Homoclinic and chaotic phenomena around L3 in the restricted 3-body problem

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Restricted planar circular 3-body problem

Consider the restricted planar circular 3-body problem.

Models the motion of a body of negligible mass under the influence of two massive bodies, the primaries.

- Restricted:

$$m_S = 1-\mu, \quad m_J = \mu, \quad m_A = 0 \quad \text{with } \mu \in \left(0, \frac{1}{2}\right].$$

- The third body moves on the same plane as the primaries.
- The primaries perform a circular motion.

Goal: perturbative study, $0 < \mu \ll 1$, for the resonance 1:1.
$\rightarrow$ The asteroid and the primaries have approximately the same period.

Hamiltonian in rotating coordinates

The Hamiltonian in rotating coordinates:

$$H(q,p;\mu) = H_0(q,p) + \mu H_1(q,p;\mu),$$

where

$$H_0(q,p) = \frac{||p||^2}{2} - q^t K p - \frac{1}{||q||}, \text{ with } K := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right),$$

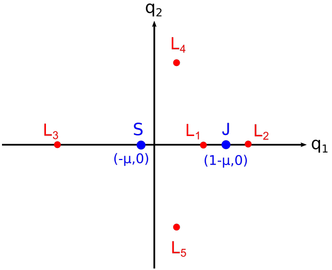
$$\mu H_1(q,p;\mu) = \frac{1}{||q||} - \frac{(1-\mu)}{||q-(-\mu,0)||} - \frac{\mu}{||q-(1-\mu,0)||}.$$

- Primaries are fixed.
- System is autonomous.
- H_0 is integrable. Corresponds to the 2-body problem.
- μH_1 is a perturbation when "far enough" of the primaries.

Equilibrium points

The 1:1 resonance contains the equilibrium points of the system.

The system has five equilibrium points, called Lagrange points.



The Lagrange point L_3 is a saddle-center with eigenvalues:

$$\mathrm{Spec} = \{\pm\sqrt{\mu}\,\rho(\mu), \pm i\,\omega(\mu)\}$$

where

$$\begin{cases} \rho(\mu) = \sqrt{21/8} + \mathcal{O}(\mu), \\ \omega(\mu) = 1 + \mathcal{O}(\mu). \end{cases}$$

The center dynamics is $\frac{1}{\sqrt{\mu}}$-faster than that of the saddle.

Change of variables

There exists a symplectic change of variables ($+$ scaling) and a change of time such that the Hamiltonian becomes

$$\widetilde{H}(\lambda, \Lambda, x, y) = \widetilde{H}_{\mathrm{p}}(\lambda, \Lambda) + \widetilde{H}_{\mathrm{osc}}(x, y) + \widetilde{H}_1(\lambda, \Lambda, x, y),$$

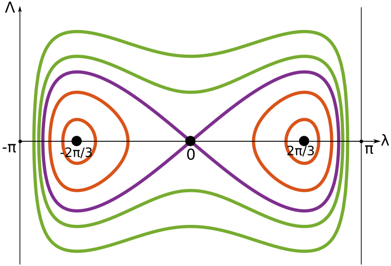
where

$$\begin{cases} \widetilde{H}_{\mathrm{p}} = -\frac{3}{2}\Lambda^2 - \cos(\lambda) - \frac{1}{\sqrt{2+2\cos(\lambda)}}, \rightarrow \text{ saddle} \\ \widetilde{H}_{\mathrm{osc}} = i\dfrac{xy}{\sqrt{\mu}}, \rightarrow \text{ center} \\ \widetilde{H}_1 \sim \mathcal{O}(\mu^{1/4}) \rightarrow \text{ non-explicit} \end{cases}$$

$\widetilde{H}_{\mathrm{p}}(\lambda, \Lambda)$ has two homoclinic connections. Non-explicit parametrization.

Singularities of the separatrix

Analysis of the separatrix of $\widetilde{H}_{\mathrm{p}}(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 - \cos(\lambda) - \frac{1}{\sqrt{2+2\cos(\lambda)}}$.



Parametrization extends analytically to

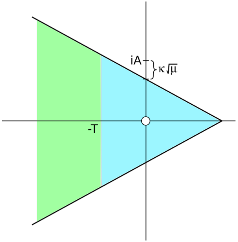
$$\Pi_A := \{t \in \mathbb{C} : |\mathrm{Im}\,(t)| < A\},$$

Only two singularities on $\partial\Pi_A$, when initial condition $(\lambda_0, 0)$.

$$A := \int_0^{\frac{-1+\sqrt{2}}{2}} \frac{2}{1-x}\sqrt{\frac{x}{3(x+1)(4x^2+4x-1)}}dx \approx 0.1778.$$

Splitting of separatrices

- A fundamental question is whether the homoclinics persist for the full problem.
- Goal: compute the difference between the stable/unstable manifolds.
- Exponentially small splitting of separatrices problem:



- Extend the region of existence of the invariant manifolds.
- Study the inner equation to obtain the correct asymptotic expression for the distance.
- Match the different approximations obtained for the invariant manifolds.

Thanks for your attention!