Chaotic resonant dynamics and exchanges of energy in Hamiltonian PDEs

Filippo Giuliani, Marcel Guàrdia, Pau Martín, Stefano Pasquali

23/10/2020, BMD

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

European Research Council
Established by the European Commission

Nonlinear Hamiltonian PDEs

- Cubic Nonlinear Schrödinger : $-iu_t + \Delta u - |u|^2 u = 0$
- Cubic Nonlinear Wave eq: $u_{tt} - \Delta u + u^3 = 0$
- Cubic Nonlinear Beam eq: $u_{tt} + \Delta^2 u + u^3 = 0$
- $u = u(t, x)$, $t \in \mathbb{R}$, $x \in \mathbb{T}^n$, $n \geq 1$
- In this talk:
  - $n = 2$
  - Small data.

Transfer of energy

Fourier series

$$u(t,x) = \sum_{n\in\mathbb{Z}^2} u_n(t)\, e^{in\cdot x}$$

$|u_n(t)|^2$ energy of the n-th mode at time t.

Fundamental question

Understand how solutions can exchange energy among Fourier modes as time evolves.

Qualitative and Quantitative issues

(i) **(Qualitative)**: In which ways the modes can exchange energy? For instance: periodically or quasi-periodically in time? In a more chaotic fashion?

(ii) **(Quantitative)**: Can these transfers of energy lead to growth of Sobolev norms for some s?

$$\|u(t)\|_{H^s(\mathbb{T}^2)} = \|u(t,\cdot)\|_{H^s(\mathbb{T}^2)} = \left(\sum_{n \in \mathbb{Z}^2} (1+|n|^2)^s |u_n(t)|^2 \right)^{1/2}$$

In this talk we discuss issue (i), but ...

Wave turbulence theory

Energy cascade

Forward (or backward) cascade: energy moves from low to very high modes (or in the opposite direction).

Question by Bourgain (2000): there exist solutions that exhibit a quantitative version of the forward energy cascade for the defocusing NLS on $\mathbb{T}^2$?

Are there solutions u of the 2D-NLS such that for $s > 1$,

$$\limsup_{t \to +\infty} \|u(t)\|_{H^s(\mathbb{T}^2)} = +\infty \, ?$$

The I-team result

Theorem (Colliander, Keel, Staffilani, Takaoka, Tao (2010))

Fix $s > 1$, $K \gg 1$ and $\delta \ll 1$. Then there exists a global solution u of NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(0)\|_{H^s(\mathbb{T}^2)} \leq \delta, \qquad \|u(T)\|_{H^s(\mathbb{T}^2)} \geq K.$$

- Other results for NLS by Bourgain, Kuksin, Kaloshin, Guardia, Hani, Procesi, Haus, Pausader, Visciglia, Tzvetkov, Maspero...
- Also results for the Szegö and Half Wave equations (Gérard, Grellier, Pocovnicu).
- No results for other equations.

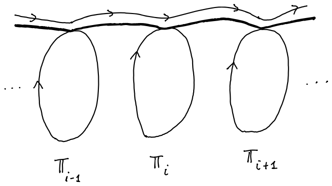
A dynamical system approach

- Consider a good approximation of the PDE for small data:
  The Birkhoff Normal Form (BNF)
- It gives the Resonant dynamics.
- Analysis of the dynamics of the **Toy model**: Search for special orbits.
- Approximation argument: an orbit of the PDE is close to one of the BNF for certain time scales.

The dynamics of the Toy Model

The toy model is a Hamiltonian system with several degrees of freedom which is hard to analyze.

- Key Point: It is integrable on certain finite dimensional invariant subspaces (as many first integrals as degrees of freedom).
- Thanks to integrability: Invariant tori connected by heteroclinic orbits.
- By perturbative arguments: construction of orbits shadowing (following closely) such structure.
- There is a correspondence between the connection of invariant objects and the transfer of energy between modes.



Some remarks

- In finite dimensional systems typically: Unstable motions (Arnold diffusion) are related to non-integrability (Chaotic dynamics, transverse homoclinic connections,...).
- The I-team result is inspired by Arnold diffusion techniques, but it does not rely on non-integrability (connections are not transversal!).
- Can we use the I-team strategy to get results for other equations?

For Wave/Beam equations

It seems hard to apply:
(i) less symmetries $\Rightarrow$ less reductions $\Rightarrow$ the analysis is harder;
(ii) The sequence of tori considered by the I-team do not lie on the same energy level, hence they are not connected!

Transfer of energy for other PDEs

To advance in this direction it would be interesting to find out different ways to transfer energy by:

(a) analyzing different invariant objects;

(b) finding more robust connections;

(c) understanding how to exploit the non-integrability to construct different transfer of energy behaviors.

To get the results that I present in this talk we dealt with these issues, but our analysis is restricted to the dynamics of resonant clusters*.

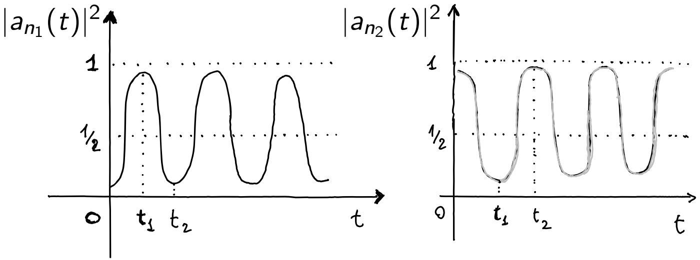
* It will not lead to growth of Sobolev norms
("local transfer of energy" – modes are somehow localized).

Local transfer of energy: Beating solutions

Periodic/Quasiperiodic transfer for NLS equations (Grebert, Haus, Paturel, Procesi, Takaoka, Thomann, Villegas-Blas ...).

$$u(t,x) = \sum_{n \in \Lambda} a_n(t) e^{inx} + \text{Small error}$$

- *Tangential set*: $\Lambda \subset \mathbb{Z}$ is a finite set.
- $|a_n(t)|^2$ are all periodic/quasiperiodic functions.
- Based on integrability of the BNF on invariant subspaces.



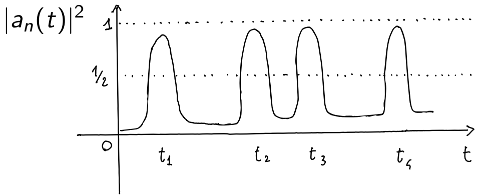
Local transfer of energy: Chaotic-like solutions

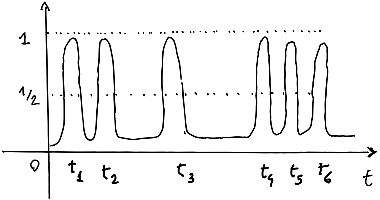
In this talk: Transfer of energy for

$$u_{tt} - \Delta u + u^3 = 0 \quad \text{and} \quad u_{tt} + \Delta^2 u + u^3 = 0, \qquad x \in \mathbb{T}^2$$

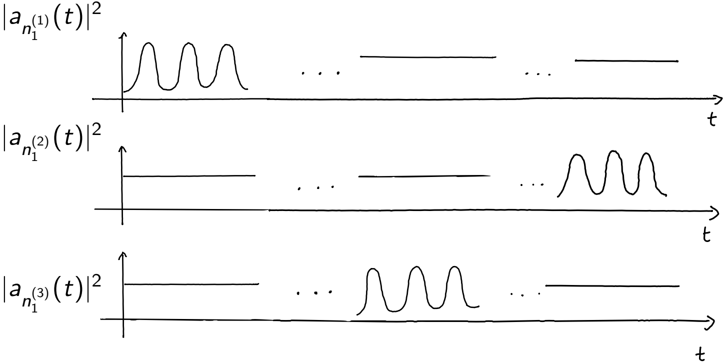
The transfer of energy among a_n is **chaotic-like**, roughly speaking this means that either:

(i) one can prescribe a sequence of times $t_1, \dots, t_k$ and find a solution whose tangential modes $\{a_n\}_{n \in \Lambda}$ exchange energy at these times;





(ii) Given a finite set of 4-tuple resonant modes $\{R_k\}_k$, $R_k := (n_1^{(k)}, n_2^{(k)}, n_3^{(k)}, n_4^{(k)})$, one can prescribe a finite sequence of R_{k_i} and find disjoint intervals of times $\mathcal{I}_{k_i}$ over which only the modes in R_{k_i} exchange energy while all the other modes are at rest.



Prescribed sequence: $1 \to 3 \to 2$

Birkhoff Normal Form

- The Hamiltonian for the Wave/Beam equation can be written as

$$H = H^{(2)} + H^{(4)}$$

  with $H^{(i)}$ is a homogeneous polynomial of degree i
- BNF: Apply a change of coordinates such that H becomes

$$K = H^{(2)} + H^{(4)}_{\text{Resonant}} + \mathcal{R}$$

  where
  - $H^{(4)}_{\text{Resonant}}$ only contains resonant terms
  - $\mathcal{R}$ can be treated as a perturbation.
- As a first step, we study the dynamics of the normal form

$$\mathcal{N} = H^{(2)} + H^{(4)}_{\text{Resonant}}$$

- NB: It is not always easy to obtain such change of coordinates for some PDEs.

Resonant Model

$$V_\Lambda := \{a_n = 0 \ \ \forall n \notin \Lambda\}$$

- $\Lambda = \bigcup_{j=1}^{N} \Lambda_i$, $N \geq 2$. The $\Lambda_i = \{n_1^{(i)}, n_2^{(i)}, n_3^{(i)}, n_4^{(i)}\}$ are disjoint resonant tuples.
- For the Wave eq:

$$n_1^{(i)} - n_2^{(i)} + n_3^{(i)} - n_4^{(i)} = 0, \qquad |n_1^{(i)}| - |n_2^{(i)}| + |n_3^{(i)}| - |n_4^{(i)}| = 0.$$

- For the Beam eq:

$$n_1^{(i)} - n_2^{(i)} + n_3^{(i)} - n_4^{(i)} = 0, \qquad |n_1^{(i)}|^2 - |n_2^{(i)}|^2 + |n_3^{(i)}|^2 - |n_4^{(i)}|^2 = 0.$$

- For well chosen Λ, the set V_Λ is invariant by the BNF.

The restriction of the normal form $\mathcal{N}$ to V_Λ defines our Resonant Model.

Resonant Dynamics, Result (1): Smale Horseshoe

Fix $N = 2$, $\Lambda = \Lambda_1 \cup \Lambda_2$. There exist sets Λ such that BNF of the Wave/Beam eq satisfies:

- Call Φ_t the flow of $\mathcal{N}$ restricted to V_Λ. Then, there exists a section Σ transverse to Φ_t such that the induced Poincaré map

$$\mathcal{P} : U = \mathring{U} \subset \Sigma \to \Sigma$$

  has an invariant set $\mathcal{S} \subset U$, $\mathcal{S} \cong \mathbb{N}^{\mathbb{Z}} \times \mathbb{T}^5$, whose dynamics is conjugated to

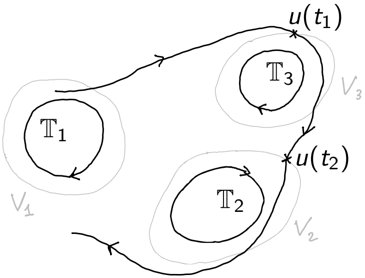
$$\widetilde{P}(\omega, \theta) = (\sigma\omega, \theta + f(\omega))$$

- Shift $\sigma : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ defined as $(\sigma\omega)_k = \omega_{k+1}$
- Namely, the map $\mathcal{P}$ (and therefore the BNF) possesses chaotic dynamics (Smale Horseshoe).

Resonant Dynamics, Result (2): Chain of invariant tori

Fix any $N \geq 2$. There exists sets Λ such that the flow Φ_t has $(2N+2)$–dimensional tori $\mathbb{T}_1, \ldots \mathbb{T}_N$ with the following property.

- Take arbitrarily small neighborhoods V_i of $\mathbb{T}_i$ and any sequence $\{p_i\}_{i\geq 1} \in \{1, \ldots, N\}^{\mathbb{N}}$.
- There exists an orbit $u(t)$ and a sequence of times $\{t_i\}_{i\geq 1} \subset \mathbb{R}$ such that $u(t_i) \in V_{p_i}$



The tori are supported in different sets of modes $\rightarrow$ The orbit transfers energy in any prescribed way for infinite time.

Some remarks

- The behaviors obtained for the BNF are for all time.
- There are “many” sets $\Lambda \subset \mathbb{Z}^2$ for which the results hold.
- Such behaviors take place at any energy level of the BNF.
- They give “chaotic transfer of energy” for the BNF:
  - Result (1): The horseshoe orbits transfer energy at random times
  - Result (2): Choosing an arbitrary sequence of tori imply activating different modes at different intervals of time.

Transfer of energy for the BNF (1)

- Fix $N = 2$: $\Lambda = \{n_1^{(1)}, n_2^{(1)}, n_3^{(1)}, n_4^{(1)}\} \cup \{n_1^{(2)}, n_2^{(2)}, n_3^{(2)}, n_4^{(2)}\}$.
- Fix $0 < \epsilon \ll 1$, $T \gg 1$.
- $\exists M_0 > 0$ such that for any sequence $\{m_j\}_{j \geq 0} \subset \mathbb{N}$, $m_j > M_0$:
- There is a solution of the BNF u with

$$u(t,x) = \sum_{n \in \Lambda} \frac{1}{|n|^\kappa} \left(a_n(t) e^{inx} + \overline{a_n}(t) e^{-inx} \right)$$

- $\kappa = 1/2$ for the Wave eq and $\kappa = 1$ for the Beam eq.
- $|a_{n_1^{(i)}}|^2 = |a_{n_3^{(i)}}|^2 = 1 - |a_{n_2^{(i)}}|^2 = 1 - |a_{n_4^{(i)}}|^2, \qquad i = 1, 2$

and...

Transfer of energy for the BNF (1)

Recall: We have $0 < \epsilon \ll 1$, $T \gg 1$, and any sequence $\{m_j\}_{j\geq 0}$, $m_j > M_0$.

- First resonant tuple:

$$|a_{n_1^{(1)}}(t)|^2 = Q(t) + \mathcal{O}(\epsilon),$$

  where Q is a $T-$periodic function with

$$\min |Q| < \epsilon \qquad \text{and} \qquad \max |Q| > 1 - \epsilon.$$

- Second resonant tuple: There exists a sequence of times $\{t_j\}_{j\geq 0}$, $t_0 = 0$,

$$t_{j+1} = t_j + T(m_j + \theta_j), \qquad \theta_j \in (0, 1).$$

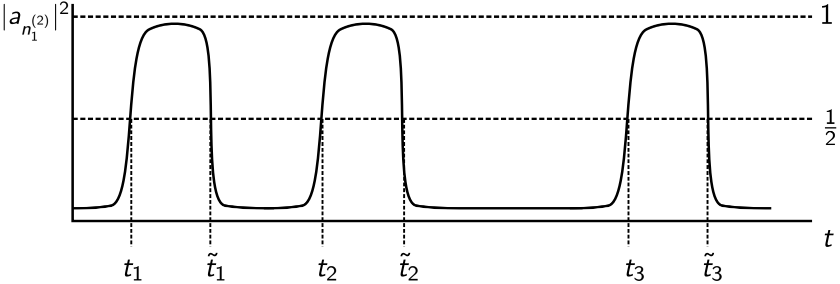
  which are the only times such that

$$|a_{n_1^{(2)}}(t)|^2 = \frac{1}{2}, \quad \frac{d}{dt}|a_{n_1^{(2)}}(t)|^2 > 0.$$

Second tuple

Namely, there exists another sequence $\{\tilde{t}_j\}_{j\geq 0}$, $t_j < \tilde{t}_j < t_{j+1}$ such that:

- For $t \in (t_j, \tilde{t}_j)$: $|a_{n_1^{(2)}}(t)|^2 > \frac{1}{2}$ and $\sup_{t\in(t_j,\tilde{t}_j)} |a_{n_1^{(2)}}(t)|^2 \geq 1-\epsilon$
- For $t \in (\tilde{t}_j, t_{j+1})$: $|a_{n_1^{(2)}}(t)|^2 < \frac{1}{2}$ and $\inf_{t\in(\tilde{t}_j,t_{j+1})} |a_{n_1^{(2)}}(t)|^2 \leq \epsilon$.



Transfer of energy for the BNF (2)

- Fix $N \geq 2$: $\Lambda = \bigcup_{i=1}^{N} \{n_1^{(i)}, n_2^{(i)}, n_3^{(i)}, n_4^{(i)}\}$.
- Fix $0 < \epsilon \ll 1$, $T \gg 1$.
- For any sequence $\{p_i\}_{i \geq 1} \in \{1, \dots, N\}^{\mathbb{N}}$, there is a solution of the BNF

$$u(t,x) = \sum_{n \in \Lambda} \frac{1}{|n|^{\kappa}} \left(a_n(t) e^{inx} + \overline{a_n}(t) e^{-inx}\right)$$

  satisfying:

  - $\kappa = 1/2$ for the Wave eq and $\kappa = 1$ for the Beam eq.
  - $|a_{n_1^{(i)}}|^2 = |a_{n_3^{(i)}}|^2 = 1 - |a_{n_2^{(i)}}|^2 = 1 - |a_{n_4^{(i)}}|^2, \qquad i = 1 \dots N$
  - and...

Transfer of energy for the BNF (2)

- There exist $[0, \infty) = I_1 \cup J_{12} \cup I_2 \cup J_{23} \cup \ldots$ with
- Beating intervals I_i: There exists $t_i > 0$ such that for $t \in I_i$:

$$sup_{t \in I_i} \left| |a_{n_1^{(p_i)}}(t)|^2 - Q(t - t_i) \right| \leq \epsilon$$

$$sup_{t \in I_i} \left| a_{n_1^{(k)}}(t) \right|^2 \leq \epsilon \qquad \text{for } k \neq p_i.$$

- Transition intervals $J_{i,i+1}$: For all $n \in \Lambda$,

$$\inf_{t \in J_{i,i+1}} |a_n(t)|^2 \leq \epsilon \qquad \text{and} \qquad \sup_{t \in J_{i,i+1}} |a_n(t)|^2 \geq 1 - \epsilon.$$

Results for the PDEs

- For $\delta \ll 1$, we construct solutions

$$u(t,x) = \delta \sum_{n\in\Lambda} \frac{1}{|n|^{\kappa}} \left(a_n(\delta^2 t) e^{inx} + \overline{a_n}(\delta^2 t) e^{-inx} \right) + R(t,x)$$

such that

  - u is defined for time $|t| \lesssim \delta^{-2}$.
  - $\|R\|_{H^s} \lesssim_s \delta^{3/2}$ for $|t| \lesssim \delta^{-2}$.
  - $\kappa = 1/2$ for the Wave eq and $\kappa = 1$ for the Beam eq.
  - $\{a_n\}_{n\in\Lambda}$ follow any of the orbits constructed by the BNF given in the previous slides

Some remarks

- We obtain chaotic-like transfer of energy at δ^{-2} time scale.
- All the results can also be obtained for the Hartree equation

$$iu_t = \Delta u + (V \star |u|^2)\, u, \quad x \in \mathbb{T}^2$$

with a convolution potential

$$V \colon \mathbb{T}^2 \to \mathbb{R}, \quad V(x) = V(-x)$$

in a suitable open set.

- It is hard to prove persistence of the Smale horseshoe for the PDEs. Most of existence results in PDEs add dissipation.

Some ideas about the proof: The resonant model

- Analyze the dynamics of the BNF on the invariant subspace

$$V_\Lambda := \{a_n = 0 \quad \forall n \notin \Lambda\}$$

- For well chosen Λ and after symplectic reduction

$$\mathcal{H}(\psi, I) = \mathcal{H}_0(\psi, I) + \varepsilon \mathcal{H}_1(\psi, I), \qquad \psi \in \mathbb{T}^N, I \in \mathbb{R}^N$$

  with $0 < \epsilon \ll 1$ and

$$\mathcal{H}_0 = \sum_{j=1}^{N} I_j(1 - I_j)(1 + 2\cos\psi_j)$$

$$\mathcal{H}_1 = \sum_{j=1}^{N} \left(a_j I_j + b_j I_j^2 + c_j I_j(1 - I_j)\cos\psi_j\right) + \sum_{i,j=1, i<j}^{N} d_{ij} I_i I_j,$$

- For $\epsilon = 0$:
  - The dynamics of each resonant tuple is decoupled.
  - The Hamiltonian $\mathcal{H}$ is integrable $\to$ chaotic behavior is not possible.

Why close to integrable?

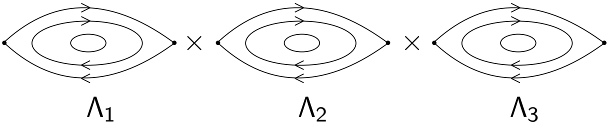
- For the cubic NLS:
  - The resonant model is $\mathcal{H}$ with $\epsilon = 0$.
  - Transfer of energy between disjoint resonant tuples is not possible.
- If the modes in Λ all have the same norm: $\mathcal{H}$ associated to Beam/Wave eq. has $\epsilon = 0$.
- We take the modes in Λ in the "thin" annulus

$$R(1-\epsilon) \leq |n_j| \leq R(1+\epsilon), \qquad \text{with } R \gg 1,\, \epsilon \ll 1.$$

- $0 < \epsilon \ll 1$: Close to integrable Hamiltonian system $\to$ perturbation methods.
- For Wave/Beam eq. and general Λ, ϵ is not small: the explained behaviors should take place but are harder to prove.

The integrable dynamics

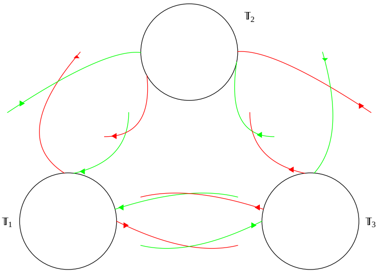
- For $\epsilon = 0$: There are saddles, periodic orbits and invariant tori.
- Their stable/unstable invariant manifolds coincide creating heteroclinic connections.



- $\epsilon = 0$ dynamics is structurally unstable in two senses. For $0 < \epsilon \ll 1$:
  - The saddles belong to different levels of energy: they cannot be connected (the saddles correspond to the tori of the I-team).
  - All stable and unstable manifolds of the periodic orbits should no longer coincide.

The perturbed dynamics: Melnikov Theory

- The stable/unstable manifolds of the periodic orbits (with same energy) intersect transversally → transverse homoclinic orbits.
- The stable/unstable manifolds of different periodic orbits (with same energy) intersect transversally → transverse heteroclinic orbits.
- The objects (periodic orbits, homoclinic connections) exist in open sets of energy level.



The perturbed dynamics

Smale Theorem $\Rightarrow$ Result (1)

Close to transverse homoclinic orbits of periodic orbits there are (infinite symbols) Smale horseshoes.

Lambda Lemma $\Rightarrow$ Result (2)

One can shadow any sequence of transverse heteroclinic orbits for infinite time.

In all these results is crucial the transversality between invariant manifolds.

THANK YOU FOR YOUR ATTENTION