Chaotic resonant dynamics and exchanges of energy in Hamiltonian PDEs

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Hamiltonian PDEs

- Cubic Nonlinear Wave eq: $u_{tt} - \Delta u + u^3 = 0$
- Cubic Nonlinear Beam eq: $u_{tt} + \Delta^2 u + u^3 = 0$
- Cubic Nonlinear Schrödinger : $-iu_t + \Delta u - |u|^2 u = 0$
- $u = u(t, x)$, $t \in \mathbb{R}$, $x \in \mathbb{T}^n$, $n \geq 1$
- In this talk:
  - $n = 2$
  - Small data.

Transfer of energy

Fourier series

$$u(t,x) = \sum_{n\in\mathbb{Z}^2} u_n(t)\, e^{in\cdot x}$$

$|u_n(t)|^2$ energy of the n-th mode at time t.

Fundamental question

Understand how solutions can exchange energy among Fourier modes as time evolves.

Qualitative and quantitative issues

(i) **(Qualitative)**: In which ways the modes can exchange energy? Periodically or quasi-periodically in time? In a more chaotic fashion?

(ii) **(Quantitative)**: Can these transfers of energy lead to growth of Sobolev norms for some s?

$$\|u(t)\|_{H^s(\mathbb{T}^2)} = \|u(t,\cdot)\|_{H^s(\mathbb{T}^2)} = \left(\sum_{n\in\mathbb{Z}^2}(1+|n|^2)^s|u_n(t)|^2\right)^{1/2}$$

In this talk we discuss issue (i), but ...

Question by Bourgain (2000)

Are there solutions u of cubic defocusing NLS on $\mathbb{T}^2$ such that for $s>1$,

$$\limsup_{t\to+\infty}\|u(t)\|_{H^s(\mathbb{T}^2)} = +\infty\,?$$

The I-team result

Theorem (Colliander, Keel, Staffilani, Takaoka, Tao (2010))

Fix $s > 1$, $K \gg 1$ and $\delta \ll 1$. Then there exists a global solution u of cubic NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(0)\|_{H^s(\mathbb{T}^2)} \leq \delta, \qquad \|u(T)\|_{H^s(\mathbb{T}^2)} \geq K.$$

- Other results for NLS by Kuksin, Kaloshin, G., Hani, Procesi, Haus, Pausader, Visciglia, Tzvetkov, Maspero...
- Also results for the Szego and Half Wave equations (Gérard, Grellier, Pocovnicu).
- No results for other equations.

The I-team result

- Consider a good approximation of the PDE for small data:
  The Birkhoff Normal Form (BNF)
- It contains the Resonant dynamics.
- Analyze the dynamics of the BNF (the I-team toy model): Transfer orbits.
- Approximation argument: an orbit of the PDE is close to one of the BNF for certain time scales.
- The toy model is a Hamiltonian system which is hard to analyze.

The dynamics of the BNF

- Key Point: The toy model is integrable on certain finite dimensional invariant subspaces (as many first integrals as degrees of freedom).
- Thanks to integrability: Invariant tori connected by heteroclinic orbits.
- Construct orbits shadowing (following closely) such structure.
- In finite dimensional systems typically: Unstable motions (Arnold diffusion) are related to non-integrability (Chaotic dynamics, transverse homoclinic connections,...).
- The I-team result does not rely on non-integrability.
- In infinite dimensions there are more routes to instability.

Transfer of energy for other PDEs

- I-team approach for the Beam/Wave eq? It does not seem easy.
- The I-team toy model is non-integrable in the invariant subspaces.
- We exploit the non-integrability to construct different transfer of energy behaviors.
- It will not lead to growth of Sobolev norms
  ("local transfer of energy" – modes are somehow localized).

Local transfer of energy: Beating solutions

Periodic/Quasiperiodic transfer (Grébert, Thomann, Haus, Procesi,Takaoka, Thomann, Villegas-Blas ...).

- Results for NLS: $u(x,t) = \sum_{n\in\Lambda} a_n(t)e^{inx} +$ Small error
- $\Lambda \subset \mathbb{Z}^2$ is a finite set.
- $|a_n(t)|$ are all periodic/quasiperiodic functions.
- Based on integrability of the BNF on invariant subspaces.

In this talk: Transfer of energy for

$$u_{tt} - \Delta u + u^3 = 0 \quad \text{and} \quad u_{tt} + \Delta^2 u + u^3 = 0, \qquad x \in \mathbb{T}^2$$

- The transfer of energy among a_n is chaotic-like.

Chaotic dynamics

- (One of) the paradigmatic chaotic dynamics is the shift – the Smale horseshoe.
- Take either $X = \{1, \dots, M\}$ or $X = \mathbb{N}$ and $X^{\mathbb{Z}}$.
- Shift $\sigma : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ defined as

$$(\sigma\omega)_k = \omega_{k+1}$$

- We embed this dynamics in the normal form $\mathcal{N}$ associated to the cubic Wave/Beam equation.

Birkhoff Normal Form

- The Hamiltonian for the Wave/Beam equation can be written as

$$H = H^{(2)} + H^{(4)}$$

  with $H^{(i)}$ is a homogeneous polynomial of degree i
- BNF: Apply a change of coordinates such that H becomes

$$K = H^{(2)} + H^{(4)}_{\mathrm{Resonant}} + \mathcal{R}$$

  where
  - $H^{(4)}_{\mathrm{Resonant}}$ only contains resonant terms
  - $\mathcal{R}$ can be treated as a perturbation.
- As a first step, we study the dynamics of the normal form

$$\mathcal{N} = H^{(2)} + H^{(4)}_{\mathrm{Resonant}}$$

- NB: It is not always easy to obtain such change of coordinates for some PDEs.

The finite set $\Lambda \subset \mathbb{Z}^2$

- $\Lambda = \bigcup_{j=1}^{N} \Lambda_i, \qquad N \geq 2.$
- $\Lambda_i = \{n_1^{(i)}, n_2^{(i)}, n_3^{(i)}, n_4^{(i)}\}$ are disjoint resonant tuples.
- For the Wave eq:

$$n_1^{(i)} - n_2^{(i)} + n_3^{(i)} - n_4^{(i)} = 0, \quad |n_1^{(i)}| - |n_2^{(i)}| + |n_3^{(i)}| - |n_4^{(i)}| = 0.$$

- For the Beam eq:

$$n_1^{(i)} - n_2^{(i)} + n_3^{(i)} - n_4^{(i)} = 0, \quad |n_1^{(i)}|^2 - |n_2^{(i)}|^2 + |n_3^{(i)}|^2 - |n_4^{(i)}|^2 = 0.$$

- For well chosen Λ, the set

$$V_\Lambda := \{a_n = 0 \;\; \forall n \notin \Lambda\}$$

is invariant by the BNF Hamiltonian $\mathcal{N}$.

Results for the BNF (1)

- Fix $N = 2$, $\Lambda = \Lambda_1 \cup \Lambda_2$.
- There exist sets Λ such that BNF of the Wave/Beam eq satisfy:
- Call Φ_t the flow of $\mathcal{N}$ restricted to V_Λ. Then, there exists a section Σ transverse to Φ_t such that the induced Poincaré map

$$\mathcal{P} : U = \mathring{U} \subset \Sigma \to \Sigma$$

has an invariant set $\mathcal{S} \subset U$, $\mathcal{S} \cong \mathbb{N}^{\mathbb{Z}} \times \mathbb{T}^5$, whose dynamics is conjugated to

$$\widetilde{P}(\omega, \theta) = (\sigma\omega, \theta + f(\omega))$$

- Namely, the map $\mathcal{P}$ (and therefore the BNF) possesses chaotic dynamics (Smale Horseshoe).

Results for the BNF (2)

- Fix any $N \geq 2$. There exists sets Λ such that:
- The flow Φ_t has $(2N+2)$–dimensional tori $\mathbb{T}_1, \dots \mathbb{T}_N$ with the following property.
- Take small neighborhoods V_i of $\mathbb{T}_i$ and any sequence $\{p_i\}_{i\geq 1} \in \{1, \dots, N\}^{\mathbb{N}}$.
- There exists an orbit $u(t)$ and a sequence of times $\{t_i\}_{i\geq 1} \subset \mathbb{R}$ such that

$$u(t_i) \in V_{p_i}$$

- The tori are supported in different sets of modes $\rightarrow$ The orbit transfers energy in any prescribed way for infinite time.

Some remarks

- The behaviors obtained for the BNF are for all time.
- There are “many” sets $\Lambda \subset \mathbb{Z}^2$ for which the results hold.
- Such behavior takes place at any energy level of the BNF.
- They give “chaotic transfer of energy” for the BNF:
  - First result: The horseshoe orbits transfer energy at random times
  - Second result: Choosing an arbitrary sequence of tori imply activating different modes at each step.

Transfer of energy for the BNF (1)

- Fix $N = 2$: $\Lambda = \{n_1^{(1)}, n_2^{(1)}, n_3^{(1)}, n_4^{(1)}\} \cup \{n_1^{(2)}, n_2^{(2)}, n_3^{(2)}, n_4^{(2)}\}$.
- Fix $0 < \epsilon \ll 1$, $T \gg 1$.
- $\exists M_0 > 0$ such that for any sequence $\{m_j\}_{j\geq 0} \subset \mathbb{N}$, $m_j > M_0$:
- There is a solution of the BNF u with

$$u(t,x) = \sum_{n\in\Lambda} \frac{1}{|n|^\kappa} \left(a_n(t)e^{inx} + \overline{a_n}(t)e^{-inx}\right)$$

- $\kappa = 1/2$ for the Wave eq and $\kappa = 1$ for the Beam eq.
- $|a_{n_1^{(i)}}| = |a_{n_3^{(i)}}| = 1 - |a_{n_2^{(i)}}| = 1 - |a_{n_4^{(i)}}|, \qquad i = 1, 2$
- and...

Transfer of energy for the BNF (1)

Recall: We have $0 < \epsilon \ll 1$, $T \gg 1$, and any sequence $\{m_j\}_{j\geq 0}$, $m_j > M_0$.

- First resonant tuple:

$$|a_{n_1^{(1)}}(t)| = Q(t) + \mathcal{O}(\epsilon),$$

  where Q is a T–periodic function with

$$\min |Q| < \epsilon \qquad \text{and} \qquad \max |Q| > 1 - \epsilon.$$

- Second resonant tuple: There exists a sequence of times $\{t_j\}_{j\geq 0}$, $t_0 = 0$,

$$t_{j+1} = t_j + T(m_j + \theta_j), \qquad \theta_j \in (0,1).$$

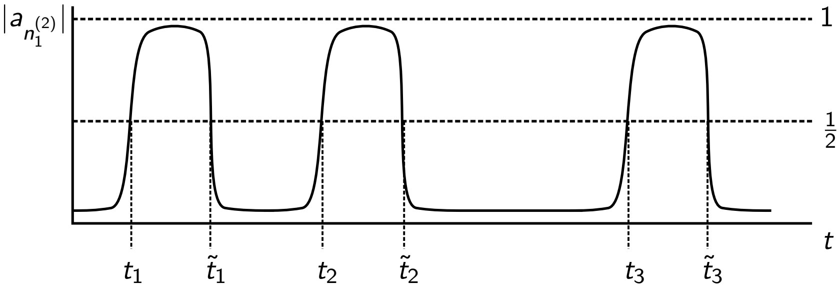
  which are the only times such that with

$$|a_{n_1^{(2)}}(t)| = \frac{1}{2}, \quad \frac{d}{dt}|a_{n_1^{(2)}}(t)| > 0.$$

Second tuple

Namely, there exists another sequence $\{\tilde{t}_j\}_{j\geq 0}$, $t_j < \tilde{t}_j < t_{j+1}$ such that:

- For $t \in (t_j, \tilde{t}_j)$: $|a_{n_1^{(2)}}(t)| > \frac{1}{2}$ and $\sup_{t\in(t_j,\tilde{t}_j)} |a_{n_1^{(2)}}(t)| \geq 1 - \epsilon$
- For $t \in (\tilde{t}_j, t_{j+1})$: $|a_{n_1^{(2)}}(t)| < \frac{1}{2}$ and $\inf_{t\in(\tilde{t}_j,t_{j+1})} |a_{n_1^{(2)}}(t)| \leq \epsilon$.



- The separation between the bumps can be chosen "randomly".

Transfer of energy for the BNF (2)

- Fix $N > 2$: $\Lambda = \bigcup_{i=1}^{N} \{n_1^{(i)}, n_2^{(i)}, n_3^{(i)}, n_4^{(i)}\}$.
- Fix $0 < \epsilon \ll 1$, $T \gg 1$.
- For any sequence $\{p_i\}_{i\geq 1} \in \{1, \dots, N\}^{\mathbb{N}}$, there is a solution of the BNF

$$u(t,x) = \sum_{n\in\Lambda} \frac{1}{|n|^{\kappa}} \left(a_n(t)e^{inx} + \overline{a_n}(t)e^{-inx}\right)$$

  satisfying:
  - $\kappa = 1/2$ for the Wave eq and $\kappa = 1$ for the Beam eq.
  - $|a_{n_1^{(i)}}| = |a_{n_3^{(i)}}| = 1 - |a_{n_2^{(i)}}| = 1 - |a_{n_4^{(i)}}|, \qquad i = 1 \dots N$
  - and...

Transfer of energy for the BNF (2)

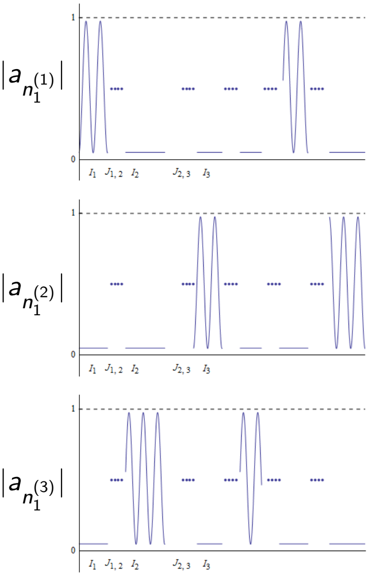
- There exist $[0,\infty) = I_1 \cup J_{12} \cup I_2 \cup J_{23} \cup \dots$ with
- Beating intervals I_i: There exists $t_i > 0$ such that for $t \in I_i$:

$$sup_{t\in I_i} \left| |a_{n_1^{(p_i)}}(t)| - Q(t-t_i) \right| \leq \epsilon$$
$$sup_{t\in I_i} \left| a_{n_1^{(k)}}(t) \right| \leq \epsilon \qquad \text{for } k \neq p_i.$$

- Transition intervals $J_{i,i+1}$: For all $n \in \Lambda$,

$$\inf_{t\in J_{i,i+1}} |a_n(t)| \leq \epsilon \qquad \text{and} \qquad \sup_{t\in J_{i,i+1}} |a_n(t)| \geq 1-\epsilon.$$

Example behavior



Orbit following the $1 \to 3 \to 2 \to 3 \to 2 \to 3$ sequence.

Results for the PDEs

- For $\delta \ll 1$, we construct solutions

$$u(t,x) = \delta \sum_{n\in\Lambda} \frac{1}{|n|^{\kappa}} \left(a_n(\delta^2 t) e^{inx} + \overline{a_n}(\delta^2 t) e^{-inx} \right) + R(t,x)$$

such that

  - u is defined for time $|t| \lesssim \delta^{-2}$.
  - $\|R\|_{H^s} \lesssim_s \delta^{3/2}$ for $|t| \lesssim \delta^{-2}$.
  - $\kappa = 1/2$ for the Wave eq and $\kappa = 1$ for the Beam eq.
  - $\{a_n\}_{n\in\Lambda}$ follow any of the orbits constructed by the BNF given in the previous slides

Some remarks

- We obtain chaotic-like transfer of energy at δ^{-2} time scale.
- All the results can also be obtained for the Hartree equation

$$iu_t = \Delta u + \left(V \star |u|^2\right) u, \quad x \in \mathbb{T}^2$$

with a convolution potential

$$V : \mathbb{T}^2 \to \mathbb{R}, \quad V(x) = V(-x)$$

in a suitable open set.

- It is hard to prove persistence of the Smale horseshoe for the PDEs. Most of existence results in PDEs add dissipation.

Some ideas about the proof: The resonant model

- Analyze the dynamics of the BNF on the invariant subspace

$$V_\Lambda := \{a_n = 0 \quad \forall n \notin \Lambda\}$$

- For well chosen Λ and after symplectic reduction

$$\mathcal{H}(\psi, I) = \mathcal{H}_0(\psi, I) + \varepsilon \mathcal{H}_1(\psi, I), \qquad \psi \in \mathbb{T}^N, I \in \mathbb{R}^N$$

with $0 < \epsilon \ll 1$ and

$$\mathcal{H}_0 = \sum_{j=1}^{N} I_j(1 - I_j)(1 + 2\cos\psi_j)$$

$$\mathcal{H}_1 = \sum_{j=1}^{N} \left(a_j I_j + b_j I_j^2 + c_j I_j(1 - I_j)\cos\psi_j\right) + \sum_{i,j=1, i<j}^{N} d_{ij} I_i I_j,$$

- For $\epsilon = 0$:
  - Each resonant tuple is decoupled
  - The Hamiltonian $\mathcal{H}$ is integrable $\rightarrow$ Chaotic behavior is not possible.

Why close to integrable?

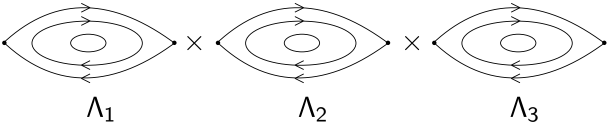
- For the cubic NLS:
  - The resonant model is $\mathcal{H}$ with $\epsilon = 0$.
  - Transfer of energy between disjoint resonant tuples is not possible.
- If the modes in Λ all have the same norm: $\mathcal{H}$ associated to Beam/Wave eq. has $\epsilon = 0$.
- We take the modes in Λ in the "thin" annulus

$$R(1-\epsilon) \leq |a_j| \leq R(1+\epsilon), \qquad \text{with } R \gg 1, \epsilon \ll 1.$$

- $0 < \epsilon \ll 1$: Close to integrable Hamiltonian system $\rightarrow$ Perturbation methods.
- For Wave/Beam eq. and general Λ, ϵ is not small: the explained behaviors should take place but are harder to prove.

The integrable dynamics

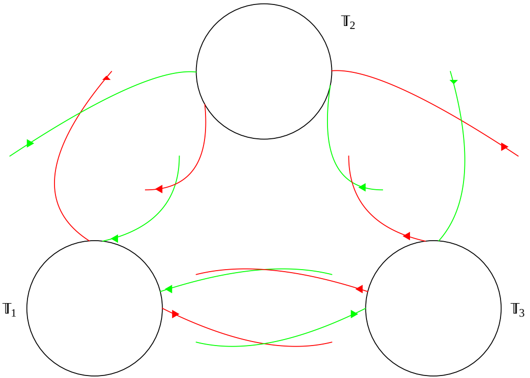
- For $\epsilon = 0$: There are saddles, periodic orbits and invariant tori.
- Their stable/unstable invariant manifolds coincide creating heteroclinic connections.



- $\epsilon = 0$ dynamics is structurally unstable in two senses. For $0 < \epsilon \ll 1$:
  - The two saddles belong to different levels of energy: they cannot be connected.
  - All stable and unstable manifolds of the periodic orbits should no longer coincide.

The perturbed dynamics: Melnikov Theory

- The stable/unstable manifolds of the periodic orbits (with same energy) intersect transversally $\rightarrow$ transverse homoclinic orbits.
- The stable/unstable manifolds of different periodic orbits (with same energy) intersect transversally $\rightarrow$ transverse heteroclinic orbits.



The perturbed dynamics

- Smale Theorem: Close to transverse homoclinic orbits of periodic orbits there are (infinite symbols) Smale horseshoes

- Lambda lemma: One can shadow any sequence of transverse heteroclinic orbits for infinite time.

- In all these results is crucial the transversality between invariant manifolds.

Towards growth of Sobolev norms for the Wave/Beam equation

- The constructed transfer of energy does not lead to Sobolev norms.
- All modes belong to a "thin" annulus in $\mathbb{Z}^2$.
- I team approach? The I-team saddles belong now to different energy levels (cannot be connected)
- The objects (periodic orbits, homoclinic connections) exist in open sets of energy levels.
- They could be "building blocks" to construct growth of Sobolev orbits for the Wave/Beam eq.

THANK YOU FOR YOUR ATTENTION