsilon\gamma^{-1}\ll 1} \|Z\|_{s+\sigma}\|\hat{\imath}\|_{s_0} + \|\mathfrak{I}_0\|_{s+\sigma}\|Z\|_{s_0+\sigma}\|\hat{\imath}\|_{s_0} + \|\mathfrak{I}_0\|_{s_0+\sigma}\|Z\|_{s_0+\sigma}\|\hat{\imath}\|_s$$

Bound for $\mathcal{E}_1$: By using the estimates on $DG_\delta^{\pm 1}$ and DG_δ^2

$$\|\mathcal{E}_1[\hat{\imath}]\|_s \lesssim_s \|\hat{\imath}\|_{s+\sigma}\|Z_\delta\|_{s_0+\sigma} + \|\hat{\imath}\|_{s_0+\sigma}\big(\|Z_\delta\|_{s+\sigma} + \|\mathfrak{I}_0\|_{s+\sigma}\|Z_\delta\|_{s_0+\sigma}\big)$$

Bound for $\mathcal{E}_2$: By using the estimates on $DG_\delta^{\pm 1}$, DG_δ^2 and for $\mathcal{R}$

$$\|\mathcal{E}_2[\hat{\imath}]\|_s \lesssim_s \|\hat{\imath}\|_{s+\sigma}\|Z\|_{s_0+\sigma} + \|\hat{\imath}\|_{s_0+\sigma}\|Z\|_{s+\sigma} + \|\mathfrak{I}_0\|_{s+\sigma}\|\hat{\imath}\|_{s_0+\sigma}\|Z\|_{s_0+\sigma}$$