TRANSFERS OF ENERGY THROUGH FAST DIFFUSION CHANNELS IN SOME RESONANT PDES ON THE CIRCLE

FILIPPO GIULIANI

ABSTRACT. In this paper we consider two classes of resonant Hamiltonian PDEs on the circle with non-convex (respect to actions) first order resonant Hamiltonian. We show that, for appropriate choices of the nonlinearities we can find time-independent linear potentials that enable the construction of solutions that undergo a prescribed growth in the Sobolev norms. The solutions that we provide follow closely the orbits of a nonlinear resonant model, which is a good approximation of the full equation. The non-convexity of the resonant Hamiltonian allows the existence of *fast diffusion channels* along which the orbits of the resonant model experience a large drift in the actions in the optimal time. This phenomenon induces a transfer of energy among the Fourier modes of the solutions which in turn is responsible for the growth of higher order Sobolev norms.

CONTENTS

1. INTRODUCTION

We consider the following resonant Hamiltonian PDEs under periodic boundary conditions:

- Nonlinear wave equations with even-power nonlinearity

$$u_{tt} - \Delta u + V_p * u + u^p = 0, \quad u = u(t,x), \quad t \in \mathbb{R}, \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z} \tag{1.1}$$

with $p \geq 4$ even.

- Nonlinear Schrödinger equations with cubic x-dependent nonlinearity

$$\mathrm{i}u_t - \Delta u + V_N * u + \cos(Nx)|u|^2 u = 0, \quad u = u(t,x), \quad t \in \mathbb{R}, \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z} \tag{1.2}$$

with $N \geq 1$.

We show that, prescribed an arbitrarily large (but finite) growth, for an opportune choice of the nonlinear terms and *time-independent* potentials $V_p(x), V_N(x)$ (supported on few harmonics) we are able to construct arbitrarily small initial data solutions of the equations (1.1), (1.2) whose Sobolev norms undergo the prescribed growth (we refer to Section 2 for the precise statement of these results). We point out that this phenomenon is purely nonlinear, since the potentials $V_i(x)$ are chosen such that the origin $u = 0$ is a resonant *elliptic* fixed point - all the linear solutions are

This paper has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 757802.

time-periodic or quasi-periodic functions with constant amplitudes.
The construction of the solutions which exhibit a growth in the norms relies on the resonant nature of the equilibrium and the degeneracy of the first order resonant Hamiltonian.
We remark that in the linear setting one can obtain stronger results, such as the existence of unbounded orbits, by adding smooth time-dependent potentials to resonant equations [12], [14], [3], [38], [30].

In a neighborhood of a resonant elliptic equilibrium the analysis of the nonlinear dynamics can be performed through Birkhoff normal form methods. We can find a change of coordinates that removes from the Hamiltonian the terms which are not relevant for the dynamics of the equations in a certain range of times. In these coordinates the Hamiltonian is said to be in normal form.
We construct solutions which are close to orbits of a finite-dimensional *resonant model*, which is obtained by restricting the Hamiltonian in normal form on an opportune submainfold of the phase space. The non-convexity of such Hamiltonian allows the existence of certain affine subspaces of the action space, called *diffusion channels*. The orbits that travel on these channels are locked in a resonance and exhibit a relevant drift in (some of) the actions in the optimal time.
A fast instability phenomenon of this kind is illustrated by the following example: consider the two-degrees of freedom Hamiltonian system

$$H(\theta_1, \theta_2, I_1, I_2) = I_1 + \varepsilon \cos(\theta_2), \quad \theta_i \in \mathbb{T}, \ I_i \in (0, r), \ i = 1, 2, \tag{1.3}$$

for some $r > 0$. When $\varepsilon = 0$ the phase space is foliated by 2-dimensional invariant resonant tori filled by periodic orbits. Hence there is stability in the actions for all time. For $\varepsilon > 0$ the equations of motion are

$$\dot{\theta}_1 = 1, \quad \dot{\theta}_2 = 0, \quad \dot{I}_1 = 0, \quad \dot{I}_2 = \varepsilon \sin(\theta_2).$$

We note that any section $\{\theta_2 = \alpha\}$, $\alpha \in [0, 2\pi)$ is invariant. If we choose as initial conditions $\theta_2(0) = \alpha$ with $\alpha \notin \{0, \pi\}$ it is easy to see that $I_2(t)$ experiences a drift of order $\mathcal{O}(1)$ in a time $T = \mathcal{O}(\varepsilon^{-1})$. Thus all the periodic orbits on a given unperturbed torus, except two, are destroyed and give rise to diffusive solutions.
The first examples of finite-dimensional nearly-integrable Hamiltonian systems exhibiting fast instability have been provided by Nekhoroshev [40] and Moser [39]. The analysis of such systems has been carried out by Biasco-Chierchia-Treschev [6], Bounemoura-Kaloshin [9] and Bounemoura [8]. In these works the authors consider two degrees of freedom systems with an unperturbed Hamiltonian that violate the Nekhoroshev's condition for stability (see [40]) and study generic properties of the perturbations that provide fast diffusion. In the present paper we are interested in studying how these fast instability phenomena may be exploited, in the context of Hamiltonian PDEs under periodic boundary conditions, to obtain different ways to transfer energy among Fourier modes of nonlinear wave solutions.
The dynamics of the linearized problem at a resonant elliptic equilibrium is in some way similar to the unperturbed dynamics of the Hamiltonian in (1.3): there is plenty of resonant invariant tori supporting motions that fill densely lower dimensional submanifolds. As observed by Poincaré, these invariant objects are usually not robust, even under small perturbations. Then we may expect that, under some degeneracy assumptions on the Hamiltonian, some of them may be partially destroyed under the effect of the nonlinear terms and accomodate unstable behaviors. As a counterpart we mention that, under assumptions of integrability and non-degeneracy of the normal form Hamiltonian, the existence of invariant tori very close to resonances [36], [43], [44], [1], [19], [16], [15] and results of long-time stability [4], [2] have been provided for completely resonant PDEs on tori.

A drift in the actions in an opportune symplectic reduction of the resonant model corresponds to a transfer of energy between resonant modes. An arbitrarily large growth of Sobolev norms can be obtained if this transition occurs across increasingly high Fourier modes.
In this paper we just consider solutions that display an exchange of energy among few modes. Similar analysis for the dynamics of single resonant clusters have been recently carried out [22], [32], [33], [21] for the search of time-recurrent exchanges of energy, usually called *beatings*. In the present paper we consider "one-way" transfers of energy between two sets of modes, say from *low* to *high* modes (see the end of Section 2 for a comparison with periodic beatings). Roughly speaking, the nonlinearities play the role of *external parameters* that modulate the size of the high modes. This allows to obtain the desired growth in the Sobolev norms by choosing appropriately the nonlinear terms. Since the construction of the solutions that we provide is relatively simple we are able to give sharp bounds for the diffusion time.

In the last decades lots of effort has been put to obtain lower bounds of Sobolev norms for solutions of nonlinear Hamiltonian PDEs on compact manifolds. The first works in this direction are due to Bourgain [11], [10] and Kuksin [35] (see also the related works [34], [37]). In 2010 Colliander-Keel-Staffilani-Takaoka-Tao (I-team) proved an outstanding result [13] concerning the H^s-instability of solutions of the cubic nonlinear Schrödinger equation on $\mathbb{T}^2$ with $s>1$ [1]. The construction of the unstable solutions is inspired by Arnold diffusion techniques: it relies on the presence of orbits of a finite dimensional good approximation of the equation (called *toy model*) that shadow a chain of invariant hyperbolic manifolds (periodic orbits in a suitable symplectic reduction). After this work many papers have been devoted to the analysis and extensions of this scheme, exclusively for NLS models [2]. In [27] Guardia-Kaloshin provide estimates for the diffusion time of solutions obtained by refining the analysis of the dynamics of the toy model, Haus-Procesi [31] and Guardia-Haus-Procesi [26] extended the result of the I-team to Schrödinger equations with all type of analytic nonlinearities. The study of the H^s-instability of different invariant objects rather than elliptic fixed points have been carried out just recently. We mention the paper [29] by Hani for a proof of H^s-instability of plane waves and Guardia-Hani-Haus-Maspero-Procesi [24], [25] for the case of finite gap solutions for $s\in(0,1)$. We point out that the I-team mechanism strongly relies on the fact that the Schrödinger equation possess several symmetries. In particular the dynamics of the resonant models considered in the aforementioned papers is far to be generic [3] and so it seems hard to implement a similar strategy for different Hamiltonian PDEs. Then it seems natural to look for alternative mechanisms and we believe that a first step in this direction should be to find out different ways of transferring energy between modes, even restricting the study to single resonant clusters. About this, we mention the very recent result [20] by Guardia-Martin-Pasquali and the author of the present paper concerning chaotic-like transfers of energy for the wave, beam and Hartree equations on $\mathbb{T}^2$.

[1]We remark that the energy provides a complete control of the H^1-norm.

[2]Regarding results of growth of Sobolev norms for different Hamiltonian PDEs see [18], [17], [41], [42].

[3]One can find arbitrarily long sequences of invariant manifolds on the same energy level. Moreover the heteroclinic connections between these manifolds are not transversal, in any possible sense.

2. MAIN RESULTS

Let $s > 0$, we introduce the Sobolev spaces

$$H^s := \left\{ u \in L^2(\mathbb{T}) : u = \sum_{j\in\mathbb{Z}} u_j\, e^{\mathrm{i}jx}\,,\ ||u||_s^2 := \sum_{j\in\mathbb{Z}} |u_j|^2 \langle j \rangle^{2s} \right\}$$

where $\langle j \rangle := \max\{1, |j|\}$. We define the following norm

$$||u||_{E^s} := ||u||_{s+1} + ||u_t||_s.$$

The first result regards a class of nonlinear wave equations with even-power nonlinearities.

Theorem 2.1. *Let $s \geq 3$. Given $\delta \ll 1$ and $K \gg 1$ there exists $p_0 = p_0(s, \delta, K)$ such that for all **even** numbers $p \geq p_0$ the equation*

$$u_{tt} - \Delta u + V_p * u + \sigma\, u^p = 0, \quad \sigma \in \{\pm 1\}, \tag{2.1}$$

where

$$V_p(x) := 1 + \cos(x) + p^2 \cos(px), \tag{2.2}$$

possesses a solution $u(t, x)$ such that

$$||u(0)||_{E^s} \leq \delta, \quad ||u(T)||_{E^s} \geq K$$

with

$$T = \mathcal{O}(2^p\, p^{-1}\, ||u(0)||_{E^s}^{1-p}).$$

With respect to the factor of growth $\mathcal{C} := K/\delta$ we have the following estimate on the diffusion time

$$T \leq \exp(\mathcal{C}^{\frac{1}{s-2}}\, |\log(\delta)|). \tag{2.3}$$

Some comments are in order:

- The nonlinearity in equation (2.1) can be replaced by any analytic function $f(u)$ with a zero of order p at the origin $u = 0$. The degree p is used as a parameter to deal with increasingly high order resonances in the first step of a Birkhoff normal form procedure. More precisely, it turns out that the modes ± 1 (low modes) and $\pm p$ (high modes) are in resonance. We remark that this choice of low modes permits to obtain an upper bound on the diffusion time (2.3) that gets better when the order of the Sobolev norm s increases.
- The degeneracy of the resonant Hamiltonian of the nonlinear term is due to the evenness of the degree p. An evidence of the unstable behavior of wave equations with even-power nonlinearities on $\mathbb{T}$ is given, for instance, by the result of non-existence of periodic solutions given in [5].
- Although the results are rather different, it is interesting to compare the estimate on the diffusion time for arbitrarily small initial data solutions obtained in [27] (see also [28]) with the bound (2.3). Using our notations, the upper bound provided in [27] is $T \leq \exp(\mathcal{C}^{\mathtt{a}})$ with $\mathtt{a} > 1$.

The next theorem concerns cubic NLS equations which are not x-translation invariant. A similar model has been considered in [23] for the search of time-periodic beating solutions.

Theorem 2.2. *Let $s > 0$. Given $\delta \ll 1$ and $K \gg 1$ there exists $N_0 = N_0(\delta, K)$ such that the following holds:*

for all $N \geq N_0$ there exists a trigonometric polynomial $V_N \colon \mathbb{T} \to \mathbb{R}$ with real Fourier coefficients such that the equation

$$\mathrm{i}u_t - \Delta u + V_N * u + \cos(Nx)|u|^2 u = 0 \tag{2.4}$$

possesses a solution $u(t,x)$ such that

$$||u(0)||_s \leq \delta, \quad ||u(T)||_s \geq K \tag{2.5}$$

with

$$T = \mathcal{O}(N^s \; ||u(0)||_s^{-2}).$$

With respect to the growth $\mathcal{C} := K/\delta$ we have the following estimate on the diffusion time

$$T \leq \mathcal{C}^2 \delta^{-2}.$$

Some comments are in order:

- We remark that in dimension one the cubic NLS is completely integrable and, by the presence of infinitely many constants of motion, the Sobolev norms are controlled for all time. In the equation (2.4) the Hamiltonian structure and the mass (or the L^2-norm) are still preserved, but the nonlinear term breaks the momentum conservation. The frequency of the cosine x-function is used as parameter to involve modes of very different size in the 4-resonant interactions. Broadly speaking, the ratio between the size of the low and high modes is given by a power of N.
- The convolution potential V_N is supported on 4 modes. Respect to the wave case we have more freedom in the choice of the Fourier support of V_N, see (4.1), (4.2).
- We remark that when $V_N = 0$ the H^1-norm of solutions with small L^2-norm is controlled for all time. This can be seen by using that $\cos(Nx)$ is uniformly bounded in N and by applying the Gagliardo-Nirenberg inequality. When $V_N \neq 0$ the H^1-norm has still an upper bound for all time, but it is not uniform in N.
- This result can be extended to higher dimensional domains $\mathbb{T}^d$, with $d \geq 1$, by considering one-dimensional solutions and following the same construction.

We point out that the convolution potentials have the role to confine the dynamics of the normalized Hamiltonian on a finite dimensional submanifold, but they still preserve the resonant nature of the equations. The solutions that we construct bifurcate from a periodic or quasi-periodic in time function $w(t,x)$ (see (3.11), (4.6)) that is obtained as a solution of the linearized problem at the origin by exciting a finite set of modes. The orbit $w(\cdot,x)$ fills densely a lower dimensional submanifold of an embedded resonant torus. The solutions $u(t,x)$ provided by the above theorems remain close to $w(t,x)$ in a weak norm for $t \in [0,T]$, but clearly not in the H^s-topology.

Comparison with beating solutions. To optimize the ratio between the Sobolev norms at time $t=0$ and at some other time $t=T$ we want to set the initial energy of the high modes almost at zero, say ε-small. Thanks to the use of diffusion channels the exchanging time turns out to be independent of ε (see for example Lemma 3.9 and Remark 3.10). This is not the case if the same amount of energy is transferred among the modes of a periodic beating solution. These solutions are usually obtained following hyperbolic periodic orbits of a resonant model which are surrounded by heteroclinic or homoclinic loops. Let us suppose for simplicity that the transfer of energy involves just two modes. Setting the initial energy of one of the modes almost at zero corresponds to consider an orbit with very large period that visits a small neighborhood $\mathcal{U}$ of a saddle point (or a hyperbolic manifold). If the size of $\mathcal{U}$ is $\mathcal{O}(\varepsilon)$ then the time spent to escape from it is $\mathcal{O}(\log(\varepsilon))$.

2.1. **Scheme of the proofs.** The proofs of Theorems 2.1, 2.2 follow the same steps. Let us briefly describe the general strategy. First we introduce the Hamiltonian structure of equations (2.1) and (2.4). Then we perform a Birkhoff normal form algorithm (see Propositions 3.7, 4.2), namely we provide a change of coordinates that remove some terms from the Hamiltonian that do not affect the dynamics in a neighborhood of the origin for a certain range of times. The terms removed in the case of the wave and Schrödinger equations are different (we discuss that in Remark 4.3). The Hamiltonian in normal form turns out to possess finite-dimensional invariant subspaces. The restriction of the Hamiltonian to such spaces defines our *resonant model*. We analyze the dynamics of the resonant model by using action-angle variables [4]. We construct an orbit that exhibit a certain drift in the actions in the optimal diffusion time (see Lemmata 3.9, 4.4). Thanks to a rescaling argument we prove that there exists a solution of the full PDE that remains close (in a weak norm) to the unstable orbit of the resonant model (see Propositions 3.12, 4.5). The last step is to prove lower bounds for the ratio of the Sobolev norms of the solution at time $t = 0$ and $t = T$, where T is the rescaled diffusion time.

Acknowledgments. The idea of applying the theory of fast instability in the context of Hamiltonian PDEs has been suggested by V. Kaloshin to L. Biasco and M. Procesi some time ago. Recently they told me about that discussion and suggested me to read the papers [6], [9] about diffusion channels. This has inspired the present work. So I would like to thank them all.
I am grateful also to M. Guardia, R. Scandone and R. Feola for many useful comments and discussions.
This paper has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 757802.

3. Proof of Theorem 2.1

3.1. **Hamiltonian structure.** In this section we introduce a useful set of coordinates and we discuss the Hamiltonian structure of the equation (2.1). To simplify the notation we drop the subindex p from the potential, namely we write $V_p = V$. Let us denote by $\Lambda := -\Delta + V*$. The wave equation (2.1) can be written as a first order system

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -\Lambda u - u^p, \end{cases} \tag{3.1}$$

that, in the following complex coordinates

$$z^+ := \frac{\Lambda^{1/2} u - \mathrm{i} v}{\sqrt{2}}, \quad z^- := \frac{\Lambda^{1/2} u + \mathrm{i} v}{\sqrt{2}}, \tag{3.2}$$

reads as

$$\begin{cases} -\mathrm{i}\dot{z}^+ = \Lambda^{1/2} z^+ + \mathrm{g}(z^+, z^-), \\ \mathrm{i}\dot{z}^- = \Lambda^{1/2} z^- + \mathrm{g}(z^+, z^-), \end{cases} \tag{3.3}$$

with

$$\mathrm{g}(z^+, z^-) := \frac{1}{\sqrt{2}} \left(\Lambda^{-1/2} \left(\frac{z^+ + z^-}{\sqrt{2}} \right) \right)^p .$$

[4]We refer to [7] for the analysis of analogous finite-dimensional models.

Remark 3.1. *We provide a solution of* (3.3) *that undergoes growth of high Sobolev norms* $||\cdot||_s$*. Then it is easy to recover the same result for the norms* $||\cdot||_{E^s}$ *by undoing the change of coordinates* (3.2).

We introduce the infinitely many coordinates

$$z^+ = \sum_{j\in\mathbb{Z}} z_j^+ e^{\mathrm{i}jx}, \quad z^- = \sum_{j\in\mathbb{Z}} z_j^- e^{-\mathrm{i}jx} \tag{3.4}$$

that transform the system (3.3) in an infinite dimensional system of ODEs in the unknowns (z_j^+, z_j^-), $j\in\mathbb{Z}$. Such system is equipped with a Hamiltonian structure given by the symplectic form $-\mathrm{i}\sum_{j\in\mathbb{Z}} dz_j^+\wedge dz_j^-$, which in turn induces the Poisson structure

$$\{F,G\} := -\mathrm{i}\sum_{j\in\mathbb{Z}} \big(\partial_{z_j^+}F\,\partial_{z_j^-}G - \partial_{z_j^-}F\,\partial_{z_j^+}G\big), \tag{3.5}$$

where F and G are two real-valued functions defined on the phase space. The Hamiltonian of (3.3) is given by

$$\begin{aligned} H(z_j^+, z_j^-) &= \sum_{j\in\mathbb{Z}} \omega(j) z_j^+ z_j^- \\ &+ \frac{1}{(p+1)\sqrt{2}^{(p+1)}} \sum_{j_1+\dots+j_{p+1}=0} \frac{(z_{j_1}^+ + z_{j_1}^-)\dots(z_{j_{p+1}}^+ + z_{j_{p+1}}^-)}{\sqrt{\omega(j_1)\dots\omega(j_{p+1})}}, \end{aligned} \tag{3.6}$$

where

$$\omega(j) := \sqrt{j^2+V_j}, \quad j\in\mathbb{Z}$$

are the linear frequencies of oscillation. We shall look for a solution mainly Fourier supported on the symmetric *tangential* set

$$S := S^+\cup S^-, \quad S^\pm := \{\pm 1, \pm p\}. \tag{3.7}$$

By the choice of the convolution potential as in (2.2) the linear frequencies of oscillation are given by

$$\omega(j) := \begin{cases} 1 & j=0, \\ \sqrt{2}|j| & j\in S, \\ |j| & j\notin S\cup\{0\}. \end{cases} \tag{3.8}$$

Remark 3.2. *We could choose* $V_0 = n$ *with an integer* $n\geq 1$*. Moreover* $\sqrt{2}$ *can be replaced by any badly approximable number. Indeed all we need is to use the fact that*

$$|\sqrt{2}n+m| \geq \frac{\gamma}{n^2} \quad \forall n,m\in\mathbb{Z}, \quad (n,m)\neq(0,0) \tag{3.9}$$

for some $\gamma\in(0,1)$.

Remark 3.3. *The frequencies* $\omega(j)$ *with* $j\in S$ *(the* tangential frequencies*) are irrational while the normal frequencies are integer numbers. This is the key property that allows to decouple the resonant dynamics of the tangential and normal modes.*

We point out that the real subspace

$$\mathtt{R} := \{\overline{z_j^+} = z_j^-\} \tag{3.10}$$

is invariant under the flow of H. Since we shall work on $\mathtt{R}$ it is convenient to adopt the following notation

$$z_j := z_j^+, \quad \overline{z_j} := z_j^- .$$

We observe that by exciting the tangential modes we obtain a solution $w(t,x)$ of the linearized problem at the origin

$$\mathrm{i}\dot{z}_j = \omega(j)\, z_j, \quad j \in \mathbb{Z},$$

of the form

$$w(t,x) = a_1\, e^{\mathrm{i}\sqrt{2}t}\, \cos(x) + a_p\, e^{\mathrm{i}\sqrt{2}pt}\, \cos(px), \quad a_1, a_p \in \mathbb{C}. \tag{3.11}$$

These linear solutions can be seen as periodic motions supported on an invariant embedded torus of dimension 4. We expect that even a small perturbation, which here is provided by the nonlinearity, may destroy these resonant manifolds and give rise to diffusive orbits.

We write the Hamiltonian (3.6) as $H = H^{(2)} + H^{(p+1)}$ where

$$\begin{aligned} H^{(2)}(z,\overline{z}) &:= \sum_{j\in\mathbb{Z}} \omega(j) z_j \overline{z_j}, \\ H^{(p+1)}(z,\overline{z}) &:= \frac{1}{(p+1)\sqrt{2}^{(p+1)}} \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^{\mathbb{Z}}\times\mathbb{N}^{\mathbb{Z}}, \\ |\alpha|+|\beta|=p+1, \\ \pi(\alpha,\beta)=0}} C_{\alpha,\beta}\, z^\alpha\, \overline{z}^\beta \end{aligned} \tag{3.12}$$

with $|\alpha| = \sum_{j\in\mathbb{Z}} \alpha_j$, $\pi(\alpha,\beta) := \sum_{j\in\mathbb{Z}} j\,(\alpha_j - \beta_j)$,

$$z^\alpha := \prod_{j\in\mathbb{Z}} z_j^{\alpha_j}, \quad \overline{z}^\beta := \prod_{j\in\mathbb{Z}} \overline{z_j}^{\beta_j}$$

and the coefficients

$$C_{\alpha,\beta} := \frac{(p+1)!}{\alpha!\beta!} \prod_{j\in\mathbb{Z}} \omega(j)^{-\frac{\alpha_j+\beta_j}{2}} \in \mathbb{R}. \tag{3.13}$$

Remark 3.4. *We observe that a monomial* $z^\alpha\, \overline{z}^\beta$ *commutes with the momentum Hamiltonian* $M(z,\overline{z}) := -\mathrm{i}\sum_{j\in\mathbb{Z}} j z_j\, \overline{z_j}$ *if and only if* $\pi(\alpha,\beta) = 0$.

The vector field of H is defined by components as

$$X_H := (X_H^{(z_j)}, X_H^{(\overline{z}_j)}), \quad X_H^{(z_j)} := \mathrm{i}\partial_{\overline{z_j}} H, \quad X_H^{(\overline{z_j})} := -\mathrm{i}\partial_{z_j} H, \quad j \in \mathbb{Z}.$$

3.2. **Birkhoff normal form.** In this section we perform a Birkhoff normal form procedure in order to highlight the terms of the Hamiltonian (3.12) which give the effective dynamics of equation (2.1) for a certain range of time. We shall work on the space of sequences

$$\ell^1 := \left\{ z\colon \mathbb{Z} \to \mathbb{C} \,|\, ||z||_{\ell^1} := \sum_{j\in\mathbb{Z}} |z_j| < \infty \right\}. \tag{3.14}$$

We point out that ℓ^1 is an algebra with respect to the convolution product. We denote by B_η the ball centered at the origin of ℓ^1 with radius $\eta > 0$, namely

$$B_\eta := \{z \in \ell^1 : ||z||_{\ell^1} \le \eta\}.$$

Definition 3.5. *Let*

$$F = F(z, \overline{z}) := \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^{\mathbb{Z}}\times\mathbb{N}^{\mathbb{Z}},\\ |\alpha|+|\beta|=q,\\ \pi(\alpha,\beta)=0}} F_{\alpha,\beta}\, z^{\alpha}\, \overline{z}^{\beta}$$

be a homogenous Hamiltonian of degree $q \geq 2$ preserving momentum. We give the following definitions:

- *Let $0 \leq k \leq q$, we denote by $F^{(q,k)}$ the projection of F on the monomials supported on*

$$\mathcal{A}_{q,k} := \{(\alpha, \beta) \in \mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}} : \pi(\alpha, \beta) = 0,\ |\alpha| + |\beta| = q,\ \#(\alpha, \beta, S^c) = k\},$$

  where $\#(\alpha, \beta, S^c)$ denotes the number of indexes $j \notin S$ such that $\alpha_j \neq 0$ or $\beta_j \neq 0$. Similarly we define $F^{(q,\leq k)}$, $F^{(q,\geq k)}$ as the projection of F on the monomials supported respectively on

$$\mathcal{A}_{q,\leq k} = \bigcup_{i=1,\ldots,k} \mathcal{A}_{q,i}, \quad \mathcal{A}_{q,\geq k} = \bigcup_{i=k,\ldots,q} \mathcal{A}_{q,i}.$$

- *We define the following norms*

$$[\![F]\!] := \sup_{(\alpha,\beta)} |F_{\alpha,\beta}|, \qquad ||X_F||_{\eta} := \sum_{j\in\mathbb{Z}} \sup_{B_\eta} |X_F^{(z_j)}|.$$

Lemma 3.6. *Let F, G be two homogenous Hamiltonians preserving momentum of degree q and $\tilde{q}$ respectively. The Poisson bracket $\{F, G\}$ defined in* (3.5) *is a homogenous Hamiltonian preserving momentum of degree $q + \tilde{q} - 2$.*
Moreover we have the following estimates

$$|F(z, \overline{z})| \leq [\![F]\!]||z||^{q}_{\ell^1}, \tag{3.15}$$

$$||X_F(z, \overline{z})||_{\ell^1} \leq q\, [\![F]\!]||z||^{q-1}_{\ell^1}, \tag{3.16}$$

$$[\![\{F, G\}]\!] \leq q\, \tilde{q}\, [\![F]\!][\![G]\!]. \tag{3.17}$$

Proof. The proof follows the same lines of the proof of Lemma 3.2 in [22]. □

We denote by Π_{Ker} and Π_{Rg} the projection on the kernel and the range, respectively, of the adjoint action of $H^{(2)}$

$$\mathrm{ad}_{H^{(2)}}[F] := \{F, H^{(2)}\}.$$

The adjoint action of $H^{(2)}$ is diagonal on the monomials $z^{\alpha}\, \overline{z}^{\beta}$ with eigenvalues

$$\Omega(\alpha, \beta) := -\mathrm{i} \sum_{j\in\mathbb{Z}} \omega(j)(\alpha_j - \beta_j).$$

Proposition 3.7. *(**Birkhoff normal form**) Recall* (3.12). *There exists $\eta > 0$ small enough such that there exists a symplectic change of coordinates $\Gamma\colon B_\eta \to B_{2\eta}$ which takes the Hamiltonian H into its (partial) Birkhoff normal form up to order $p + 1$, namely*

$$H \circ \Gamma = H^{(2)} + H_{\mathrm{res}} + H^{(p+1,\geq 4)} + R \tag{3.18}$$

where:
(i) the resonant Hamiltonian is given by

$$H_{\mathrm{res}} := \Pi_{\mathrm{Ker}} H^{(p+1,0)} = \frac{1}{\sqrt{2}^{p+1}}\big(2\mathrm{Re}\,(z_1^p\, \overline{z_p}\,) + 2\mathrm{Re}\,(z_{-1}^p\, \overline{z_{-p}}\,)\big). \tag{3.19}$$

(ii) The remainder R is such that

$$||X_R||_\eta \leq C_1\gamma^{-1}\eta^{2p-1} + \widetilde{C_1}\gamma^{-2}\eta^{3p-2} \tag{3.20}$$

with

$$C_1 = 2^{p-1}p^3((p+1)!)^2, \quad \widetilde{C_1} = (3p-1)\, p^5 2^{\frac{3}{2}p-\frac{5}{2}}\, ((p+1)!)^3.$$

Moreover the map Γ is invertible and close to the identity

$$||\Gamma^{\pm 1} - \mathrm{Id}||_\eta \leq C_0\, \gamma^{-1}\, \eta^p \tag{3.21}$$

with

$$C_0 = \sqrt{2}^{p-1} p^2 (p+1)!$$

and it preserves the real subspace $\mathtt{R}$.

Remark 3.8. *With an abuse of notation we have renamed the Birkhoff coordinates as the original ones $(z_j)_j$.*

Proof. We consider the following homogenous Hamiltonian

$$F = \sum_{\mathcal{A}_{p+1,\leq 3}} F_{\alpha,\beta} z^\alpha\, \overline{z}^\beta$$

with

$$F_{\alpha,\beta} := \begin{cases} \dfrac{C_{\alpha,\beta}}{\Omega(\alpha,\beta)\sqrt{2}^{p+1}(p+1)} & \text{if } \Omega(\alpha,\beta) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that the coefficient $F_{\alpha,\beta}$ are uniformly bounded (we prove this claim later, see (3.26)), then the Hamiltonian F is well defined and by using Young's inequality is easy to prove that the associated equation is locally well posed in ℓ^1. By a standard bootstrap argument one can prove that, provided η is small enough, the flow Φ_F^t maps B_η into $B_{2\eta}$ for $t \in [0,1]$. We define $\Gamma := \Phi_F^1$. By the definition of $F_{\alpha,\beta}$ and the fact that $\omega(j) \in \mathbb{R}$ for all $j \in \mathbb{Z}$ (see (3.8)) the vector field X_F preserves the real subspace $\mathtt{R}$.

By definition F satisfies the following homological equation

$$\{F, H^{(2)}\} + H^{(p+1,\leq 3)} = \Pi_{\mathrm{Ker}} H^{(p+1,\leq 3)}. \tag{3.22}$$

We claim that

$$\Pi_{\mathrm{Ker}} H^{(p+1,k)} = 0, \quad k = 1,2,3. \tag{3.23}$$

If $(\alpha,\beta) \in \mathcal{A}_{p+1,1}$ then there exists $j \notin S$ such that (note that $\omega(j) = \omega(-j)$ for all $j \in \mathbb{Z}$)

$$\Omega(\alpha,\beta) = \omega(1)(\alpha_1 + \alpha_{-1} - \beta_1 - \beta_{-1}) + \omega(p)(\alpha_p + \alpha_{-p} - \beta_p - \beta_{-p}) + |j|(\alpha_j - \beta_j)$$

with $(\alpha_j, \beta_j) = (1,0)$ or $(\alpha_j,\beta_j) = (0,1)$ and $\sum_{i\in S} \alpha_i + \beta_i = p$. Then by (3.8) we have that $\Omega(\alpha,\beta) = \sqrt{2}n + m$ for some $n, m \in \mathbb{Z}$ with $|n| \leq p$, hence (see (3.9))

$$|\Omega(\alpha,\beta)| \geq \frac{\gamma}{p^2} > 0. \tag{3.24}$$

Reasoning in the same way one can get the bound (3.24) for $\Omega(\alpha,\beta)$ when $(\alpha,\beta) \in \mathcal{A}_{p+1,k}$ for $k = 2,3$. This proves the claim (3.23). Now we prove the estimate (3.21) on the map Γ. By (3.13), (3.8) we have

$$|C_{\alpha,\beta}| \leq (p+1)!. \tag{3.25}$$

If $(\alpha,\beta)\in\mathcal{A}_{p+1,0}$ and $\Omega(\alpha,\beta)\neq 0$ then by the definition of the tangential frequencies in (3.8)

$$|\Omega(\alpha,\beta)|\geq\sqrt{2}.$$

Hence by Lemma 3.6-(3.16) we have

$$[\![F]\!]\leq\gamma^{-1}\sqrt{2}^{-(p+1)}p^2p!,\qquad ||X_F||_\eta\leq\gamma^{-1}\sqrt{2}^{-(p+1)}p^2(p+1)!\eta^p. \tag{3.26}$$

By the mean value theorem and using that $\Phi_F^t\colon B_\eta\to B_{2\eta}$ for $t\in[0,1]$ we have

$$||\Gamma(z)-z||_\eta\leq\sup_{t\in[0,1]}||X_F(\Phi_F^t(z))||_\eta\leq\gamma^{-1}\sqrt{2}^{p-1}p^2(p+1)!\eta^p.$$

This gives the bound (3.21) for Γ. If η is small enough we can invert Γ by Neumann series and get a similar bound for the inverse.

Now we prove formula (3.19). By (3.23) we focus on monomials of the following form

$$\prod_{i\in S}z_i^{\alpha_i}\,\overline{z_i}^{\beta_i}$$

where

$$\sum_{i\in S}\alpha_i+\sum_{i\in S}\beta_i=p+1,\qquad\alpha_i,\beta_i\geq 0. \tag{3.27}$$

These monomials are resonant if (recall (3.8))

$$\big(\alpha_1+\alpha_{-1}-\beta_1-\beta_{-1}\big)+p(\alpha_p+\alpha_{-p}-\beta_p-\beta_{-p})=0. \tag{3.28}$$

While the momentum conservation reads as

$$\big(\alpha_1-\alpha_{-1}-\beta_1+\beta_{-1}\big)+p(\alpha_p-\alpha_{-p}-\beta_p+\beta_{-p})=0. \tag{3.29}$$

First we observe that (3.28) implies

$$p|\alpha_p+\alpha_{-p}-\beta_p-\beta_{-p}|=|\alpha_1+\alpha_{-1}-\beta_1-\beta_{-1}|. \tag{3.30}$$

If $|\alpha_p+\alpha_{-p}-\beta_p-\beta_{-p}|\neq 0$ then by (3.27) we have $p+1\geq|\alpha_1+\alpha_{-1}-\beta_1-\beta_{-1}|\geq p$.
The case $|\alpha_1+\alpha_{-1}-\beta_1-\beta_{-1}|=p+1$ is clearly impossible since the left hand side of (3.30) is even. Thus

$$|\alpha_1+\alpha_{-1}-\beta_1-\beta_{-1}|=p,\quad|\alpha_p+\alpha_{-p}-\beta_p-\beta_{-p}|=1. \tag{3.31}$$

By the latter equality we deduce that there is exactly one integer number in $\{\alpha_{\pm p},\beta_{\pm p}\}$ which is equal to 1, while all the others are zero. Then (3.29) implies that

$$|\alpha_1-\alpha_{-1}-\beta_1+\beta_{-1}|=p. \tag{3.32}$$

It is easy to see that (3.31) and (3.32) imply $\alpha_1=\beta_1$ or $\alpha_{-1}=\beta_{-1}$. Without loss of generality suppose that $\alpha_1=\beta_1$, then $|\alpha_{-1}-\beta_{-1}|=p$. This means that $(\alpha_{-1},\beta_{-1})=(p,0)$ or $(\alpha_{-1},\beta_{-1})=(0,p)$ (and so by (3.27) $\alpha_1=\beta_1=0$). The resonant monomials corresponding to these cases are

$$z_p\overline{z_1}^p,\qquad z_{-p}\overline{z_{-1}}^p$$

and their complex conjugate . We are left with the case $|\alpha_p+\alpha_{-p}-\beta_p-\beta_{-p}|=0$. By (3.30) we have $|\alpha_1+\alpha_{-1}-\beta_1-\beta_{-1}|=0$. By the fact that $\alpha_i,\beta_i\geq 0$ this implies that

$$\alpha_j+\alpha_{-j}=\beta_j+\beta_{-j},\qquad j=1,p.$$

Then (3.27) becomes $2(\alpha_1+\alpha_{-1}+\alpha_p+\alpha_{-p})=p+1$, which is a contradiction since $p+1$ is odd. Now we prove (4.7). The new Hamiltonian is obtained by Taylor expanding $H\circ\Phi_F^t$ at $t=0$. We have

$$\begin{aligned}H\circ\Gamma &= H+\{F,H\}+\frac{1}{2}\int_0^1(1-t)\{F,\{F,H\}\}\circ\Phi_F^t\,dt\\ &= H^{(2)}+\{F,H^{(2)}\}+H^{(p+1,\leq 3)}+H^{(p+1,\geq 4)}+\{F,H^{(p+1)}\}\\ &+\frac{1}{2}\int_0^1(1-t)\{F,\{F,H^{(2)}\}\}\circ\Phi_F^t\,dt+\frac{1}{2}\int_0^1(1-t)\{F,\{F,H^{(p+1)}\}\}\circ\Phi_F^t\,dt\\ &\overset{(3.22),(3.23)}{=} H^{(2)}+\Pi_{\mathrm{Ker}}H^{(p+1,0)}+H^{(p+1,\geq 4)}+R\end{aligned}$$

where

$$R:=\{F,H^{(p+1)}\}-\frac{1}{2}\int_0^1(1-t)\{F,\Pi_{\mathrm{Rg}}H^{(p+1,\leq 3)}\}\circ\Phi_F^t\,dt+\frac{1}{2}\int_0^1(1-t)\{F,\{F,H^{(p+1)}\}\}\circ\Phi_F^t\,dt.$$

By using Lemma 3.6-(3.17) and the bounds (3.26), (3.25) we obtain the following bounds

$$\begin{aligned}[\![H^{(p+1)}]\!]&\leq p!\sqrt{2}^{-(p+1)},\\ [\![\{F,H^{(p+1)}\}]\!]&\leq\gamma^{-1}2^{-(p+1)}p^2((p+1)!)^2,\\ [\![\{F,\{F,H^{(p+1)}\}\}]\!]&\leq\gamma^{-2}p^5 2^{-\frac{3}{2}p-\frac{1}{2}}((p+1)!)^3.\end{aligned}$$

Then using again Lemma 3.6-(3.16) we get the estimate (3.20). ☐

3.3. **Dynamics of the resonant model.** We introduce the rotating coordinates

$$z_j=r_j\,e^{\mathrm{i}\omega(j)t}$$

in order to remove the quadratic part of the Hamiltonian $H\circ\Gamma$. If z is a solution of (4.7) then r satisfies the equation associated to the Hamiltonian

$$\mathcal{H}=H_{\mathrm{res}}+\mathcal{Q}(t)+\mathcal{R}(t) \tag{3.33}$$

where

$$\begin{aligned}\mathcal{Q}\big((r_j)_{j\in\mathbb{Z}},t\big)&:=H^{(p+1,\geq 4)}\big(r_j\,e^{\mathrm{i}\omega(j)t}\big),\\ \mathcal{R}\big((r_j)_{j\in\mathbb{Z}},t\big)&:=R\big(r_j\,e^{\mathrm{i}\omega(j)t}\big).\end{aligned} \tag{3.34}$$

The next step is to study the dynamics of the resonant Hamiltonian H_{res}. We observe that the finite dimensional submanifold

$$\mathcal{U}_S:=\{r\colon\mathbb{Z}\to\mathbb{C}\,|\,r_j=0\ \ j\notin S\}$$

is invariant by the flow of H_{res}. We introduce the following action-angle variables on $\mathcal{U}_S$

$$r_j=\sqrt{I_j}\,e^{\mathrm{i}\theta_j}\quad j\in S.$$

The reduced Hamiltonian now reads as (recall (3.19))

$$\mathcal{G}:=\mathcal{G}^++\mathcal{G}^-,\quad \mathcal{G}^{\pm}=\sqrt{2}^{1-p}\,I_{\pm 1}^{p/2}\,\sqrt{I_{\pm p}}\,\cos(p\,\theta_{\pm 1}-\theta_{\pm p}). \tag{3.35}$$

We observe that $\mathcal{G}$ is the sum of two uncoupled integrable Hamiltonians, indeed both $\mathcal{G}^{\pm}$ have one degree of freedom in an opportune reduction. Then it makes sense to analyze just $\mathcal{G}^+$. The following lemma provides the existence of an orbit that exhibit a large drift in one of its actions.

Lemma 3.9. *Let $\varepsilon > 0$ be arbitrarily small and $\mathsf{c} > 0$. There exists an orbit of $\mathcal{G}^+$*

$$g^+_{\varepsilon,\mathsf{c}}(t) = (\theta_1(t), \theta_p(t), I_1(t), I_p(t))$$

such that

$$\begin{aligned} I_1(0) &= \mathsf{c} - p\,\varepsilon, \qquad I_p(0) = \varepsilon, \\ I_k(T_0) &= \mathsf{c}(1-p^{-1}), \quad I_p(T_0) = \frac{\mathsf{c}}{p^2}, \end{aligned} \tag{3.36}$$

with

$$\frac{\sqrt{2}^{p+1}}{\mathsf{c}^{\frac{p-1}{2}}}\, p^{-1} \leq T_0 \leq \frac{\sqrt{2}^{p+1}}{\mathsf{c}^{\frac{p-1}{2}}\,(1-p^{-1})^{p/2}}\, p^{-1}. \tag{3.37}$$

Proof. We consider the following linear symplectic change of coordinates

$$\varphi_1 = \theta_1, \quad \varphi_p = -p\,\theta_1 + \theta_p, \quad J_1 = I_1 + p\,I_p, \quad J_p = I_p.$$

The new Hamiltonian reads as

$$\mathcal{G}_* = \lambda\,(J_1 - pJ_p)^{p/2}\sqrt{J_p}\,\cos(\varphi_p).$$

Since $J_1 = I_1 + pI_p$ is a constant of motion for $\mathcal{G}_*$ we can look for solutions on the diffusion channel

$$\{(\mathsf{c} - pI_p, I_p) : I_p \in (0, \mathsf{c}/p)\} = \{J_1 = I_1 + p\,I_p = \mathsf{c}\}. \tag{3.38}$$

We fix the section $\{\varphi_p = \pi/2\}$, which is invariant by the flow of $\mathcal{G}_*$. The dynamics of $J_p = I_p$ is determined by the equation

$$\dot{J}_p = \lambda\ (\mathsf{c} - pJ_p)^{p/2}\sqrt{J_p} =: f(J_p).$$

The function f is Lipschitz continuous and strictly positive in the interval $[\varepsilon, \mathsf{c}/p)$. Hence we can easily conclude that there exists an orbit with initial condition $J_p(0) = \varepsilon$ which is monotone increasing and reach the value c/p^2 at time

$$T_0 := \sqrt{2}^{p-1}\int_\varepsilon^{\frac{\mathsf{c}}{p^2}} \frac{1}{(\mathsf{c} - p\,J_p)^{p/2}\sqrt{J_p}}\,dJ_p.$$

By using the fact that

$$\mathsf{c} - p\varepsilon \geq \mathsf{c} - p\,J_p \geq \mathsf{c}(1-p^{-1})$$

on the interval of integration we get the bounds (3.37).

□

Remark 3.10. *The time of diffusion T_0 does not depend on ε, see the first line in* (3.36)*.*

Remark 3.11. *By* (3.38) *and the fact that $J_p = I_p$ is monotone increasing in the time interval $[0, T_0]$ we have that*

$$\sup_{t\in[0,T_0]} I_1(t) = I_1(0) < \mathsf{c}.$$

By the discussion above it is clear that Lemma 3.9 provides also an orbit $g^-_{\varepsilon,\mathsf{c}}(t)$ for $\mathcal{G}^-$ with the same evolution of the actions I_{-1}, I_{-p} as in (3.36).
We consider the family of solutions $g^\pm_{\varepsilon,\mathsf{c}}(t)$ given by the Lemma 3.9 for the values of the parameters

$$\begin{aligned} &\varepsilon = \varepsilon_0\,p^{-2s}, \quad \text{for some } \varepsilon_0 > 0 \text{ small enough}, \\ &\mathsf{c} \in (4, p^\nu) \quad \text{for an opportune } \nu > 0 \text{ (not too large)}. \end{aligned} \tag{3.39}$$

We define $b(\varepsilon, \mathtt{c}; t, x) = b(t,x) = \sum_{j\in\mathbb{Z}} b_j(t) e^{\mathrm{i}jx}$ with

$$b_j(t) := \begin{cases} \sqrt{I_j(t)}\, e^{\mathrm{i}\theta_j(t)} & j \in S \\ 0 & \text{otherwise.} \end{cases}$$

Then the function $b(t,x)$ is a solution of $H_{\rm res}$ such that

$$\begin{aligned} |b_{\pm1}(0)|^2 &= \mathtt{c} - \varepsilon\, p, \quad |b_{\pm p}(0)|^2 = \varepsilon, \\ |b_{\pm1}(T_0)|^2 &= \mathtt{c}(1-p^{-1}), \quad |b_{\pm p}(T_0)|^2 = \frac{\mathtt{c}}{p^2}. \end{aligned} \tag{3.40}$$

For p large enough we have

$$2p^{-1}\left(\frac{2}{\mathtt{c}}\right)^{p/2} \le T_0 \le 4p^{-1}\left(\frac{2}{\mathtt{c}}\right)^{p/2}. \tag{3.41}$$

3.4. **Approximation argument.** In this section we show that solutions of the Hamiltonian (3.33) with initial condition ℓ^1-close enough to $b(0,x)$ remain ℓ^1-close to $b(t,x)$ for all the time that we need to appreciate the drift in the actions (3.40). Since we need to work in a sufficiently small neighborhood of the origin we use a rescaling argument.

The solutions $u(t,x)$ of $H_{\rm res}$ are invariant under the rescaling

$$u(t,x) \to \mu^{-1} u(\mu^{-p+1} t, x).$$

Given $\mu > 0$ we consider the rescaled solution

$$r^{\mu}(t,x) := \mu^{-1} b(\mu^{-p+1} t, x). \tag{3.42}$$

The diffusion time is rescaled in the following way

$$T := \mu^{p-1} T_0, \tag{3.43}$$

hence we need to ensure a good approximation of $r^{\mu}(t)$ by solutions $r(t)$ of (3.33) at least in the range of time $[0,T]$. By Remark 3.11 we have

$$||r^{\mu}(t)||_{\ell^1} \le 4\sqrt{\mathtt{c}}\mu^{-1} \lesssim \mu^{-1} \qquad \text{for } t \in [0,T]. \tag{3.44}$$

Let us define

$$\mu_0 := p^{\mathtt{a}}\gamma^{-\mathtt{b}}, \quad \mathtt{a} > 0, \quad \mathtt{b} := \frac{1}{p - 5/2}. \tag{3.45}$$

Proposition 3.12. *Let p and $\mathtt{a}$ be large enough. For all $\mu \ge \mu_0$ in* (3.45) *we have that if $r(t)$ is a solution of* (3.33) *such that*

$$||r(0) - r^{\mu}(0)||_{\ell^1} \le \mu^{-\sigma_1}, \quad \sigma_1 := \frac{3p}{4} + 2, \tag{3.46}$$

then

$$||r(t) - r^{\mu}(t)||_{\ell^1} \le \mu^{-\sigma_2}, \quad \sigma_2 := \sigma_1 - \frac{1}{2}, \quad \textit{for } t \in [0,T]. \tag{3.47}$$

Proof. We set $\xi := r - r^{\mu}$ and we study the evolution of its ℓ^1-norm. We have that $\dot{\xi} = Z_0(t) + Z_1(t)\,\xi + Z_2(t,\xi)$ with (recall (3.34))

$$\begin{aligned} Z_0 &:= X_{\mathcal{R}}(r_{\mu}), \\ Z_1 &:= DX_{H_{\rm res}}(r^{\mu}), \\ Z_2 &:= X_{H_{\rm res}}(r) - X_{H_{\rm res}}(r^{\mu}) - DX_{H_{\rm res}}(r^{\mu})\xi + X_{\mathcal{R}}(r) - X_{\mathcal{R}}(r_{\mu}) + X_{\mathcal{Q}}(r) - X_{\mathcal{Q}}(r_{\mu}). \end{aligned}$$

By the differential form of Minkowsky's inequality we get

$$\frac{d}{dt}||\xi||_{\ell^1} \leq ||Z_0(t)||_{\ell^1} + ||Z_1(t)\,\xi||_{\ell^1} + ||Z_2(t)||_{\ell^1}.$$

Now we give bounds on the terms of the right hand side of the above inequality.
Bound for Z_0: By (3.20), (3.44) we have

$$||Z_0||_{\ell^1} \lesssim C_1\gamma^{-1}\mu^{-2p+1} + \widetilde{C_1}\gamma^{-2}\mu^{-3p+2}.$$

By (3.45) and the fact that $\mu \geq \mu_0$ we have, for p large enough,

$$C_1\gamma^{-1}\mu^{-2p+1} \leq \mu^{-(7/4)p-1}, \quad \widetilde{C_1}\gamma^{-2}\mu^{-3p+2} \leq \mu^{-(7/4)p-1}. \tag{3.48}$$

Hence

$$||Z_0||_{\ell^1} \lesssim \mu^{-(7/4)p-1}.$$

Bound for Z_1: Taking into account the definition of (3.19), the bound (3.44) and Lemma 3.6 we have, for p large enough,

$$||Z_1(t)\,\xi||_{\ell^1} \lesssim (p+1)\sqrt{2}^{-(p+1)}\mu^{-p+1}||\xi||_{\ell^1} \lesssim \mu^{-p+1}||\xi||_{\ell^1}.$$

Bound for Z_2: We use a bootstrap argument. Let us define T_* as the sup of the times t such that

$$||\xi(t)||_{\ell^1} \leq \mu^{-\sigma_2}. \tag{3.49}$$

We observe that for $t = 0$ we have $||\xi(0)||_{\ell^1} \leq \mu^{-\sigma_1}$ and since $\sigma_1 > \sigma_2$ we have $T_* > 0$. A posteriori we shall prove that $T_* > T > 0$. We call

$$Z_{2,1} := X_{H_{\rm res}}(r) - X_{H_{\rm res}}(r^\mu) - DX_{H_{\rm res}}(r^\mu)\xi,$$
$$Z_{2,2} := X_{\mathcal{R}}(r) - X_{\mathcal{R}}(r_\mu), \quad Z_{2,3} := X_{\mathcal{Q}}(r) - X_{\mathcal{Q}}(r_\mu).$$

By the definition of (3.19)

$$||Z_{2,1}||_{\ell^1} \leq (p+1)\,\sqrt{2}^{-(p+1)}\sum_{j=2}^{p}||r^\mu||_{\ell^1}^{p-j}||\xi||_{\ell^1}^{j} \overset{(3.49),(3.44)}{\lesssim} (p+1)\,\sqrt{2}^{-(p+1)}\sum_{j=2}^{p}\mu^{(j-p)-\sigma_2(j-1)}||\xi||_{\ell^1}.$$

Since $\sigma_2 \geq 1$ we have, for p large enough,

$$||Z_{2,1}||_{\ell^1} \lesssim (p+1)\,\sqrt{2}^{-(p+1)}\mu^{-p+1}||\xi||_{\ell^1} \lesssim \mu^{-p+1}||\xi||_{\ell^1}.$$

Now recall the bound (3.20). We have

$$||Z_{2,2}||_{\ell^1} \leq C_1\gamma^{-1}\sum_{j=1}^{2p-1}||r^\mu||_{\ell^1}^{2p-1-j}||\xi||_{\ell^1}^{j} + \widetilde{C_1}\gamma^{-2}\sum_{j=1}^{3p-2}||r^\mu||_{\ell^1}^{3p-2-j}||\xi||_{\ell^1}^{j}.$$

Reasoning as for the bound on $Z_{2,1}$ and using (3.44), (3.48), (3.49) we get

$$||Z_{2,2}||_{\ell^1} \lesssim \mu^{-(3/2)p+1}||\xi||_{\ell^1}.$$

The most problematic term is $H^{(p+1,\geq 4)}$, because it has the same degree of $H_{\rm res}$. However we recall that the monomials of $H^{(p+1,\geq 4)}$ are Fourier supported on at least four normal modes. Then by the (3.34) and the bound (3.25)

$$||Z_{2,3}||_{\ell^1} \leq p!\sqrt{2}^{-(p+1)}\sum_{j=3}^{p}||r^\mu||_{\ell^1}^{p-j}||\xi||_{\ell^1}^{j} \overset{(3.49)}{\leq} (p!\mu^{-\sigma_2})\sqrt{2}^{-(p+1)}||\xi||_{\ell^1}\sum_{j=3}^{p}||r^\mu||_{\ell^1}^{p-j}||\xi||_{\ell^1}^{j-2}.$$

Since

$$p!\mu^{-\sigma_2} \le p!\mu_0^{-\sigma_2} \overset{(3.45)}{\lesssim} p!p^{-p-1}$$

and reasoning as for the bound of $Z_{2,1}$ we get, for p large enough,

$$||Z_{2,3}||_{\ell^1} \lesssim \mu^{-p+1}\,||\xi||_{\ell^1}.$$

By collecting all the previous estimates we obtained

$$\frac{d}{dt}||\xi||_{\ell^1} \lesssim \mu^{-(7/4)p-1} + \mu^{-p+1}||\xi||_{\ell^1}.$$

Then by Gronwall lemma

$$||\xi(t)||_{\ell^1} \lesssim p^{-\sigma_1}\,\exp(\mu^{-p+1}t) \quad \text{for } t\in[0,T_*].$$

For times $t \in [0, \mu^{p-1}\log(\mu)]$ we have that $||\xi|| \lesssim \mu^{1-\sigma_1} \le \mu^{-\sigma_2}$, then $T_* > \mu^{p-1}\log(\mu)$. We conclude by noting that $\log(\mu) \ge \log(\mu_0) > T_0$ (recall (3.41), (3.39)). Thus $T_* > T$ and we can drop the bootstrap assumption.

□

3.5. **Conclusion of the proof.** In this section we conclude the proof of Theorem 2.1 by showing that a solution $z(t)$ with initial datum $z(0) = r^{\mu}(0)$, with an opportune choice of μ, undergoes the prescribed growth of its Sobolev norms.

We consider (recall the definition of μ_0 in (3.45))

$$\delta_0 \ll \sqrt{2\mathtt{c}}\mu_0^{-1}$$

and we fix $\delta \in (0, \delta_0)$. Recalling the rescaling (3.42) we choose μ such that

$$\sqrt{2\mathtt{c}}\mu^{-1} = \delta. \tag{3.50}$$

Let us consider $r(t)$ solution of (3.33) with $r(0) = \Gamma^{-1}r^{\mu}(0)$. The (3.50) implies that $\mu \ge \mu_0$, hence we can apply the approximation argument in Proposition 3.12. Let us call

$$z(t) = \Gamma\big((r_j e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big).$$

Now we give a lower bound for $||z(T)||_s$. It turns out that it is sufficient to estimate $|z_{\pm p}(T)|$. We give a lower bound for $z_p(T)$, the one for $z_{-p}(T)$ is obtained in the same way. We have

$$\begin{aligned} |z_p(T)| &\ge |r_p(T)| - |\Gamma_p\big((r_j e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big) - r_p(T)e^{\mathrm{i}\omega(p)T}| \\ &\ge |r_p^{\mu}(T)| - |r_p(T) - r_p^{\mu}(T)| - |\Gamma_p\big((r_j e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big) - r_p(T)e^{\mathrm{i}\omega(p)T}|. \end{aligned} \tag{3.51}$$

First we need a lower bound for $|r_p^{\mu}(T)|$. By (3.40) and the rescaling (3.42)

$$|r_p^{\mu}(T)| \ge \sqrt{\mathtt{c}}p^{-1}\,\mu^{-1}. \tag{3.52}$$

Now we give an upper bound for $|r_p(T) - r_p^{\mu}(T)|$. By the estimates (3.21) and (3.44) we have that

$$||r(0) - r^{\mu}(0)||_{\ell^1} \le C_0\gamma^{-1}\mu^{-p}.$$

By the definition of μ_0 in (3.45) and the fact that $\mu \ge \mu_0$ we have, for p large enough, $||r(0) - r^{\mu}(0)||_{\ell^1} \le p^{-\sigma_1}$ (recall σ_1 in (3.46)). Then by Proposition 3.12 (recall σ_2 in (3.47))

$$||r(t) - r^{\mu}(t)||_{\ell^1} \le \mu^{-\sigma_2} \quad \text{for } t\in[0,T]. \tag{3.53}$$

Hence

$$|r_p(T) - r_p^{\mu}(T)| \le \mu^{-\sigma_2}. \tag{3.54}$$

We are left with an upper bound for $|\Gamma_p\big((r_j e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big) - r_p(T)e^{\mathrm{i}\omega(p)T}|$. By (3.53) and (3.44) we have

$$|r_p(T)| \le ||r(T)||_{\ell^1} \le ||r^\mu(T)||_{\ell^1} + ||r(T) - r^\mu(T)||_{\ell^1} \le 8\sqrt{\mathsf{c}}\mu^{-1}.$$

Then by using the estimate (3.21) and (3.45) we get, for p large enough,

$$|\Gamma_p\big((r_j e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big) - r_p(T)e^{\mathrm{i}\omega(p)T}| \le 8^p\, C_0\gamma^{-1}\mu^{-p} \le \mu^{-\sigma_1}. \tag{3.55}$$

By (3.51) and collecting the bounds (3.52), (3.54), (3.55) we obtained

$$|z_p(T)| \ge \sqrt{\mathsf{c}}\, p^{-1}\,\mu^{-1} - \mu^{-\sigma_2} - \mu^{-\sigma_1} \ge \sqrt{\mathsf{c}}\frac{p^{-1}}{2}\mu^{-1}.$$

This implies that

$$||z(T)||_s^2 \ge \big(|z_p(T)|^2 + |z_{-p}(T)|^2\big)\, p^{2s} \ge 2^{-1}\mathsf{c}\,\mu^{-2}p^{2(s-2)}. \tag{3.56}$$

Regarding the Sobolev norm at time zero of $z(t)$ we have by (3.40) and choosing ε_0 in (3.39) small enough

$$||z(0)||_s^2 = ||r^\mu(0)||_s^2 = 2\mu^{-2}\big((\mathsf{c} - \varepsilon\, p) + \varepsilon p^{2s}\big) \le 2\mathsf{c}\mu^{-2} = \delta^2. \tag{3.57}$$

By (3.56) and (3.57) the ratio between the Sobolev norms at time $t = T$ and $t = 0$ has the following lower bound

$$\frac{||z(T)||_s}{||z(0)||_s} \ge \frac{p^{s-2}}{2}. \tag{3.58}$$

Hence since $s > 2$ we can use p as a parameter to get the prescribed growth of the H^s norm. By (3.43) and (3.50) we have that

$$T = \mu^{p-1}T_0 = (\sqrt{2\mathsf{c}})^{p-1}\delta^{1-p}T_0 \overset{(3.41)}{\le} \frac{2^{p+2}}{\sqrt{\mathsf{c}}p}\,\delta^{1-p}. \tag{3.59}$$

From (4.19) and (3.59) we get (2.3).

4. Proof of Theorem 2.2

We follow the same steps of the proof of the Theorem 2.1 shown in Section 3.
First we build the convolution potential V_N. To simplify the notation we write $V_N = V$. We consider the tangential set

$$S := \{k_1, k_2, k_3, k_4\} \subset \mathbb{Z} \tag{4.1}$$

with

$$k_1, k_3 > 0, \quad k_2 < 0, \quad k_4 := k_1 - k_2 + k_3 + N. \tag{4.2}$$

Let us also assume that

$$k_3 := \max\{k_1, |k_2|, k_3\} \le \sqrt{N}. \tag{4.3}$$

Let us consider $\mathsf{q} := (\mathsf{q}_1, \mathsf{q}_2, \mathsf{q}_3) \in [1, 2]^3$ such that

$$|\mathsf{q}\cdot\ell + k| \ge \frac{\gamma}{\langle\ell\rangle^\tau} \quad \forall \ell \in \mathbb{Z}^3, \quad 0 < |\ell| \le 9, \quad \forall k \in \mathbb{Z}, \quad (\ell, k) \ne (0, 0) \tag{4.4}$$

with $\gamma \in (0, 1)$ and $\tau > 0$ large enough. We set

$$V_j := \begin{cases} \mathsf{q}_i - k_i^2, & j = k_i, \quad i = 1, 2, 3, \\ \mathsf{q}_1 - \mathsf{q}_2 + \mathsf{q}_3 - k_4^2, & j = k_4, \\ 0 & \text{otherwise.} \end{cases}$$

We consider the Fourier expansion $u = \sum_{j\in\mathbb{Z}} u_j\, e^{\mathrm{i}jx}$ with $u_j := \frac{1}{2\pi}\int_{\mathbb{T}} u\, e^{-\mathrm{i}jx}\, dx$. The equation (2.4) is Hamiltonian with respect to the symplectic structure $-\mathrm{i}\sum_{j\in\mathbb{Z}} du_j \wedge d\overline{u_j}$. The Hamiltonian is $H = H^{(2)} + H^{(4)}$ where

$$H^{(2)}(u_j, \overline{u_j}) := \sum_{j\in\mathbb{Z}} \omega(j) u_j\, \overline{u_j}, \quad H^{(4)}(u_j, \overline{u_j}) := \sum_{j_1-j_2+j_3-j_4=\pm N} u_{j_1}\, \overline{u_{j_2}}\, u_{j_3}\, \overline{u_{j_4}} \tag{4.5}$$

with

$$\omega(j) := \begin{cases} \mathtt{q}_j & j = k_i, \quad i = 1,2,3, \\ \mathtt{q}_1 - \mathtt{q}_2 + \mathtt{q}_3, & j = k_4, \\ j^2 & \text{otherwise.} \end{cases}$$

Remark 4.1. *As in the case of the wave equation (see Remark 3.3), the tangential frequencies are irrational real numbers while the normal ones are integers.*

The equation (2.4) can be written as an infinite dimensional system of ODEs for the Fourier coefficients

$$-\mathrm{i}\dot{u}_j = \omega(j) u_j + \sum_{j_1-j_2+j_3-j=\pm N} u_{j_1}\, \overline{u_{j_2}}\, u_{j_3}, \qquad j \in \mathbb{Z}.$$

We consider the solution of the linear problem

$$-\mathrm{i}\dot{u}_j = \omega(j) u_j, \quad j \in \mathbb{Z},$$

obtained by exciting the modes in S, namely

$$w(t,x) = \sum_{k\in S} a_k\, e^{\mathrm{i}(\omega(k)t + k\,x)}. \tag{4.6}$$

By the definition of the linear frequencies of oscillation and by (4.4) the orbit $w(\cdot, x)$ is conjugated to a quasi-periodic motion on an embedded 4-d resonant torus that fills densely a lower dimensional manifold.

As in Section 3.2 we construct a change of coordinates that puts (partially) in normal form the Hamiltonian (4.5). Again we work with the ℓ^1-topology. Recall (3.14) and Definition 3.5.

Proposition 4.2. *Recall* (4.5). *There exists $\eta > 0$ small enough such that there exists a symplectic change of coordinates $\Gamma\colon B_\eta \to B_{2\eta}$ which takes the Hamiltonian H into its (partial) Birkhoff normal form up to order 4, namely*

$$H \circ \Gamma = H^{(2)} + H_{\mathrm{res}} + H^{(4,\geq 2)} + R \tag{4.7}$$

where:
(i) the resonant Hamiltonian is given by

$$H_{\mathrm{res}} := \Pi_{\mathrm{Ker}} H^{(4,0)} = 2\mathrm{Re}\left(u_{k_1}\, \overline{u_{k_2}}\, u_{k_3}\, \overline{u_{k_4}}\right). \tag{4.8}$$

(ii) The remainder R is such that

$$||X_R||_\eta \lesssim \gamma^{-1}\eta^5 + \gamma^{-2}\eta^7. \tag{4.9}$$

Moreover the map Γ is invertible and close to the identity

$$||\Gamma^{\pm 1} - \mathrm{Id}||_\eta \lesssim \gamma^{-1}\eta^3. \tag{4.10}$$

Proof. The proof follows the same lines of the proof of Proposition 3.7. Actually the proof is easier since the parameter N, that we need to control, does not enter in the estimates for the map Γ and the remainder R. The only thing that we need to prove is that $\Pi_{\rm Ker} H^{(4,1)} = 0$ and formula (4.8). If $(\alpha, \beta) \in \mathcal{A}_{4,1}$ then

$$\Omega(\alpha, \beta) = \mathtt{q} \cdot \ell \pm j^2$$

for some $\ell = \ell(\alpha, \beta) \in \mathbb{Z}^3$ with $3 \le |\ell| \le 9$ and $j \notin S$. Since j^2 is an integer, by (4.4) we have $\Omega(\alpha, \beta) \ge \gamma\, 9^{-\tau} > 0$. This proves that $\Pi_{\rm Ker} H^{(4,1)} = 0$. If $(\alpha, \beta) \in \mathcal{A}_{4,0}$ by (4.4) and the choice of the potential the only resonant monomials are $u_{k_1}\, \overline{u_{k_2}}\, u_{k_3}\, \overline{u_{k_4}}$ and its complex conjugate.

□

Remark 4.3. *The normal form procedure performed in Proposition 4.2 removes less terms respect to the one of Proposition 3.7. This is because in the case of the wave equation* (2.1) *we need better estimates on the non-normalized vector field of degree* $\mathtt{p}$*. Indeed these bounds get worse when the parameter* $\mathtt{p}$ *is increasing (the coefficients of the associated Hamiltonian grow like* $\mathtt{p}!$*, see for instance the bound* (3.13)*) and the approximation time* T *is not short enough to close the approximation argument (see* (3.43) *and* (3.41)*).*

We introduce the rotating coordinates

$$u_j = r_j\, e^{\mathrm{i}\omega(j) t}$$

in order to remove the quadratic part of the Hamiltonian $H \circ \Gamma$. Then r satisfies the equation associated to the Hamiltonian

$$\mathcal{H} = H_{\rm res} + \mathcal{Q}(t) + \mathcal{R}(t) \tag{4.11}$$

where

$$\begin{aligned} \mathcal{Q}\big((r_j)_{j\in\mathbb{Z}}, t\big) &:= H^{(4,\ge 2)}\big((r_j\, e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big), \\ \mathcal{R}\big((r_j)_{j\in\mathbb{Z}}, t\big) &:= R\big((r_j\, e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big). \end{aligned} \tag{4.12}$$

We study the dynamics of the resonant Hamiltonian $H_{\rm res}$. We observe that the finite dimensional subspace

$$\mathcal{U}_S := \{ r \colon \mathbb{Z} \to \mathbb{C} \,|\, r_j = 0 \ \ j \notin S \}$$

is invariant by the flow of $H_{\rm res}$. We introduce the following action-angle variables on $\mathcal{U}_S$

$$r_{k_j} = \sqrt{I_j}\, e^{\mathrm{i}\theta_j}, \quad j = 1, 2, 3, 4.$$

The Hamiltonian $H_{\rm res}$ now reads as

$$\mathcal{G} := 2\sqrt{I_1\, I_2\, I_3\, I_4}\cos(\theta_1 - \theta_2 + \theta_3 - \theta_4).$$

Lemma 4.4. *Let* $\varepsilon > 0$ *be arbitrarily small and let* $\mathtt{c} > 0$*. There exists an orbit of* $\mathcal{G}$

$$g_{\varepsilon,\mathtt{c}}(t) = (\theta_1(t), \dots, \theta_4(t), I_1(t), \dots, I_4(t))$$

such that

$$I_1(0) = I_2(0) = I_3(0) = \frac{\mathtt{c} - \varepsilon}{3}, \quad I_4(0) = \varepsilon$$

$$I_1(T_0) = I_3(T_0) = \frac{\mathtt{c}}{6} + \frac{2\varepsilon}{3}, \quad I_2(T_0) = \frac{\mathtt{c}}{2} - \frac{4\varepsilon}{3}, \quad I_4(T_0) = \frac{\mathtt{c}}{6}$$

with

$$T_0 \le \frac{6}{\mathtt{c}}.$$

Proof. We apply the following linear symplectic change of variables

$$\varphi = A\theta, \quad J = A^{-T} I, \quad A := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

We observe that the matrix $A \in SL(4,\mathbb{Z})$, hence this defines a linear automorphism of the torus $\mathbb{T}$. The new Hamiltonian is given by

$$\mathcal{G}_* = 2\sqrt{(J_1 - J_4)(J_2 + J_4)(J_3 - J_4)\, J_4}\cos(\varphi_4).$$

We observe that J_1, J_2, J_3 are constants of motion. Then we fix

$$J_i = \alpha := \frac{\mathsf{c} - \varepsilon}{3} + \varepsilon \quad i = 1, 3, \qquad J_2 = \beta := \frac{\mathsf{c} - \varepsilon}{3} - \varepsilon$$

and we look for solutions traveling along the following diffusion channel

$$\{(\alpha - I_4, \beta + I_4, \alpha - I_4, I_4) : I_4 \in (0, \alpha)\},$$

which is contained in the mass level

$$\{J_1 + J_2 + J_3 = I_1 + I_2 + I_3 + I_4 = \mathsf{c}\}.$$

If we restrict to the invariant section $\{\varphi_4 = \pi/2\}$ the equation of motion for $J_4 = I_4$ is

$$\dot{J}_4 = 2\,(\alpha - J_4)\,\sqrt{(\beta + J_4)\ \, J_4}.$$

Reasoning as in Lemma 3.9 we can conclude that there exists an orbit such that

$$J_4(0) = \varepsilon, \quad J_4(T_0) = \frac{\mathsf{c}}{6}$$

with

$$T_0 = \frac{1}{2}\int_{\varepsilon}^{\mathsf{c}/6} \frac{1}{(\alpha - J_4)\sqrt{(\beta + J_4)\, J_4}}\, dJ_4 \leq \frac{3}{\mathsf{c}}\int_0^{\mathsf{c}/6} \frac{1}{\sqrt{(\frac{\mathsf{c}}{2} + J_4)\, J_4}}\, dJ_4 \leq \frac{6}{\mathsf{c}}.$$

□

We set

$$\varepsilon = \varepsilon_0 k_4^{-2s}, \quad \text{for some } \varepsilon_0 > 0 \text{ small enough.} \tag{4.13}$$

We define $b(\varepsilon, \mathsf{c}; t, x) = b(t,x) = \sum_{j \in \mathbb{Z}} b_j(t) e^{\mathrm{i} j x}$ with

$$b_j(t) := \begin{cases} \sqrt{I_j(t)}\, e^{\mathrm{i}\theta_j(t)} & j \in S \\ 0 & \text{otherwise.} \end{cases}$$

Then the function $b(t,x)$ is a solution of H_{res} such that

$$|b_{k_1}(0)|^2 = |b_{k_2}(0)|^2 = |b_{k_3}(0)|^2 = \frac{\mathsf{c} - \varepsilon}{3}, \quad |b_{k_4}(0)|^2 = \varepsilon,$$

$$|b_{k_1}(T_0)|^2 = |b_{k_3}(T_0)|^2 = \frac{\mathsf{c}}{6} + \frac{2\varepsilon}{3}, \quad |b_{k_2}(T_0)|^2 = \frac{\mathsf{c}}{2} - \frac{4\varepsilon}{3}, \quad |b_{k_4}(T_0)|^2 = \frac{\mathsf{c}}{6}.$$

In particular from the proof of Lemma 4.4 we deduce that

$$\sup_{t \in [0, T_0]} |b_{k_i}(t)|^2 < \mathsf{c}, \quad i = 1, 2, 3, 4. \tag{4.14}$$

We follow the same procedure of Section 3.4. The solutions $u(t,x)$ of H_{res} are invariant under the rescaling

$$u(t,x) \to \mu^{-1} u(\mu^{-2} t, x).$$

Then we consider the rescaled solution

$$r^{\mu}(t,x) = \mu^{-1} b(\mu^{-2} t, x). \tag{4.15}$$

The diffusion time is rescaled in the following way

$$T = \mu^2 T_0 \leq \frac{6\,\mu^2}{\mathsf{c}}\,. \tag{4.16}$$

By (4.14) we have

$$||r^{\mu}||_{\ell^1} \leq 4\sqrt{\mathsf{c}}\mu^{-1}. \tag{4.17}$$

Proposition 4.5. *There exists $\mu_0 > 0$ large enough enough such that for all $\mu \geq \mu_0$ we have that if $r(t)$ is a solution of* (3.33) *such that*

$$||r(0) - r^{\mu}(0)||_{\ell^1} \leq \mu^{-5/2}$$

then

$$||r(t) - r^{\mu}(t)||_{\ell^1} \leq \mu^{-3/2}, \quad \textit{for } t \in [0,T].$$

Proof. The proof is very close to the proof of Proposition 3.12. Actually it is easier because the coefficients of the vector fields and the time T do not depend on the parameter N. One can also follow the proof of Theorem 2.9 in [26] taking into account the bound (4.17) on the ℓ^1-norm of r^{μ}. □

We fix δ small enough and we choose μ such that

$$\sqrt{\mathsf{c}}\, k_3^s\, \mu^{-1} = C\delta \tag{4.18}$$

for some pure constant $C > 0$. Let us consider $r(t)$ solution of (4.11) with $r(0) = \Gamma^{-1} r^{\mu}(0)$. Let us call

$$z(t) = \Gamma\big((r_j e^{\mathrm{i}\omega(j)t})_{j\in\mathbb{Z}}\big).$$

Reasoning as in Section 3.5 we can obtain the following lower bound for the Sobolev norms of z at time $t = T$

$$||z(T)||_s^2 \geq \big(|z_{k_4}(T)|^2\big)\, k_4^{2s} \gtrsim \mu^{-2} \mathsf{c}\, k_4^{2s}.$$

Regarding the Sobolev norm at time zero we have by (3.40) and the choice of ε in (4.13) (recall (4.3))

$$||z(0)||_s^2 = ||r^{\mu}(0)||_s^2 \lesssim \mu^{-2}\, \mathsf{c}\, k_3^{2s}.$$

We choose the constant C in (4.18) to get $||z(0)||_s^2 \leq \delta^2$. The ratio between the Sobolev norms at time $t = T$ and $t = 0$ gives

$$\frac{||z(T)||_s}{||z(0)||_s} \gtrsim \left(\frac{k_4}{k_3}\right)^s \gtrsim N^{s/2}. \tag{4.19}$$

We conclude by giving the estimate on the diffusion time respect to the growth $\mathcal{C} := K/\delta$.

$$T \leq \mu^2 T_0 \lesssim N^s \delta^{-2} = \mathcal{C}^2 \delta^{-2}.$$

References

[1] P. Baldi, M. Berti, and R. Montalto, *Kam for autonomous quasi-linear perturbations of kdv*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire, 33 (2016), pp. 1589 – 1638.

[2] D. Bambusi, *Nekhoroshev theorem for small amplitude solutions in nonlinear Schrödinger equations*, Math. Z., 230 (1999), pp. 345–387.

[3] D. Bambusi, B. Grébert, A. Maspero, and R. Didier, *Reducibility of the quantum harmonic oscillator in d-dimensions with polynomial time-dependent perturbation*, Anal. PDE, 11 (2018), pp. 775–799.

[4] D. Bambusi and N. N. Nekhorošev, *Long time stability in perturbations of completely resonant PDEs*, Acta Applicandae Mathematica, 70 (2002), pp. 1–22.

[5] M. Berti and M. Procesi, *Quasi-periodic solutions of completely resonant forced wave equations*, Comm. Partial Differential Equations, 31 (2006), pp. 959–985.

[6] L. Biasco, L. Chierchia, and D. Treschev, *Stability of nearly integrable, degenerate Hamiltonian systems with two degrees of freedom*, J. Nonlinear Sci., 16 (2006), pp. 79–107.

[7] L. Biasco, J. E. Massetti, and M. Procesi, *An abstract birkhoff normal form theorem and exponential type stability of the 1d nls*, Communications in Mathematical Physics, 375 (2020), pp. 2089–2153.

[8] A. Bounemoura, *Generic perturbations of linear integrable Hamiltonian systems*, Regul. Chaotic Dyn., 21 (2016), pp. 665–681.

[9] A. Bounemoura and V. Kaloshin, *Generic fast diffusion for a class of non-convex Hamiltonians with two degrees of freedom*, Mosc. Math. J., 14 (2014), pp. 181–203, 426.

[10] J. Bourgain, *Aspects of long time behaviour of solutions of nonlinear Hamiltonian evolution equations*, Geom. Funct. Anal., 5 (1995), pp. 105–140.

[11] J. Bourgain, *On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE*, Internat. Math. Res. Notices, (1996), pp. 277–304.

[12] J. Bourgain, *Growth of Sobolev norms in linear Schrödinger equations with quasi-periodic potential*, Comm. Math. Phys., 204 (1999), pp. 207–247.

[13] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation*, Invent. Math., 181 (2010), pp. 39–113.

[14] J.-M. Delort, *Growth of Sobolev norms of solutions of linear Schrödinger equations on some compact manifolds*, Int. Math. Res. Not. IMRN, (2010), pp. 2305–2328.

[15] R. Feola and F. Giuliani, *Quasi-periodic traveling waves on an infinitely deep fluid under gravity*, (2020). Preprint available at `https://arxiv.org/abs/2005.08280`.

[16] R. Feola, F. Giuliani, and M. Procesi, *Reducible kam tori for the degasperis–procesi equation*, Communications in Mathematical Physics, 377 (2020), pp. 1681–1759.

[17] P. Gérard and S. Grellier, *The cubic Szegö equation*, Ann. Sci. Éc. Norm. Supér. (4), 43 (2010), pp. 761–810.

[18] ——, *Effective integrable dynamics for a certain nonlinear wave equation*, Anal. PDE, 5 (2012), pp. 1139–1155.

[19] F. Giuliani, *Quasi-periodic solutions for quasi-linear generalized kdv equations*, Journal of Differential Equations, 262 (2017), pp. 5052 – 5132.

[20] F. Giuliani, M. Guardia, P. Martin, and S. Pasquali, *Chaotic-like transfers of energy in hamiltonian pdes*, Preprint, (2020).

[21] B. Grébert, É. Paturel, and L. Thomann, *Beating effects in cubic Schrödinger systems and growth of Sobolev norms*, Nonlinearity, 26 (2013), pp. 1361–1376.

[22] B. Grébert and L. Thomann, *Resonant dynamics for the quintic nonlinear Schrödinger equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 29 (2012), pp. 455–477.

[23] B. Grébert and C. Villegas-Blas, *On the energy exchange between resonant modes in nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 28 (2011), pp. 127–134.

[24] M. Guardia, Z. Hani, E. Haus, A. Maspero, and M. Procesi, *A note on growth of Sobolev norms near quasiperiodic finite-gap tori for the 2D cubic NLS equation*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 30 (2019), pp. 865–880.

[25] M. Guardia, E. Haus, Z. Hani, A. Maspesro, and M. Procesi, *Strong nonlinear instability and growth of Sobolev norms near quasiperiodic finite-gap tori for the 2D cubic NLS equation*. Preprint available at `http://arxiv.org/abs/1810.03694`, 2019.

[26] M. Guardia, E. Haus, and M. Procesi, *Growth of Sobolev norms for the analytic NLS on* $\mathbb{T}^2$, Adv. Math., 301 (2016), pp. 615–692.
[27] M. Guardia and V. Kaloshin, *Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation*, J. Eur. Math. Soc. (JEMS), 17 (2015), pp. 71–149.
[28] ——, *Erratum to "Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation" [MR3312404]*, J. Eur. Math. Soc. (JEMS), 19 (2017), pp. 601–602.
[29] Z. Hani, *Long-time instability and unbounded Sobolev orbits for some periodic nonlinear Schrödinger equations*, Arch. Ration. Mech. Anal., 211 (2014), pp. 929–964.
[30] E. Haus and A. Maspero, *Growth of sobolev norms in time dependent semiclassical anharmonic oscillators*, Journal of Functional Analysis, 278 (2020), p. 108316.
[31] E. Haus and M. Procesi, *Growth of Sobolev norms for the quintic NLS on* T^2, Anal. PDE, 8 (2015), pp. 883–922.
[32] ——, *KAM for beating solutions of the quintic NLS*, Comm. Math. Phys., 354 (2017), pp. 1101–1132.
[33] E. Haus and L. Thomann, *Dynamics on resonant clusters for the quintic non linear Schrödinger equation*, Dyn. Partial Differ. Equ., 10 (2013), pp. 157–169.
[34] S. Kuksin, *Growth and oscillations of solutions of nonlinear Schrödinger equation*, Comm. Math. Phys., 178 (1996), pp. 265–280.
[35] ——, *Oscillations in space-periodic nonlinear Schrödinger equations*, Geom. Funct. Anal., 7 (1997), pp. 338–363.
[36] S. Kuksin and J. Pöschel, *Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation*, Ann. of Math. (2), 143 (1996), pp. 149–179.
[37] S. B. Kuksin, *On turbulence in nonlinear Schrödinger equations*, Geom. Funct. Anal., 7 (1997), pp. 783–822.
[38] A. Maspero, *Lower bounds on the growth of sobolev norms in some linear time dependent schrödinger equations*, Math. Res. Lett., In press (2018).
[39] J. Moser, *On the elimination of the irrationality condition and birkhoff?s concept of complete stability*, Bol. Soc. Mat. Mexicana, 2 (1960), pp. 167–175.
[40] N. N. Nekhorošev, *An exponential estimate of the time of stability of nearly integrable Hamiltonian systems*, Uspehi Mat. Nauk, 32 (1977), pp. 5–66, 287.
[41] O. Pocovnicu, *Explicit formula for the solution of the Szegö equation on the real line and applications*, Discrete Contin. Dyn. Syst., 31 (2011), pp. 607–649.
[42] ——, *First and second order approximations for a nonlinear wave equation*, J. Dynam. Differential Equations, 25 (2013), pp. 305–333.
[43] M. Procesi, *A normal form for beam and non-local nonlinear Schrödinger equations*, J. Phys. A, 43 (2010), pp. 434028, 13.
[44] X. Yuan, *Quasi-periodic solutions of completely resonant nonlinear wave equations*, Journal of Differential Equations, 230 (2006), pp. 213 – 274.

UPC, Barcelona
E-mail address: filippo.giuliani@upc.edu