METASTABILITY PHENOMENA IN TWO-DIMENSIONAL RECTANGULAR LATTICES WITH NEAREST-NEIGHBOUR INTERACTION

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Abstract. We study analytically the dynamics of two-dimensional rectangular lattices with periodic boundary conditions. We consider anisotropic initial data supported on one low-frequency Fourier mode. We show that, in the continuous approximation, the resonant normal form of the system is given by integrable PDEs. We exploit the normal form result in order to prove the existence of metastability phenomena for the lattices. More precisely, we show that the energy spectrum of the normal modes attains a distribution in which the energy is shared among a packet of low-frequencies modes; such distribution remains unchanged up to the time-scale of validity of the continuous approximation.

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Contents

1. Introduction

In this paper we present an analytical study of the dynamics of two-dimensional rectangular lattices with nearest-neighbour interaction and periodic boundary conditions, for initial data with only one low-frequency Fourier mode initially excited. We give some rigorous results concerning the relaxation to a metastable state, in which energy sharing takes place among low-frequency modes only.

The study of metastability phenomena for lattices started with the numerical result by Fermi, Pasta and Ulam (FPU) [FPU95], who investigated the dynamics of a one-dimensional chain of particles with nearest neighbour interaction. In the original simulations all the energy was initially given to a single low-frequency Fourier mode with the aim of measuring the time of relaxation of the system to the 'thermal equilibrium' by looking at the evolution of the Fourier spectrum. Classical statistical mechanics prescribes that the energy spectrum corresponding to the thermal equilibrium is a plateau (the so-called theorem of equipartition of energy). Despite the authors believed that the approach to such an equilibrium would have occurred in a short time-scale, the outcoming Fourier spectrum was far from being flat and they observed two features of the dynamics that were in contrast with their expectations: the lack of thermalization displayed by the energy spectrum and the recurrent behaviour of the dynamics.

Both from a physical and a mathematical point of view, the studies on FPU-like systems have a long and active history: a concise survey of this vast literature is discussed in the monograph [Gal07]. For a more recent account on analytic results on the 'FPU paradox' we refer to [BCMM15].

In particular, we mention the papers [BP06] and [Bam08], in which the authors used the techniques of canonical perturbation theory for PDEs in order to show that the FPU α model (respectively, β model) can be rigorously described by a system of two uncoupled KdV (resp. mKdV) equations, which are obtained as a resonant normal form of the continuous approximation of the FPU model; moreover, this result allowed to deduce a rigorous result about the energy sharing among the Fourier modes, up to the time-scales of validity of the approximation. If we denote by N the number of degrees of freedom for the lattice and by $\mu \sim \frac{1}{N} \ll 1$ the wave-number of the initially excited mode, if we assume that the specific energy $\epsilon \sim \mu^4$ (resp. $\epsilon \sim \mu^2$ for the FPU β model), then the dynamics of the KdV (resp. mKdV) equations approximates the solutions of the FPU model up to a time of order $\mathcal{O}(\mu^{-3})$. However, the relation between the specific energy and the number of degrees of freedom implies that the result does not hold in the thermodynamic limit regime, namely for large N and for fixed specific energy ϵ (such a regime is the one which is relevant for statistical mechanics).

Unlike the extensive research concerning the one-dimensional systems, it seems to the authors that the behaviour of the dynamics of two-dimensional lattices is far less clear and it is expected that the interplay between the geometry of the lattice and the specific energy regime could lead to different results.

On the one hand, Benettin and collaborators [BVT80] [Ben05] [BG08] studied a two-dimensional FPU lattice with triangular cells and different boundary conditions in order to estimate the equipartition time-scale, and they found out that in the thermodynamic limit regime the equipartition is reached faster than in the one-dimensional case. The authors decided not to consider model with square cells in order to have a spectrum of linear frequencies which is different with respect to the one of the one-dimensional model; they also added (see [BG08], section B.(iii))

There is a good chance, however, that models with square lattice, and perhaps a different potential so as to avoid instability, behave differently from models with triangular lattice, and are instead more similar to one-dimensional models. This would correspond to an even stronger lack of universality in the two-dimensional FPU problem.

On the other hand, up to the authors' knowledge, the only analytical results on the dynamics of two-dimensional lattices in this framework concern the existence of breathers [Wat94] [BW06] [BW07] [YWSC09] [WJ14] [BPP10].

In this paper we study two-dimensional rectangular lattices with $(2N_1+1)\times(2N_2+1)$ sites, square cell, nearest-neighbour interaction and periodic boundary conditions, and we show the existence of metastability phenomena as in [BP06]. More precisely, under some suitable assumptions on the ratio between the sides of the lattice and on the type of small-amplitude solution we want to describe, we obtain for a 2D Electrical Transmission lattice (ETL) either a system of two uncoupled KdV equations or a system of two uncoupled KP-II equations as a resonant normal form for the continuous approximation of the lattice, while for the 2D Klein-Gordon lattice with quartic defocusing nonlinearity we obtain a one-dimensional cubic defocusing NLS equation. Since all the above PDEs are integrable, we can exploit integrability to deduce a mathematically rigorous result on the formation of the metastable packet.

Up to the authors' knowledge, this is the first analytical result about metastable phenomena in two-dimensional Hamiltonian lattices with periodic boundary conditions; in particular, this is the first rigorous result for two-dimensional lattices in which the dynamics of the lattice in a genuinely two-dimensional regime is described by a system of two-dimensional integrable PDEs.

Some comments are in order:

i. denoting by $\mu \ll 1$ the wave-number of the Fourier mode initially excited, we have that the time-scale of validity of our result is of order $\mathcal{O}(\mu^{-3})$ for the 2D ETL lattice, and of order $\mathcal{O}(\mu^{-2})$ for the 2D Klein-Gordon lattice;
ii. the ansatz about the small amplitude solutions we are describing gives a relation between the specific energy of the system ϵ and the wave-number $\mu \sim \frac{1}{N_1}$ of the Fourier mode initially excited. More precisely, we obtain $\epsilon \sim \mu^4$ for the 2D ETL lattice as in [BP06], and $\epsilon \sim \mu^2$ for the 2D Klein-Gordon lattice. This implies that the result does not hold in the thermodynamic limit regime;
iii. our result can be easily generalized to higher-dimensional lattices (see Remark 2.8 and Remark 2.9), and in particular it applies to the physical case of three-dimensional rectangular lattices with cubic cells.

To prove our results we follow the strategy of [BP06]. The first step consists in the approximation of the dynamics of the lattice with the dynamics of a continuous system. As a second step we perform a normal form canonical transformation and we obtain that the effective dynamics is given by a system of integrable PDEs (KdV, KP-II, NLS depending on the lattice and the relation between N_1 and N_2). Next, we exploit the dynamics of these integrable PDEs in order to construct approximate solutions of the original discrete lattices, and we estimate the error with repect to a true solution with the corresponding initial datum. Finally, we use the known results about the dynamics of the above mentioned integrable PDEs in order to estimate the specific energies for the approximate solutions of the original lattices.

The novelties of this work are: on the one side, a mathematically rigorous proof of the approximation of the dynamics of the ETL lattice by the dynamics of certain integrable PDEs (among these integrable PDEs, there is one which is *genuinely* two-dimensional, the KP-II equation) and of the dynamics of the two-dimensional KG lattice by the dynamics of the one-dimensional nonlinear Schrödinger equation; on the other side, there are two technical differences with respect to previous works, namely the normal form theorem (which is a variant of the technique used in [BCP02] [Bam05] [Pas19]) and the estimates for bounding the error between the approximate solution and the true solution of the lattice (which need a more careful study than the ones appearing in [SW00] [BP06] for the one-dimensional case).

2. Main Results

2.1. **The Electrical transmission lattice.**

We describe a lossless periodic two-dimensional electrical transmission lattice (ETL), given by a rectangular configuration of repeating units, each made up of two linear inductors and a nonlinear capacitor; in the non-periodic setting the model has been studied in [BW06]. We define lattice nodes by the locations of capacitors. We denote

$$\mathbb{Z}^2_{N_1,N_2} := \{(j_1,j_2) : j_1,j_2 \in \mathbb{Z}, |j_1| \le N_1, |j_2| \le N_2\}; \tag{1}$$

we also write $e_1 := (1,0)$, $e_2 := (0,1)$ and $\mathbb{Z}^2_N := \mathbb{Z}^2_{N,N}$.

The variable $V_j(t)$, $j \in \mathbb{Z}^2_{N_1,N_2}$, denotes the voltage across the j-th capacitor, $Q_j(t)$ denotes the charge stored on the j-th capacitor and $I_j(t)$ denotes the current through the j-th inductor along direction e_1. To derive the equations for the voltage V_j and the charge Q_j in the lattice one can proceed as follows. Considering a section of the lattice and applying Faraday's law and Lenz's law, the difference in shunt voltage at site j and site $j+e_1$ is given by

$$V_{j+e_1} - V_j = -L\frac{\mathrm{d}I_j}{\mathrm{d}t}, \tag{2}$$

where L is the inductance, which we assume to be constant. Assuming the capacitance C to be an analytic function of the voltage V we can expand it in Taylor series, obtaining for small voltages

$$C_j(V) \sim C_0(1 + 2aV_j + 3bV_j^2), \tag{3}$$

where $C_0 := C_j(0)$, a and b are real constants determined by the physical realisation of the network. Using standard relations between electrical quantities we finally obtain a closed equation for the charge

$$\frac{\mathrm{d}^2 Q_j}{\mathrm{d}t^2} = \frac{1}{LC_0}(\Delta_1(Q + \alpha Q^2 + \beta Q^3))_j, \tag{4}$$

$$(\Delta_1 Q)_j := (Q_{j+e_1} - 2Q_j + Q_{j-e_1}) + (Q_{j+e_2} - 2Q_j + Q_{j-e_2}). \tag{5}$$

where α, β are real parameters related to a and b. Up to a rescaling of time, we can set $LC_0 = 1$ without loss of generality. The Hamiltonian associated to (4) is given by

$$H(Q,P) = \sum_{j \in \mathbb{Z}^2_{N_1,N_2}} -\frac{1}{2}P_j\,(\Delta_1 P)_j + (F(Q))_j, \tag{6}$$

$$(F(Q))_j = \frac{Q_j^2}{2} + \alpha\frac{Q_j^3}{3} + \beta\frac{Q_j^4}{4}. \tag{7}$$

We will refer to (6) as α model (respectively, β model and $\alpha+\beta$ model) if $\beta = 0$ (respectively $\alpha = 0$, and $\alpha, \beta \neq 0$). With the above Hamiltonian formulation we obtain the equations of motion associated to (6):

$$\begin{cases} \dot{Q}_j = -(\Delta_1 P)_j \\ \dot{P}_j = -(F'(Q))_j \end{cases};$$

$$\ddot{Q}_j = (\Delta_1 F'(Q))_j. \tag{8}$$

We also introduce the Fourier coefficients of Q via the following standard relation,

$$Q_j := \frac{1}{\sqrt{(2N_1+1)(2N_2+1)}} \sum_{k \in \mathbb{Z}^2_{N_1,N_2}} \hat{Q}_k e^{i\frac{j\cdot k\,2\pi}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}}},\quad j \in \mathbb{Z}^2_{N_1,N_2}, \tag{9}$$

and similarly for P_j. We denote by

$$E_k := \frac{\omega_k^2|\hat{P}_k|^2 + |\hat{Q}_k|^2}{2}, \tag{10}$$

$$\omega_k^2 := 4\sin^2\left(\frac{k_1\,\pi}{2N_1+1}\right) + 4\sin^2\left(\frac{k_2\,\pi}{2N_2+1}\right), \tag{11}$$

the energy and the square of the frequency of the mode at site $k = (k_1, k_2) \in \mathbb{Z}^2_{N_1,N_2}$. For states described by real functions, one has $E_{(k_1,k_2)} = E_{(-k_1,k_2)}$ and $E_{(k_1,k_2)} = E_{(k_1,-k_2)}$ for all $k = (k_1, k_2)$, so we will consider only indexes in

$$\mathbb{Z}^2_{N_1,N_2,+} := \{(k_1,k_2) \in \mathbb{Z}^2_{N_1,N_2} : k_1, k_2 \geq 0\}.$$

It is also convenient to introduce the following specific quantities,

$$\kappa := \kappa(k) = \left(\frac{k_1}{N_1+\frac{1}{2}}, \frac{k_2}{N_2+\frac{1}{2}}\right), \tag{12}$$

$$\mathcal{E}_\kappa := \frac{E_k}{\left(N_1+\frac{1}{2}\right)\left(N_2+\frac{1}{2}\right)}, \tag{13}$$

where (13) is the specific energy of the normal mode with index κ.

We want to study the behaviour of small amplitude solutions of (8), with initial data in which only one low-frequency Fourier mode is excited.

We assume $N_1 \leq N_2$, and we introduce the quantities

$$\mu := \frac{2}{2N_1+1}, \tag{14}$$

$$\sigma := \log_{N_1+\frac{1}{2}}\left(N_2 + \frac{1}{2}\right), \tag{15}$$

which play the role of parameters in our construction.

We study the α model of (8) in the following regimes of these parameters:

- (KdV) the *very weakly transverse regime*, where the effective dynamics is described by a system of two *uncoupled* Korteweg-de Vries (KdV) equations. This regime corresponds to the choices $\mu \ll 1$ and $2 < \sigma < 5$;
- (KP) the *weakly transverse regime* where the effective dynamics is a described by a system of two *uncoupled* Kadomtsev-Petviashvili (KP) equation. This regime corresponds to the choices $\mu \ll 1$ and $\sigma = 2$.

From now on, we denote by $\kappa_0 := \left(\frac{1}{N_1+\frac{1}{2}}, \frac{1}{(N_1+\frac{1}{2})^\sigma}\right) = (\mu, \mu^\sigma)$.

Theorem 2.1. *Consider* (8) *with* $\alpha \neq 0$, $2 < \sigma < 5$.

Fix $1 \leq \gamma \leq \frac{7-\sigma}{2}$ *and two positive constants* C_0 *and* T*, then there exist positive constants* μ_0*,* C_1 *and* C_2 *(depending only on* γ*,* C_0 *and on* T*) such that the following holds. Consider an initial datum with*

$$\mathcal{E}_{\kappa_0}(0) = C_0\mu^4, \ \mathcal{E}_\kappa(0) = 0 \ \ \forall \kappa = (\kappa_1, \kappa_2) \neq \kappa_0, \tag{16}$$

and assume that $\mu < \mu_0$*. Then there exists* $\rho > 0$ *such that along the corresponding solution one has*

$$\mathcal{E}_\kappa(t) \leq C_1\, \mu^4 e^{-\rho|(\kappa_1/\mu, \kappa_2/\mu^\sigma)|} + C_2\, \mu^{4+\gamma}, \ \ |t| \leq \frac{T}{\mu^3} \tag{17}$$

for all κ*. Moreover, for any* n_2 *with* $0 \leq n_2 \leq N_2$ *there exists a sequence of almost-periodic functions* $(F_n)_{n=(n_1,n_2)\in\mathbb{Z}^2_{N_1,N_2,+}}$ *such that, if we denote*

$$\mathcal{F}_{\kappa_0} = \mu^4 F_n, \ \mathcal{F}_\kappa = 0 \ \ \forall \kappa \neq n\kappa_0 \tag{18}$$

then

$$|\mathcal{E}_\kappa(t) - \mathcal{F}_\kappa(t)| \leq C_2\, \mu^{4+\gamma}, \ \ |t| \leq \frac{T}{\mu^3}. \tag{19}$$

Theorem 2.2. *Consider* (8) *with* $\alpha \neq 0$, $\sigma = 2$.

Fix $1 \leq \gamma \leq \frac{5}{2}$ *and two positive constants* C_0 *and* T*, then there exist positive constants* μ_0*,* C_1 *and* C_2 *(depending only on* γ*,* C_0 *and on* T*) such that the following holds. Consider an initial datum with*

$$\mathcal{E}_{\kappa_0}(0) = C_0\mu^4, \ \mathcal{E}_\kappa(0) = 0 \ \ \forall \kappa = (\kappa_1, \kappa_2) \neq \kappa_0, \tag{20}$$

and assume that $\mu < \mu_0$*. Then there exists* $\rho > 0$ *such that along the corresponding solution one has*

$$\mathcal{E}_\kappa(t) \leq C_1\, \mu^4 e^{-\rho|(\kappa_1/\mu, \kappa_2/\mu^\sigma)|} + C_2\, \mu^{4+\gamma}, \ \ |t| \leq \frac{T}{\mu^3} \tag{21}$$

for all κ*.*

Remark 2.3. *In Theorem 2.2 we do not mention the existence of the sequence of almost-periodic functions approximating the specific energies of the modes, because this is related to the construction of action-angle/Birkhoff coordinates for the KP equation, which is an important open problem in the theory of integrable PDEs.*

2.2. **The 2D Klein-Gordon lattice.** Among the lattices that have received a great amount of attention, we mention the class of Klein-Gordon (KG) lattices, which combine the nearest-neighbour potential with an on-site one. The Hamiltonian of the system with $2N+1$ particles in the one-dimensional case is

$$H(r,s) = \sum_{j=-N}^{N} \frac{s^2}{2} + \frac{(r_{j+1}-r_j)^2}{2} + U(r_j), \tag{22}$$

$$U(x) = m^2\frac{x^2}{2} + \beta\frac{x^{2p+2}}{2p+2}, \quad m>0, \ p\geq 1. \tag{23}$$

to KG latices lack the translational invariance and they have an additional parameter whose scaling must be determined.

We now pass to two-dimensional KG lattices: the scalar model

$$H(Q,P) = \sum_{j\in\mathbb{Z}^2_{N_1,N_2}} \frac{P_j^2}{2} + \frac{1}{2}\sum_{\substack{j,k\in\mathbb{Z}^2_{N_1,N_2}\\ |j-k|=1}} \frac{(Q_j-Q_k)^2}{2} + \sum_{j\in\mathbb{Z}^2_{N_1,N_2}} U(Q_j), \tag{24}$$

$$U(x) = m^2\frac{x^2}{2} + \beta\frac{x^{2p+2}}{2p+2}, \quad m>0, \ \ \beta>0, \ p\geq 1, \tag{25}$$

can be used to describe rigid rotating molecules in the lattice plane (Q being the angle of rotation), where each molecule interacts with its neighbors and with the periodic substrate potential U; alternatively, Q can represent the transverse motion of a planar lattice [Ros03].

Using the operator Δ_1 introduced in (5), the Hamiltonian (24) can be rewritten as

$$H(Q,P) = \sum_{j\in\mathbb{Z}^2_{N_1,N_2}} \frac{P_j^2}{2} + \frac{1}{2}\sum_{j\in\mathbb{Z}^2_{N_1,N_2}} Q_j\,(-\Delta_1 Q)_j + \sum_{j\in\mathbb{Z}^2_{N_1,N_2}} U(Q_j), \tag{26}$$

the associated equations of motion are

$$\ddot{Q}_j = (\Delta_1 Q)_j - m^2 Q_j - \beta Q_j^{2p+1}, \ \ j\in\mathbb{Z}^2_{N_1,N_2}. \tag{27}$$

If we take $p=1$, we obtain a generalization of the one-dimensional ϕ^4 model.

We also introduce the Fourier coefficients of Q via the following relation,

$$Q_j := \frac{1}{\sqrt{(2N_1+1)(2N_2+1)}} \sum_{k\in\mathbb{Z}^2_{2N+1}} \hat{Q}_k e^{i\frac{j\cdot k\,2\pi}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}}}, \ \ j\in\mathbb{Z}^2_{N_1,N_2}, \tag{28}$$

and similarly for P_j, and we denote by

$$E_k := \frac{|\hat{P}_k|^2 + \omega_k^2|\hat{Q}_k|^2}{2}, \tag{29}$$

$$\omega_k^2 := m^2 + 4\sin^2\left(\frac{k_1\,\pi}{2N_1+1}\right) + 4\sin^2\left(\frac{k_2\,\pi}{2N_2+1}\right), \tag{30}$$

the energy and the square of the frequency of the mode at site $k=(k_1,k_2)\in\mathbb{Z}^2_{N_1,N_2}$.

We study the two-dimensional KG lattice (24) in the following regimes:

(1D NLS) the *very weakly transverse regime*, where the effective dynamics is described by a cubic one-dimensional nonlinear Schrödinger (NLS) equation. This regime corresponds to the choices $\mu \ll 1$ and $1<\sigma<7$.

Theorem 2.4. *Consider* (24) *with* $\beta\neq 0$, $1<\sigma<7$.

Fix $0<\gamma\leq\frac{7-\sigma}{2}$ *and two positive constants* C_0 *and* T*, then there exist positive constants* μ_0*,* C_1 *and* C_2 *(depending only on* γ*,* C_0 *and on* T*) such that the following holds. Consider an initial datum with*

$$\mathcal{E}_{\kappa_0}(0) = C_0\mu^2, \ \mathcal{E}_\kappa(0)=0, \ \forall\kappa=(\kappa_1,\kappa_2)\neq\kappa_0, \tag{31}$$

and assume that $\mu < \mu_0$. *Then there exists* $\rho > 0$ *such that along the corresponding solution one has*

$$\mathcal{E}_\kappa(t) \le C_1\, \mu^2 e^{-\rho|(\kappa_1/\mu, \kappa_2/\mu^\sigma)|} + C_2\, \mu^{2+\gamma}, \ \ |t| \ \le \ \frac{T}{\mu^2} \tag{32}$$

for all κ. *Moreover, for any* n_2 *with* $0 \le n_2 \le N_2$ *there exists a sequence of almost-periodic functions* $(F_n)_{n=(n_1,n_2)\in\mathbb{Z}^2_{N_1,N_2,+}}$ *such that, if we denote*

$$\mathcal{F}_{\kappa_0} \ = \ \mu^2\, F_n, \ \ \mathcal{F}_\kappa \ = \ 0 \ \ \forall \kappa \neq n\kappa_0 \tag{33}$$

then

$$|\mathcal{E}_\kappa(t) - \mathcal{F}_\kappa(t)| \le C_2\, \mu^{2+\gamma}, \ \ |t| \ \le \ \frac{T}{\mu^2}. \tag{34}$$

2.3. **Further remarks.**

Remark 2.5. *The specific choice of the direction of longitudinal propagation in the regimes that we have considered is not relevant.*

Remark 2.6. *Using the definition of* σ *and* μ *in* (15), (14) *we can read Theorems 2.1, 2.2 using, as parameter, the total number of sites in the lattice* N. *The statement should read as follows:*

Consider (8) *with* $\alpha \neq 0$ *and* $2 \le \sigma < 5$. *Fix* $1 \le \gamma \le \frac{7-\sigma}{2}$ *and two positive constants* C_0 *and* T, *then there exists positive constants* N_0, C_1 *and* C_2 *(depending only on* γ, C_0 *and* T*) such that if we consider an initial datum with*

$$\mathcal{E}_{\kappa_0}(0) = \frac{C_0}{N^{\frac{4}{1+\sigma}}}, \quad \mathcal{E}_\kappa(0) = 0 \quad \forall \kappa \neq \kappa_0 \tag{35}$$

with $N > N_0$. *There exists* $\rho > 0$ *such that along the corresponding solution one has*

$$\mathcal{E}_\kappa(t) \le \frac{C_1}{N^{\frac{4}{1+\sigma}}} e^{-\rho|(N_1\kappa_1, N_2\kappa_2)|} + \frac{C_2}{N^{\frac{4+\gamma}{1+\sigma}}}, \quad |t| \le T N^{\frac{3}{1+\sigma}}\,. \tag{36}$$

for all κ.

Remark 2.7. *We point out that the time of validity of Theorem 2.4 for the KG lattice is of order* $\mathcal{O}(\mu^{-2})$, *which is shorter than the time of validity of Theorem 2.1 and Theorem 2.2 for the FPU lattice. We recall that in the one-dimensional case it has been observed that, for a fixed value of specific energy* ϵ *and for long-wavelength modes initially excited, the* ϕ^4 *model reached equipartition faster than the FPU* β *model (see* [LLPR07], *sec. 2.1.8).*

Remark 2.8. *Theorem 2.1 and Theorem 2.2 can be generalized to higher dimensional lattices. Indeed, let* $d \le 4$, *define*

$$\mathbb{Z}^d_{N_1,\dots,N_d} := \{(j_1,\dots,j_d) : j_1,\dots,j_d \in \mathbb{Z}, |j_1| \le N_1, \dots, |j_d| \le N_d\}, \tag{37}$$

and consider the d*-dimensional ETL*

$$H(Q,P) = \sum_{j\in\mathbb{Z}^d_{N_1,\dots,N_d}} -\frac{1}{2}P_j\,(\Delta_1 P)_j + (F(Q))_j, \tag{38}$$

$$(F(Q))_j = \frac{Q_j^2}{2} + \alpha\frac{Q_j^3}{3} + \beta\frac{Q_j^4}{4}, \ \ j \in \mathbb{Z}^d_{N_1,\dots,N_d}. \tag{39}$$

We assume $N_1 \le N_2, \dots, N_d$, *and we introduce the quantities*

$$\mu := \frac{2}{2N_1+1}, \tag{40}$$

$$\sigma_i := \log_{N_1+\frac{1}{2}}\left(N_{i+1} + \frac{1}{2}\right), \ \ i = 1,\dots,d-1. \tag{41}$$

Then we can describe the following regimes:

- (KdV-d) *the* α *model in the very weakly transverse regime with* $\mu \ll 1$, $2 < \sigma_1, \dots, \sigma_{d-1} < 5$*;*
- (KP-d) *the* α *model, in the weakly transverse regime with* $\mu \ll 1$ *and* $\sigma_1 = 2$, $2 < \sigma_2, \dots, \sigma_{d-1} < 5$.

Moreover, in order to obtain Theorem 2.1 and Theorem 2.2 we will have to assume that

$$2\gamma + \sum_{i=1}^{d-1} \sigma_i \le 7. \tag{42}$$

which, together with the fact that $\sigma_i > 2$ for all $i = 1, \dots, d-1$, is consistent with the assumption $d \le 4$.

Remark 2.9. *Theorem 2.4 can be generalized to higher dimensional lattices. Indeed, let $d \le 6$, define $\mathbb{Z}^d_{N_1,\dots,N_d}$ as in (37) and consider the d-dimensional NLKG lattice*

$$H(Q,P) = \sum_{j\in\mathbb{Z}^d_{N_1,\dots,N_d}} \frac{P_j^2}{2} + \frac{1}{2} \sum_{\substack{j,k\in\mathbb{Z}^d_{N_1,\dots,N_d}\\ |j-k|=1}} \frac{(Q_j - Q_k)^2}{2} + \sum_{j\in\mathbb{Z}^d_{N_1,\dots,N_d}} U(Q_j), \tag{43}$$

$$U(x) = m^2\frac{x^2}{2} + \beta\frac{x^{2p+2}}{2p+2}, \quad m > 0, \;\; \beta > 0, \; p \ge 1, \tag{44}$$

We assume $N_1 \le N_2, \dots, N_{d-1}$, and we introduce the quantities μ and σ_i ($1 \le i \le d-1$) as in (40) and (41).

Then we can describe the following regime:

(1DNLS-d) *the model (43) with $m = 1$ and $p = 1$ in the very weakly transverse regime, with $\mu \ll 1$, $1 < \sigma_1, \dots, \sigma_{d-1} < 7$;*

Moreover, in order to obtain Theorem 2.4 we will have to assume that

$$2\gamma + \sum_{i=1}^{d-1} \sigma_i \le 7. \tag{45}$$

which, together with the fact that $\sigma_i > 1$ for all $i = 1, \dots, d-1$, is consistent with the assumption $d \le 6$.

Remark 2.10. *There are other interesting regimes for (8) and (27) especially for their relation with the modified KdV equation and two-dimensional Non-Linear Schrödinger equation respectively. These will be discussed in Sec. D.1 and D.2 respectively.*

3. Galerkin Averaging

3.1. An Averaging Theorem.

Following [Pas19] (see also [BP06] and [Bam05]) we use a Galerkin averaging method in order to compute the difference between the solutions of the full equation (76) and the equation in normal form.

To this end we first have to introduce a topology in the phase space. This is conveniently done in terms of Fourier coefficients.

Definition 3.1. *Fix two constants $\rho \ge 0$ and $s \ge 0$. We will denote by $\ell^2_{\rho,s}$ the Hilbert space of complex sequences $v = (v_n)_{n\in\mathbb{Z}^2\setminus\{0\}}$ with obvious vector space structure and with scalar product*

$$\langle v, w\rangle_{\rho,s} := \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \overline{v_n} w_n e^{2\rho|n|}|n|^{2s} . \tag{46}$$

and such that

$$\|v\|^2_{\rho,s} := \langle v, v\rangle_{\rho,s} = \sum_{n\in\mathbb{Z}^2\setminus\{0\}} |v_n|^2 e^{2\rho|n|}|n|^{2s} \tag{47}$$

is finite.

We will denote by ℓ^2 the space $\ell^2_{0,0}$. We will identify a 2-periodic function v with the sequence of its Fourier coefficients $\{\hat{v}_n\}_n$,

$$v(y) = \frac{1}{2}\sum_{n\in\mathbb{Z}^2} \hat{v}_n e^{i\pi\, n\cdot y},$$

and we will say that $v \in \ell^2_{\rho,s}$ if the sequence of its Fourier coefficients belong to $\ell^2_{\rho,s}$.

Now fix $\rho \geq 0$, and consider the scale of Hilbert spaces $\mathcal{H}^{\rho,s} := \ell^2_{\rho,s} \times \ell^2_{\rho,s} \ni \zeta = (\xi, \eta)$ $(s \geq 1)$, endowed with one of the following symplectic forms:

$$\Omega_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \Omega_2 := \begin{pmatrix} \partial_{x_1}^{-1} & 0 \\ 0 & -\partial_{x_1}^{-1} \end{pmatrix}. \tag{48}$$

Ω_2 is well defined on the space of zero-average x_1 functions, i.e. on those functions $\zeta(x_1, x_2)$ such that for every x_2 we have $\int_{-1}^{1} \zeta(x_1, x_2)\, \mathrm{d}x_1 = 0$.

If we fix $a \in \{1, 2\}$, s and $U_s \subset \ell^2_{\rho,s}$ open, we define the gradient of $K \in C^\infty(U_s, \mathbb{R})$ with respect to $\xi \in \ell^2_{\rho,s}$ as the unique function s.t.

$$\langle \nabla_\xi K, h \rangle = \mathrm{d}_\xi K h, \ \ \forall h \in \ell^2_{\rho,s};$$

similarly, for an open set $\mathcal{U}_s \subset \mathcal{H}^{\rho,s}$ the Hamiltonian vector field of the Hamiltonian function $H \in C^\infty(\mathcal{U}_s, \mathbb{R})$ is given by

$$X_H(\xi, \eta) = \Omega_a^{-1} \nabla_{(\xi,\eta)} H(\xi, \eta).$$

The open ball of radius R and center 0 in $\ell^2_{\rho,s}$ will be denoted by $B_{\rho,s}(R)$; we write $\mathcal{B}_{\rho,s}(R) := B_{\rho,s}(R) \times B_{\rho,s}(R) \subset \mathcal{H}^{\rho,s}$.

Now, we introduce the Fourier projection operators $\hat{\pi}_j : \ell^2_{\rho,s} \to \ell^2_{\rho,s}$

$$\hat{\pi}_j((v_n)_{n \in \mathbb{Z}^2 \setminus \{0\}}) := (v_n)_{j-1 \leq |n| \leq j}, \ \ j \geq 1, \tag{49}$$

the operators $\pi_j : \mathcal{H}^{\rho,s} \to \mathcal{H}^{\rho,s}$

$$\pi_j((\zeta_n)_{n \in \mathbb{Z}^2 \setminus \{0\}}) := \left((\xi_n)_{j-1 \leq |n| \leq j}, (\eta_n)_{j-1 \leq |n| \leq j} \right), \ \ j \geq 1, \tag{50}$$

and the operators $\Pi_M : \mathcal{H}^{\rho,s} \to \mathcal{H}^{\rho,s}$

$$\Pi_M((\zeta_n)_{n \in \mathbb{Z}^2 \setminus \{0\}}) := ((\xi_n)_{|n| \leq M}, (\eta_n)_{|n| \leq M}), \ \ M \geq 0. \tag{51}$$

Lemma 3.2. *The projection operators defined in* (50) *and* (51) *satisfy the following properties:*

i. for any $j \geq 0$ and for any $\zeta \in \mathcal{H}^{\rho,s}$

$$\zeta = \sum_{j \geq 0} \pi_j \zeta;$$

ii. for any $j \geq 0$ there exist a positive constants C' (possibly depending on s) such that

$$\|\Pi_M \zeta\|_{\mathcal{H}^{\rho,s}} \leq C' \, \|\zeta\|_{\mathcal{H}^{\rho,s}} \ \ \forall \zeta \in \mathcal{H}^{\rho,s};$$

iii. there exist positive constants C_1'', C_2'' (possibly depending on s) such that

$$C_1'' \|\zeta\|_{\mathcal{H}^{\rho,s}} \leq \left\| \left[\sum_{j \in \mathbb{N}} j^{2s} |\pi_j \zeta|^2 \right]^{1/2} \right\|_{\mathcal{H}^{\rho,0}} \leq C_2'' \|\zeta\|_{\mathcal{H}^{\rho,s}} \ \ \forall \zeta \in \mathcal{H}^{\rho,s}. \tag{52}$$

Proof. Properties *i.* and *ii.* follow from the definitions of the operators.

As for property *iii.*, we just observe that there exists $C_2'' > 0$ such that

$$\begin{aligned} \left\| \left[\sum_{j \in \mathbb{N}} j^{2s} |\pi_j \zeta|^2 \right]^{1/2} \right\|^2_{\mathcal{H}^{\rho,0}} &\leq C_2'' \left\| \left[\sum_{j \in \mathbb{N}} j^{2s} e^{2\rho j} |\pi_j \zeta|^2 \right]^{1/2} \right\|_{\ell^2 \times \ell^2} \\ &= C_2'' \|\zeta\|_{\mathcal{H}^{\rho,s}}, \end{aligned}$$

and similarly for the other estimate. □

Now we consider a Hamiltonian system of the form

$$H = h_0 + \delta F, \tag{53}$$

where we assume that

(PER) h_0 generates a linear periodic flow Φ^τ with period 2,

$$\Phi^{\tau+2} = \Phi^\tau \ \ \forall \tau.$$

which is analytic $\Phi^\tau : \mathcal{H}^{\rho,s} \to \mathcal{H}^{\rho,s}$ for any $s \geq 1$;

(INV) for any $s \geq 1$, Φ^τ leaves invariant the space $\Pi_j \mathcal{H}^{\rho,s}$ for any $j \geq 0$. Furthermore, for any $j \geq 0$

$$\pi_j \circ \Phi^\tau = \Phi^\tau \circ \pi_j.$$

Next, we assume that the vector field of F admits an asymptotic expansion in δ of the form

$$F \sim \sum_{j \geq 1} \delta^{j-1} F_j, \tag{54}$$

$$X_F \sim \sum_{j \geq 1} \delta^{j-1} X_{F_j}, \tag{55}$$

and that the following property is satisfied

(HVF) There exists $R^* > 0$ such that for any $j \geq 1$

- X_{F_j} is analytic from $\mathcal{B}_{\rho,s+2j}(R^*)$ to $\mathcal{H}^{\rho,s}$.

Moreover, for any $r \geq 1$ we have that

- $X_{F - \sum_{j=1}^{r} \delta^{j-1} F_j}$ is analytic from $\mathcal{B}_{\rho,s+2(r+1)}(R^*)$ to $\mathcal{H}^{\rho,s}$.

The main result of this section is the following theorem.

Theorem 3.3. *Fix $R > 0$, $s_1 \gg 1$. Consider* (53)*, and assume (PER), (INV) and (HVF). Then $\exists$ $s_0 > 0$ with the following properties: for any $s \geq s_1$ there exists $\delta_s \ll 1$ such that for any $\delta < \delta_s$ there exists $\mathcal{T}_\delta^{(1)} : \mathcal{B}_{\rho,s}(R) \to \mathcal{B}_{\rho,s}(2R)$ analytic canonical transformation such that*

$$H_1 := H \circ \mathcal{T}_\delta^{(1)} = h_0 + \delta \mathcal{Z}_1 + \delta^2\, \mathcal{R}^{(1)}, \tag{56}$$

where $\mathcal{Z}_1$ is in normal form, namely

$$\{\mathcal{Z}_1, h_0\} = 0, \tag{57}$$

and there exists a positive constant C_s' such that

$$\sup_{\mathcal{B}_{\rho,s+s_0}(R)} \|X_{\mathcal{Z}_1}\|_{\mathcal{H}^{\rho,s}} \leq C_s',$$

$$\sup_{\mathcal{B}_{\rho,s+s_0}(R)} \|X_{\mathcal{R}^{(1)}}\|_{\mathcal{H}^{\rho,s}} \leq C_s', \tag{58}$$

$$\sup_{\mathcal{B}_{\rho,s}(R)} \|\mathcal{T}_\delta^{(1)} - id\|_{\mathcal{H}^{\rho,s}} \leq C_s'\, \delta. \tag{59}$$

In particular,

$$\mathcal{Z}_1(\xi,\eta) = \langle F_1 \rangle (\xi,\eta), \tag{60}$$

where $\langle F_1 \rangle (\xi,\eta) := \int_0^2 F_1 \circ \Phi^\tau(\xi,\eta) \frac{d\tau}{2}$.

Remark 3.4. *By using the same arguments of* [Bam05] *and* [Pas19] *one can prove a more general version of Theorem 3.3, in which the Hamiltonian is put in normal form up to order r, for any $r \geq 1$. In this latter case, both δ_s and s_0 will also depend on r.*

3.2. **Proof of the Averaging Theorem.** The proof of Theorem 3.3 is actually an application of the techniques used in [Pas19] and [BP06]).

First notice that by assumption (INV) the Hamiltonian vector field of h_0 generates a continuous flow Φ^τ which leaves $\Pi_M \mathcal{H}^{\rho,s}$ invariant.

Now we set $H = H_{1,M} + \mathcal{R}_{1,M} + \mathcal{R}_1$, where

$$H_{1,M} := h_0 + \delta\, F_{1,M}, \tag{61}$$

$$F_{1,M} := F_1 \circ \Pi_M, \tag{62}$$

and

$$\mathcal{R}_{1,M} := h_0 + \delta F_1 - H_{1,M}, \tag{63}$$

$$\mathcal{R}_1 := \delta\,(F - F_1)\,. \tag{64}$$

The system described by the Hamiltonian (61) is the one that we will put in normal form. In the following we will use the notation $a \lesssim b$ to mean: there exists a positive constant K independent of M and R (but eventually on s), such that $a \leq Kb$. We exploit the following intermediate results:

Lemma 3.5. *For any $s \geq s_1$ there exists $R > 0$ such that $\forall\ \sigma > 0$, $M > 0$*

$$\sup_{\mathcal{B}_{\rho,s+\sigma+4}(R)} \|X_{\mathcal{R}_{1,M}}(\xi,\eta)\|_{\mathcal{H}^{\rho,s}} \lesssim \frac{\delta}{(M+1)^{\sigma}}, \tag{65}$$

$$\sup_{\mathcal{B}_{\rho,s+4}(R)} \|X_{\mathcal{R}_1}(\xi,\eta)\|_{\mathcal{H}^{\rho,s}} \lesssim \delta^2. \tag{66}$$

Proof. We recall that $\mathcal{R}_{1,M} = h_0 + \delta F_j - H_{1,M}$.
We first notice that $\|id - \Pi_M\|_{\mathcal{H}^{\rho,s+\sigma}\to\mathcal{H}^{\rho,s}} \lesssim (M+1)^{-\sigma}$, indeed

$$\begin{aligned}\Big\|\sum_{j\geq M+1}\pi_j f\Big\|_{\mathcal{H}^{\rho,s}} &\lesssim \left\|\left[\sum_{j\geq M+1}|j^s\pi_j f|^2\right]^{1/2}\right\|_{\mathcal{H}^{\rho,0}}\\ &\lesssim (M+1)^{-\sigma}\left\|\left[\sum_{j\geq M+1}|j^{s+\sigma}\pi_j f|^2\right]^{1/2}\right\|_{\mathcal{H}^{\rho,0}}\\ &\lesssim (M+1)^{-\sigma}\|f\|_{\mathcal{H}^{\rho,s+\sigma}},\end{aligned}$$

Inequality (65) follows from the next chain of inequalities:

$$\begin{aligned}\sup_{(\xi,\eta)\in\mathcal{B}_{\rho,s+4+\sigma}(R)}\|X_{\mathcal{R}_{1,M}}(\xi,\eta)\|_{\mathcal{H}^{\rho,s}} &\lesssim \|dX_{\delta F_1}\|_{L^\infty(\mathcal{B}_{\rho,s+4}(R),\mathcal{H}^{\rho,s})}\|id-\Pi_M\|_{L^\infty(\mathcal{B}_{\rho,s+4+\sigma}(R),\mathcal{B}_{\rho,s+4}(R))}\\ &\lesssim \delta\,(M+1)^{-\sigma}.\end{aligned}$$

Estimate (66) is an immediate consequence of (HVF).

□

Lemma 3.6. *For any $s \geq s_1$ there exists $R > 0$ such that*

$$\sup_{\mathcal{B}_{\rho,s}(R)}\|X_{F_{1,M}}(\xi,\eta)\|_{\mathcal{H}^{\rho,s}} \leq K^{(F)}_{1,s}M^2,$$

where

$$K^{(F)}_{1,s} := \sup_{\mathcal{B}_{\rho,s}(R)}\|X_{F_1}(\xi,\eta)\|_{\mathcal{H}^{\rho,s-2}}.$$

Proof. It follows from

$$\begin{aligned}\sup_{(\xi,\eta)\in\mathcal{B}_{\rho,s}(R)}\Big\|\sum_{h\leq M}\pi_h X_{F_{1,M}}(\xi,\eta)\Big\|_{\mathcal{H}^{\rho,s}} &\lesssim \sup_{(\xi,\eta)\in\mathcal{B}_{\rho,s}(R)}\left\|\left[\sum_{h\leq M}|h^s\pi_h X_{F_{1,M}}(\xi,\eta)|^2\right]^{1/2}\right\|_{\mathcal{H}^{\rho,0}} &(67)\\ &\leq M^2\sup_{(\xi,\eta)\in\mathcal{B}_{\rho,s}(R)}\left\|\left[\sum_{h\leq M}|h^{s-2}\pi_h X_{F_{1,M}}(\xi,\eta)|^2\right]^{1/2}\right\|_{\mathcal{H}^{\rho,0}} &(68)\\ &\lesssim M^2\sup_{(\xi,\eta)\in\mathcal{B}_{\rho,s}(R)}\|X_{F_{1,M}}(\xi,\eta)\|_{\mathcal{H}^{\rho,s-2}} &(69)\\ &= K^{(F)}_{1,s}\,M^2. &(70)\end{aligned}$$

$\square$

To normalize (61) we need a slight reformulation of Theorem 4.4 in [Bam99]. Here we report a statement of the result adapted to our context which is proved in Appendix A.

Lemma 3.7. *Let $s \geq s_1 + 2$, $R > 0$, and consider the system* (61). *Assume that $\delta < M^{-4}$, and that*

$$K_{1,s}^{(F)} M^2 \delta < 2^{-9} e^{-1} \pi^{-1} R, \tag{71}$$

where

$$K_{1,s}^{(F)} := \sup_{(\xi,\eta)\in\mathcal{B}_{\rho,s}(R)} \|X_{F_1}(\xi,\eta)\|_{\mathcal{H}^{\rho,s-2}}.$$

Then there exists an analytic canonical transformation $\mathcal{T}_{\delta,M}^{(1)} : \mathcal{B}_{\rho,s}(R) \to \mathcal{B}_{\rho,s}(2R)$ such that

$$\sup_{\mathcal{B}_{\rho,s}(R/2)} \|\mathcal{T}_{\delta,M}^{(1)}(\xi,\eta) - (\xi,\eta)\|_{\mathcal{H}^{\rho,s}} \leq 4\pi K_{1,s}^{(F)} M^2 \delta, \tag{72}$$

and that puts (61) *in normal form up to a small remainder,*

$$H_{1,M} \circ \mathcal{T}_{\delta,M}^{(1)} = h_0 + \delta Z_M^{(1)} + \delta^2 \mathcal{R}_M^{(1)}, \tag{73}$$

with $Z_M^{(1)}$ is in normal form, namely $\{h_{0,M}, Z_M^{(1)}\} = 0$, and

$$\begin{aligned} \sup_{\mathcal{B}_{\rho,s}(R/2)} \|X_{Z_M^{(1)}}(\xi,\eta)\|_{\mathcal{H}^{\rho,s}} &\leq 4\,M^2\,\delta\,K_{1,s}^{(F)}\,M^2 K_{1,s}^{(F)} \\ &= 4(K_{1,s}^{(F)})^2 M^4 \delta, \end{aligned} \tag{74}$$

$$\begin{aligned} &\sup_{\mathcal{B}_{\rho,s}(R/2)} \|X_{\mathcal{R}_M^{(1)}}(\xi,\eta)\|_{\mathcal{H}^{\rho,s}} \\ &\leq 2^{12} e \frac{1}{R^2} K_{1,s}^{(F)} M^2 \left(2^{10} 3^2 e \frac{1}{R} K_{1,s}^{(F)} K_{1,s}^{(F)} M^4 \delta + 5 K_{1,s}^{(F)}\,M^2 \right) \end{aligned} \tag{75}$$

Now we conclude with the proof of the Theorem 3.3.

Proof. If we define $\delta_s := 2^{-9} e^{-1} \pi^{-1} R (K_{1,s}^{(F)})^{-1}$ and we choose

$$\begin{aligned} s_0 &= \sigma + 4, \\ M &= \mu^{-2/\sigma}, \\ \sigma &> 4, \end{aligned}$$

then the transformation $\mathcal{T}_{\delta,M}^{(1)}$ defined by Lemma 3.7 satisfies (56) because of (73).

Next, Eq. (57) follows from Lemma 3.7, Eq. (58) follows from (74) and (75), while (59) is precisely (72). Finally, (60) can be deduced by applying Lemma A.7 to $G = F_1$. $\square$

4. APPLICATIONS TO TWO-DIMENSIONAL LATTICES

4.1. **The KdV regime for the ETL lattice.**

We want to study the behaviour of small amplitude solutions of (8), with initial data in which only one low-frequency Fourier mode is excited.

As a first step, we introduce an interpolating function $Q = Q(t,x)$ such that

- (A1) $Q(t,j) = Q_j(t)$, for all $j \in \mathbb{Z}^2_{N_1,N_2}$;
- (A2) Q is periodic with period $2N_1 + 1$ in the x_1-variable, and periodic with period $2N_2 + 1$ in the x_2-variable;
- (A3) Q has zero average, $\int_{[-(N_1+\frac{1}{2}),N_1+\frac{1}{2}]\times[-(N_2+\frac{1}{2}),N_2+\frac{1}{2}]} Q(t,j)\mathrm{d}j = 0\ \forall t$;
- (A4) Q fulfills

$$\ddot{Q} = \Delta_1 (Q + \alpha Q^2 + \beta Q^3), \tag{76}$$

$$\Delta_1 := 4\sinh^2\left(\frac{\partial_{x_1}}{2}\right) + 4\sinh^2\left(\frac{\partial_{x_2}}{2}\right). \tag{77}$$

It is easy to verify that (76) is Hamiltonian with Hamiltonian function

$$H(Q,P)=\int_{\left[-\frac{1}{\mu},\frac{1}{\mu}\right]\times\left[-\frac{1}{\mu^\sigma},\frac{1}{\mu^\sigma}\right]}\frac{-P\,\Delta_1 P+Q^2}{2}+\alpha\frac{Q^3}{3}+\beta\frac{Q^4}{4}\mathrm{d}x, \tag{78}$$

where P is a periodic function which has zero average and is canonically conjugated to Q.

First we consider (76), with $\alpha\neq 0$, and we look for small amplitude solutions of the form

$$Q(t,x)=\mu^2 q(\mu t,\mu x_1,\mu^\sigma x_2), \tag{79}$$

where $q:\mathbb{R}\times\mathbb{T}^2\to\mathbb{R}$ is a periodic function and μ, σ defined in (14)-(15). We introduce the rescaled variables $\tau=\mu t$, $y_1=\mu x_1$, $y_2=\mu^\sigma x_2$, and we denote

$$I:=[-1,1]^2. \tag{80}$$

Plugging (79) into (76), we get

$$q_{\tau\tau}=\frac{\Delta_{\mu,y_1,\sigma}}{\mu^2}\left(q+\mu^2\alpha q^2\right), \tag{81}$$

$$\Delta_{\mu,y_1,\sigma}:=4\sinh^2\left(\frac{\mu\partial_{y_1}}{2}\right)+4\sinh^2\left(\mu^\sigma\frac{\partial_{y_2}}{2}\right), \tag{82}$$

which is a Hamiltonian PDE corresponding to the Hamiltonian functional

$$K_1(q,p)=\int_I\frac{-p\,\Delta_{\mu,y_1,\sigma}p}{2\mu^2}+\frac{q^2}{2}+\alpha\mu^2\frac{q^3}{3}\mathrm{d}y, \tag{83}$$

and p is the variable canonically conjugated to q.

Now, observe that the the operator $\Delta_{\mu,y_1,\sigma}$ admits the following asymptotic expansion,

$$\frac{\Delta_{\mu,y_1,\sigma}}{\mu^2}\sim\partial_{y_1}^2+\mu^{2(\sigma-1)}\partial_{y_2}^2+\sum_{m\geq 1}c_m\left(\mu^{2m}\partial_{y_1}^{2(m+1)}+\mu^{2[(m+1)\sigma-1]}\partial_{y_2}^{2(m+1)}\right), \tag{84}$$

$$c_m:=\frac{2}{(2m)!}, \tag{85}$$

which, up to terms of order $\mathcal{O}(\mu^4)$, reads

$$\frac{\Delta_{\mu,y_1,\sigma}}{\mu^2}\sim\partial_{y_1}^2+\frac{\mu^2}{12}\partial_{y_1}^4+\mathcal{O}(\mu^4), \tag{86}$$

(recall that $\sigma>2$). Therefore the Hamiltonian (83) admits the following asymptotic expansion

$$K_1(q,p)\sim\hat{h}_0(q,p)+\mu^2\hat{F}_1(q,p)+\mu^4\hat{\mathcal{R}}(q,p), \tag{87}$$

$$\hat{h}_0(q,p)=\int_I\frac{-p\,(\partial_{y_1}^2 p)+q^2}{2}\mathrm{d}y, \tag{88}$$

$$\hat{F}_1(q,p)=\int_I-\frac{p\,\partial_{y_1}^4 p}{24}+\alpha\frac{q^3}{3}\mathrm{d}y. \tag{89}$$

Note that the nonlinearity of degree 4 does not affect the Hamiltonian up to order $\mathcal{O}(\mu^4)$. Following the approach of [BP06], we can introduce the following non-canonical change of coordinates

$$\xi:=\frac{1}{\sqrt{2}}(q+\partial_{y_1}p), \tag{90}$$

$$\eta:=\frac{1}{\sqrt{2}}(q-\partial_{y_1}p). \tag{91}$$

Since the previous transformation is not canonical, the Poisson tensor in these new coordinates is

$$J=\partial_{y_1}\begin{pmatrix}-1&0\\0&1\end{pmatrix}, \tag{92}$$

and Hamilton equations associated to a Hamiltonian K_1 are

$$\partial_\tau \xi = -\partial_{y_1} \frac{\delta K_1}{\delta \xi}$$
$$\partial_\tau \eta = \partial_{y_1} \frac{\delta K_1}{\delta \eta}.$$

Remark 4.1. *The explicit expression of the Poisson tensor* (92) *let us compute straightforwardly Casimir invariants associated to J, which are*

$$C(\xi, \eta) = A + B \int_{-1}^{1} \xi(\tau, y_1, y_2) \mathrm{d}y_1 + C \int_{-1}^{1} \eta(\tau, y_1, y_2) \mathrm{d}y_1, \tag{93}$$

where A, B and C are arbitrary real constants.

Since Casimir invariants are constants of motion, we can restrict our analysis on the subspace defined by

$$\int_{-1}^{1} \xi(\tau, y_1, y_2) - \eta(\tau, y_1, y_2) \mathrm{d}y_1 = 0 \ \ \forall \tau \in \mathbb{R}, \ |y_2| \leq 1. \tag{94}$$

However, by recalling (90)-(91) *one sees that* (94) *implies*

$$\int_{-1}^{1} \partial_{y_1} p(\tau, y_1, y_2) \mathrm{d}y_1 = 0 \ \ \forall \tau \in \mathbb{R}, \ |y_2| \leq 1, \tag{95}$$

which is true due to periodic boundary conditions.

Moreover, if we write K_1 in (ξ, η) coordinates we have

$$K_1(\xi, \eta) \sim h_0(\xi, \eta) + \mu^2 F_1(\xi, \eta) + \mu^4 \mathcal{R}(\xi, \eta), \tag{96}$$

$$h_0(\xi, \eta) = \int_I \frac{\xi^2 + \eta^2}{2} \mathrm{d}y, \tag{97}$$

$$F_1(\xi, \eta) = \int_I -\frac{[\partial_{y_1}(\xi - \eta)]^2}{48} + \alpha \frac{(\xi + \eta)^3}{3 \cdot 2^{3/2}} \mathrm{d}y. \tag{98}$$

Now we apply the averaging Theorem 3.3 to the Hamiltonian (96), with $\delta = \mu^2$: observe that the equations of motion of h_0 have the following simple form:

$$\begin{cases} \xi_\tau &= -\partial_{y_1} \xi \\ \eta_\tau &= \partial_{y_1} \eta \end{cases} ; \quad \begin{cases} \xi(\tau, y) &= \xi_0(y_1 - \tau, y_2) \\ \eta(\tau, y) &= \eta_0(y_1 + \tau, y_2) \end{cases} . \tag{99}$$

Proposition 4.2. *The average of F_1 in* (96) *with respect to the flow of h_0 in* (97) *is given by*

$$\langle F_1 \rangle (\xi, \eta) = -\int_I \frac{(\partial_{y_1} \xi)^2 + (\partial_{y_1} \eta)^2}{48} \mathrm{d}y + \frac{\alpha}{3 \cdot 2^{3/2}} ([\xi^3] + [\eta^3]), \tag{100}$$

where we denote by $[f^j]$ the average $\int_I f^j(y) \frac{\mathrm{d}y}{4}$.

Proof. We compute as an example the average of the term proportional to $\int_I (\partial_{y_1} \xi)(\partial_{y_1} \eta) \mathrm{d}y$. Using Taylor expansion at $y_2 = 0$ we obtain

$$\left\langle \int_I (\partial_{y_1} \xi)(\partial_{y_1} \eta) \mathrm{d}y \right\rangle = \int_0^2 \int_I \partial_{y_1} \xi(y_1 - \tau, y_2) \, \partial_{y_1} \eta(y_1 + \tau, y_2) \mathrm{d}y \, \frac{\mathrm{d}\tau}{2} \tag{101}$$

$$= \int_0^2 \int_I \partial_{y_1} \xi(y_1 - \tau, 0) \, \partial_{y_1} \eta(y_1 + \tau, 0) \mathrm{d}y \, \frac{\mathrm{d}\tau}{2} \tag{102}$$

$$+ \int_0^2 \int_I y_2 \, \partial_{y_2} \partial_{y_1} \xi(y_1 - \tau, 0) \, \partial_{y_1} \eta(y_1 + \tau, y_2) \mathrm{d}y \frac{\mathrm{d}\tau}{2} \tag{103}$$

$$+ \int_0^2 \int_I y_2 \, \partial_{y_1} \xi(y_1 - \tau, y_2) \, \partial_{y_2} \partial_{y_1} \eta(y_1 + \tau, 0) \mathrm{d}y \frac{\mathrm{d}\tau}{2} \tag{104}$$

$$+ \mathcal{R}(\xi, \eta), \tag{105}$$

where

$$\begin{aligned}\mathcal{R}(\xi,\eta) = &\int_0^2 \int_I \left(\int_0^{y_2} (y_2-z)\partial_{y_2}^2\partial_{y_1}\xi(y_1-\tau,z)\mathrm{d}z\right)\partial_{y_1}\eta(y_1+\tau,y_2)\mathrm{d}y\,\frac{\mathrm{d}\tau}{2}\\ &+\int_0^2\int_I \partial_{y_1}\xi(y_1-\tau,y_2)\left(\int_0^{y_2}(y_2-z)\partial_{y_2}^2\partial_{y_1}\eta(y_1+\tau,z)\mathrm{d}z\right)\mathrm{d}y\,\frac{\mathrm{d}\tau}{2}.\end{aligned}\tag{106}$$

If we consider the integral in (102), we get

$$\int_0^2\int_I \partial_{y_1}\xi(y_1-\tau,0)\,\partial_{y_1}\eta(y_1+\tau,0)\frac{\mathrm{d}y}{4}\,\mathrm{d}\tau = \frac{1}{4}\int_{-2}^2\int_0^4 \partial_{y_1}\xi(s_1,0)\,\partial_{y_1}\eta(s_2,0)\mathrm{d}s_1\,\mathrm{d}s_2 = 0,\tag{107}$$

while we deal with the integral in (103) in the following way,

$$\begin{aligned}&\int_0^2\int_I y_2\,\partial_{y_2}\partial_{y_1}\xi(y_1-\tau,0)\,\partial_{y_1}\eta(y_1+\tau,y_2)\mathrm{d}y\,\mathrm{d}\tau\\ &=\int_{-1}^1 y_2\left[\int_0^2\int_{-1}^1 \partial_{y_2}\partial_{y_1}\xi(y_1-\tau,0)\partial_{y_1}\eta(y_1+\tau,y_2)\mathrm{d}y_1\,\mathrm{d}\tau\right]\mathrm{d}y_2,\\ &=\int_{-1}^1 y_2\left[\frac{1}{4}\int_{-2}^2\int_0^4 \partial_{y_2}\partial_{s_1}\xi(s_1,0)\partial_{s_2}\eta(s_2,y_2)\mathrm{d}s_1\,\mathrm{d}s_2\right]\mathrm{d}y_2,\\ &=\frac{1}{4}\left(\int_0^4 \partial_{y_2}\partial_{s_1}\xi(s_1,0)\,\mathrm{d}s_1\right)\int_{-1}^1 y_2\left[\int_{-2}^2 \partial_{s_2}\eta(s_2,y_2)\,\mathrm{d}s_2\right]\mathrm{d}y_2\\ &=\frac{1}{4}\left(\int_0^4 \partial_{s_1}(\partial_{y_2}\xi)(s_1,0)\,\mathrm{d}s_1\right)\int_{-1}^1 y_2\left[\int_{-2}^2 \partial_{s_2}\eta(s_2,y_2)\,\mathrm{d}s_2\right]\mathrm{d}y_2,\end{aligned}\tag{108}$$

which vanishes because $\partial_{y_2}\xi$ is periodic in the s_1-variable.

A verbatim repetition of the argument shows that (104) vanishes.

Finally, we observe that the remainder (106) also vanishes, because

$$\begin{aligned}&\int_0^2\int_I\left(\int_0^{y_2}(y_2-z)\partial_{y_2}^2\partial_{y_1}\xi(y_1-\tau,z)\mathrm{d}z\right)\partial_{y_1}\eta(y_1+\tau,y_2)\mathrm{d}y\,\mathrm{d}\tau\\ &=\int_{-1}^1\int_0^{y_2}(y_2-z)\left[\frac{1}{2}\int_0^4 \partial_{y_2}^2\partial_{s_1}\xi(s_1,z)\mathrm{d}s_1\right]\left[\frac{1}{2}\int_{-2}^2\partial_{s_2}\eta(s_2,y_2)\mathrm{d}s_2\right]\mathrm{d}z\,\mathrm{d}y_2,\end{aligned}\tag{109}$$

which vanishes because

$$\int_0^4 \partial_{y_2}^2\partial_{s_1}\xi(s_1,z)\,\mathrm{d}s_1 = \int_0^4 \partial_{s_1}(\partial_{y_2}^2\xi)(s_1,z)\,\mathrm{d}s_1 = 0$$

by periodicity of $\partial_{y_2}^2\xi$ with respect to the s_1-variable, and similarly

$$\begin{aligned}&\int_0^2\int_I \partial_{y_1}\xi(y_1-\tau,y_2)\left(\int_0^{y_2}(y_2-z)\partial_{y_2}^2\partial_{y_1}\eta(y_1+\tau,z)\mathrm{d}z\right)\mathrm{d}y\,\mathrm{d}\tau\\ &=\int_{-1}^1\int_0^{y_2}(y_2-z)\left[\frac{1}{2}\int_{-2}^2\partial_{s_1}\xi(s_1,y_2)\mathrm{d}s_1\right]\left[\frac{1}{2}\int_0^4\partial_{y_2}^2\partial_{s_2}\eta(s_2,z)\mathrm{d}s_2\right]\mathrm{d}z\,\mathrm{d}y_2 = 0,\end{aligned}$$

by exploiting the periodicity of $\partial_{y_2}^2\eta$ with respect to the s_2-variable. □

Corollary 4.3. *The equations of motion associated to $h_0(\xi,\eta)+\mu^2\langle F_1\rangle(\xi,\eta)$ are given by*

$$\begin{cases}\xi_\tau &= -\partial_{y_1}\xi-\frac{\mu^2}{24}\partial_{y_1}^3\xi-\frac{\mu^2\alpha}{2\sqrt{2}}\,\partial_{y_1}(\xi^2)\\ \eta_\tau &= \partial_{y_1}\eta+\frac{\mu^2}{24}\partial_{y_1}^3\eta+\frac{\mu^2\alpha}{2\sqrt{2}}\,\partial_{y_1}(\eta^2)\end{cases}.\tag{110}$$

The latter is a system of two uncoupled KdV equations in translating frames with respect to the direction y_1, for each fixed value of the coordinate y_2.

Remark 4.4. *If one considers a square lattice, namely*

$$H(Q,P)=\sum_{j\in\mathbb{Z}_N^2} -\frac{1}{2}P_j\,(\Delta_1 P)_j+(F(Q))_j,\tag{111}$$

with $F(Q)$ as in (7)*, with its continuous approximation*

$$H(Q,P)=\int_{\left[-\frac{1}{\mu},\frac{1}{\mu}\right]^2}\frac{-P\,\Delta_1 P+Q^2}{2}+\alpha\frac{Q^3}{3}+\beta\frac{Q^4}{4}\mathrm{d}x, \tag{112}$$

and makes the ansatz (79) *about the solution, one gets the rescaled Hamiltonian*

$$K_1(q,p)=\int_{I_{\mu,\sigma}}\frac{-p\,\Delta_{\mu,y_1,\sigma}p}{2\mu^2}+\frac{q^2}{2}+\alpha\mu^2\frac{q^3}{3}+\beta\mu^4\frac{q^4}{4}\mathrm{d}y, \tag{113}$$

$$\Delta_{\mu,y_1,\sigma}:=4\sinh^2\left(\frac{\mu\partial_{y_1}}{2}\right)+4\sinh^2\left(\mu^\sigma\frac{\partial_{y_2}}{2}\right), \tag{114}$$

$$I_{\mu,\sigma}:=[-1,1]\times[-\mu^{\sigma-1},\mu^{\sigma-1}], \tag{115}$$

which, combined with the fact that

$$\int_{-1}^{1}\xi(\tau,y_1,y_2)-\eta(\tau,y_1,y_2)\mathrm{d}y_1=0\ \ \forall\tau\in\mathbb{R},\ |y_2|\le\mu^{\sigma-1}, \tag{116}$$

leads to the system (110) *of two uncoupled KdV equations in translating frames with respect to the direction y_1.*

4.2. **The KP regime for the ETL lattice.** For this regime we consider (76), with $\alpha\neq 0$, and we look for small amplitude solutions of the form

$$Q(t,x)=\mu^2 q(\mu t,\mu x_1,\mu^2 x_2), \tag{117}$$

with μ as in (14). We introduce the rescaled variables $\tau=\mu t$, $y_1=\mu x_1$, $y_2=\mu^2 x_2$.

Plugging (117) into (76), leads to

$$q_{\tau\tau}=\frac{\Delta_{\mu,y_1}}{\mu^2}\left(q+\mu^2\alpha q^2\right), \tag{118}$$

$$\Delta_{\mu,y_1}:=4\sinh^2\left(\frac{\mu\partial_{y_1}}{2}\right)+4\sinh^2\left(\mu^2\frac{\partial_{y_2}}{2}\right), \tag{119}$$

which is a Hamiltonian PDE corresponding to the Hamiltonian functional,

$$K_3(q,p)=\int_I\frac{-p\,\Delta_{\mu,y_1}p}{2\mu^2}+\frac{q^2}{2}+\alpha\mu^2\frac{q^3}{3}+\beta\mu^4\frac{q^4}{4}\mathrm{d}y, \tag{120}$$

where I is as in (80), and p is the variable canonically conjugated to q.

Now, observe that the the operator Δ_{μ,y_1} admits the following asymptotic expansion up to terms of order $\mathcal{O}(\mu^4)$,

$$\frac{\Delta_{\mu,y_1}}{\mu^2}\sim\partial_{y_1}^2+\mu^2\partial_{y_2}^2+\frac{\mu^2}{12}\partial_{y_1}^4+\mathcal{O}(\mu^4), \tag{121}$$

Therefore the Hamiltonian (120) admits the following asymptotic expansion

$$K_3(q,p)\sim\hat{h}_0(q,p)+\mu^2\hat{F}_1(q,p)+\mu^4\hat{\mathcal{R}}(q,p), \tag{122}$$

$$\hat{h}_0(q,p)=\int_I\frac{-p\,(\partial_{y_1}^2 p)+q^2}{2}\mathrm{d}y, \tag{123}$$

$$\hat{F}_1(q,p)=\int_I-\frac{p\,\partial_{y_1}^4 p}{24}-\frac{p\,\partial_{y_2}^2 p}{2}+\alpha\frac{q^3}{3}\mathrm{d}y. \tag{124}$$

By exploiting again the non-canonical change of coordinates $(q,p)\mapsto(\xi,\eta)$ introduced in (90)-(91) and the Poisson tensor (92), and

$$\int_{-1}^{1}\xi(\tau,y_1,y_2)-\eta(\tau,y_1,y_2)\mathrm{d}y_1=0\ \ \forall\tau\in\mathbb{R},\ |y_2|\le 1, \tag{125}$$

we obtain

$$K_3(\xi,\eta) \sim h_0(\xi,\eta) + \mu^2 F_1(\xi,\eta) + \mu^4 \mathcal{R}(\xi,\eta), \tag{126}$$

$$h_0(\xi,\eta) = \int_I \frac{\xi^2+\eta^2}{2}\mathrm{d}y, \tag{127}$$

$$F_1(\xi,\eta) = \int_I -\frac{[\partial_{y_1}(\xi-\eta)]^2}{48} + \frac{[\partial_{y_2}\partial_{y_1}^{-1}(\xi-\eta)]^2}{4} + \alpha\frac{(\xi+\eta)^3}{3\cdot 2^{3/2}}\mathrm{d}y, \tag{128}$$

where (128) is well defined because of (125).

Now we apply the averaging Theorem 3.3 to the Hamiltonian (126), with $\delta = \mu^2$.

Proposition 4.5. *The average of F_1 in* (126) *with respect to the flow of h_0 in* (126) *is given by*

$$\langle F_1\rangle(\xi,\eta) = \int_I -\frac{(\partial_{y_1}\xi)^2+(\partial_{y_1}\eta)^2}{48} + \frac{(\partial_{y_2}\partial_{y_1}^{-1}\xi)^2+(\partial_{y_2}\partial_{y_1}^{-1}\eta)^2}{4}\mathrm{d}y + \frac{\alpha}{3\cdot 2^{3/2}}([\xi^3]+[\eta^3]) \tag{129}$$

where we denote by $[f^j]$ the average $\int_I f^j(y)\frac{\mathrm{d}y}{4}$.

Corollary 4.6. *The equations of motion associated to $h_0(\xi,\eta) + \mu^2\langle F_1\rangle(\xi,\eta)$ are given by*

$$\begin{cases} \xi_{\mathcal{T}} &= -\partial_{y_1}\xi - \frac{\mu^2}{24}\partial_{y_1}^3\xi - \frac{\mu^2}{2}\partial_{y_1}^{-1}\partial_{y_2}^2\xi - \frac{\alpha\mu^2}{2\sqrt{2}}\partial_{y_1}(\xi^2) \\ \eta_{\mathcal{T}} &= \partial_{y_1}\eta + \frac{\mu^2}{2}\partial_{y_1}^{-1}\partial_{y_2}^2\eta + \frac{\mu^2}{24}\partial_{y_1}^3\eta + \frac{\alpha\mu^2}{2\sqrt{2}}\partial_{y_1}(\eta^2) \end{cases}. \tag{130}$$

More explicitly, we observe that (130) is a system of two uncoupled KP equations on a two-dimensional torus in translating frames.

4.3. **The one-dimensional NLS regime for the KG Lattice.** We want to study the behaviour of small amplitude solutions of (27) , with initial data in which only one low-frequency Fourier mode is excited.

Analogously to the procedure of the previous sections, the first step is to introduce an interpolating function $Q = Q(t,x)$ such that

(B1) $Q(t,j) = Q_j(t)$, for all $j \in \mathbb{Z}^2_{N_1,N_2}$;
(B2) Q is periodic with period $2N_1+1$ in the x_1-variable, and periodic with period $2N_2+1$ in the x_2-variable;
(B3) Q fulfills

$$\ddot{Q} = \Delta_1 Q - m^2 Q - \beta Q^{2p+1}, \tag{131}$$

where Δ_1 is the operator defined in (77).

It is easy to verify that (131) is Hamiltonian with Hamiltonian function

$$H(Q,P) = \int_{[-\frac{1}{\mu},\frac{1}{\mu}]\times[-\frac{1}{\mu^\sigma},\frac{1}{\mu^\sigma}]} \frac{P^2}{2} + m^2\frac{Q^2}{2} - \frac{Q\,\Delta_1 Q}{2} + \beta\frac{Q^{2p+2}}{2p+2}\mathrm{d}x, \tag{132}$$

where P is a periodic function and is canonically conjugated to Q.

Starting from the Hamiltonian (24), where $p=1$, we look for small amplitude solutions of the form

$$Q(t,x) = \mu\, q(\mu^2 t, \mu x_1, \mu^\sigma x_2)\,. \tag{133}$$

where $q : \mathbb{R}\times\mathbb{T}^2 \to \mathbb{R}$ is a periodic function and σ, μ are defined respectively in (15)-(14).

We introduce the rescaled variable $y_1 = \mu x_1$ and $y_2 = \mu^\sigma x_2$, and we define I as in (80). The Hamiltonian (24) in the rescaled variable is given by

$$K_4(q,p) = \int_I \frac{p^2}{2} + \frac{q^2}{2} - \frac{q\,\Delta_{\mu,y_1,\sigma}q}{2} + \beta\mu^2\frac{q^4}{4}\mathrm{d}y, \tag{134}$$

with the operator $\Delta_{\mu,y_1,\sigma}$ as in (82), and p is the variable canonically conjugated to q. The corresponding equation of motion are given by

$$q_{tt} = -q + \Delta_{\mu,y_1,\sigma}q - \beta\mu^2 q^3. \tag{135}$$

Recall that

$$\frac{\Delta_{\mu,y_1,\sigma}}{\mu^2} \sim \partial_{y_1}^2 + \mu^{2(\sigma-1)}\partial_{y_2}^2 + \frac{\mu^2}{12}\partial_{y_1}^4 + \mathcal{O}(\mu^{2(2\sigma-1)}),$$

hence the Hamiltonian (134) admits the following asymptotic expansion

$$K_4(q,p) \sim \hat{h}_0(q,p) + \mu^2\hat{F}_1(q,p) + \mu^{2(2\sigma-1)}\hat{\mathcal{R}}(q,p), \tag{136}$$

$$\hat{h}_0(q,p) = \int_I \frac{p^2+q^2}{2}\mathrm{d}y, \tag{137}$$

$$\hat{F}_1(q,p) = \int_I -\frac{q\,\partial_{y_1}^2 q}{2} + \beta\frac{q^4}{4}\mathrm{d}y, \tag{138}$$

and the equation of motion associated to h_0+F_1 is given by the following cubic one-dimensional nonlinear Klein-Gordon (NLKG) equation,

$$q_{tt} = -(q - \mu^2\partial_{y_1}^2 q) - \mu^2\beta q^3. \tag{139}$$

We now exploit the change of coordinates $(q,p)\mapsto(\psi,\bar{\psi})$ given by

$$\psi = \frac{1}{\sqrt{2}}(q - ip), \tag{140}$$

therefore the inverse change of coordinates is given by

$$q = \frac{1}{\sqrt{2}}(\psi+\bar{\psi}), \tag{141}$$

$$p = \frac{1}{\sqrt{2}}i(\psi-\bar{\psi}), \tag{142}$$

while the Poisson tensor is given by $-i\mathrm{d}\psi\wedge\mathrm{d}\bar{\psi}$. With this change of variables the Hamiltonian takes the form

$$K_4(\psi,\bar{\psi}) \sim h_0(\psi,\bar{\psi}) + \mu^2 F_1(\psi,\bar{\psi}) + \mu^{2(2\sigma-1)}\mathcal{R}(\psi,\bar{\psi}), \tag{143}$$

$$h_0(\psi,\bar{\psi}) = \int_I \psi\,\bar{\psi}\mathrm{d}y, \tag{144}$$

$$F_1(\psi,\bar{\psi}) = \int_I -\frac{(\psi+\bar{\psi})\,[-\partial_{y_1}^2(\psi+\bar{\psi})]}{4} + \beta\frac{(\psi+\bar{\psi})^4}{16}\mathrm{d}y. \tag{145}$$

Now we apply the averaging Theorem 3.3 to the Hamiltonian (143), with $\delta=\mu^2$. Observe that h_0 generates a periodic flow,

$$-i\partial_t\psi = \psi;$$

$$\psi(t,y) = e^{it}\psi_0(y). \tag{146}$$

Proposition 4.7. *Then the average of F_1 in* (143) *with respect to the flow of h_0* (137) *is given by*

$$\langle F_1\rangle(\psi,\bar{\psi}) = \int_I \frac{\bar{\psi}\,(-\partial_{y_1}^2\psi)}{2}\mathrm{d}y + \frac{3\beta}{8}\int_I |\psi|^4\mathrm{d}y. \tag{147}$$

Corollary 4.8. *The equations of motion associated to $h_0(\psi,\bar{\psi}) + \mu^2\langle F_1\rangle(\psi,\bar{\psi})$ are given by a cubic one-dimensional nonlinear Schrödinger equation for each fixed value of y_2,*

$$-i\psi_t = \psi - \mu^2\,\partial_{y_1}^2\psi + \mu^2\frac{3\beta}{4}\,|\psi|^2\psi. \tag{148}$$

5. DYNAMICS OF THE NORMAL FORM EQUATION

5.1. **The KdV equation.** In this section we recall some known facts on the dynamics of the KdV equation with periodic boundary conditions. The interested reader can find more detailed explanations and proofs in [KP03].

Consider the KdV equation

$$\xi_\tau = -\frac{1}{24}\partial_{y_1}^3\xi - \frac{\alpha}{2\sqrt{2}}\partial_{y_1}(\xi^2), \ \ y_1 \in [0,2]. \tag{149}$$

Through the Lax pair formulation of the evolution problem (149) one get that the periodic eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ of the Sturm-Liouville operator

$$L_\xi := -\partial_{y_1}^2 + 6\sqrt{2}\xi(\tau, y_1) \tag{150}$$

are conserved quantities under the evolution of the KdV equation (149). Moreover, if we define the gaps of the spectrum $\gamma_m := \lambda_{2m} - \lambda_{2m-1}$ $(m \geq 1)$, it is well known that the squared spectral gaps $(\gamma_m^2)_{m\geq 1}$ form a complete set of constants of motion for (149).

The following relation between the sequence of the spectral gaps and the regularity of the corresponding solution to the KdV equation holds (see Theorem 9, Theorem 10 and Theorem 11 in [KP08]; see also [Pös11])

Theorem 5.1. *Assume that $\xi \in L^2$, then $\xi \in \ell^2_{0,s}$ if and only if its spectral gaps satisfy*

$$\sum_{m\geq 1} m^{2s}|\gamma_m|^2 < +\infty.$$

Moreover if $\xi \in \ell^2_{\rho,s}$, then

$$\sum_{m\geq 1} m^{2s}e^{2\rho m}|\gamma_m|^2 < +\infty; \tag{151}$$

conversely, if (151) *holds, then $\xi \in \ell^2_{\rho',0}$ for some $\rho' > 0$.*

Finally, Kappeler and Pöschel constructed the following global Birkhoff coordinates (see Theorem 1.1 in [KP03])

Theorem 5.2. *There exists a diffeomorphism $\Omega : L^2 \to \ell^2_{0,1/2} \times \ell^2_{0,1/2}$ such that:*

- *Ω is bijective, bianalytic and canonical;*
- *for each $s \geq 0$, the restriction of Ω to $\ell^2_{0,s}$, namely the map*
  $$\Omega : \ell^2_{0,s} \to \ell^2_{0,s+1/2} \times \ell^2_{0,s+1/2}$$
  is bijective, bianalytic and canonical;
- *the coordinates $(x,y) \in \ell^2_{0,3/2} \times \ell^2_{0,3/2}$ are Birkhoff coordinates for the KdV equation, namely they form a set of canonically conjugated coordinates in which the Hamiltonian of the KdV equation* (149) *depends only on the action $I_m := \frac{x_m^2 + y_m^2}{2}$ $(m \geq 1)$.*

The dynamics of the KdV equation (149) in terms of the variables (x,y) is trivial: it can be immediately seen that any solution is periodic, quasiperiodic or almost periodic, depending on the number of spectral gaps (equivalently, depending on the number of actions) initially different from zero.

5.2. **The KP equation.**

In this section we recall some known facts on the dynamics of the KP equation on the two-dimensional torus

$$\xi_\tau = -\frac{1}{24}\partial_{y_1}^3\xi - \frac{1}{2}\partial_{y_1}^{-1}\partial_{y_2}^2\xi - \frac{\alpha}{2\sqrt{2}}\,\partial_{y_1}(\xi^2), \ \alpha = \pm 1, \ y \in \mathbb{T}^2 := \mathbb{R}^2/(2\pi\mathbb{Z})^2. \tag{152}$$

The KP equation has been introduced in order to describe weakly-transverse solutions of the water waves equations; it has been considered as a two-dimensional analogue of the KdV equation, since also the KP equation admits an infinite number of constants of motions [LC82] [CLL83] [CL87]. It is customary to call KP-I equation (152) with $\alpha = -1$, and KP-II equation (152) with $\alpha = 1$.

The global-well posedness for the KP-II equation on the standard two-dimensional torus has been discussed by Bourgain in [Bou93b]. The main point of the result by Bourgain consists in extending the local well-posedness result to a global one, even though the L^2-norm is the only constant of motion for the KP-II equation that allows an a-priori bound for the solution (see Theorem 8.10 and Theorem 8.12 in [Bou93b]).

Theorem 5.3. *Consider* (152) *with* $\alpha = 1$.

Let $\rho \geq 0$ *and* $s \geq 0$, *and assume that the initial datum* $\xi(0,\cdot,\cdot) = \xi_0 \in \ell^2_{\rho,s}$. *Then* (152) *is globally well-posed in* $\ell^2_{\rho,s}$. *Moreover, the* ℓ^2 *norm of the solution is conserved,*

$$\|\xi(t)\|_{\ell^2} = \|\xi_0\|_{\ell^2}, \tag{153}$$

while

$$\|\xi(t)\|_{\ell^2_{0,s}} \leq e^{C|t|}\|\xi_0\|_{\ell^2_{0,s}}, \tag{154}$$

where C *depends on* s.

Remark 5.4. *As pointed out by Bourgain in Sec. 10.2 of* [Bou93b], *a global well-posedness result for sufficiently smooth solution of the KP-I equation (namely,* (152) *with* $\alpha = -1$*) on the two-dimensional torus can be obtained by generalizing the argument in* [SJ87] *for small data and by using the a-priori bounds given by the constants of motion for the KP-I equation.*

For the KP equation the construction of action-angle/Birkhoff coordinates is still an open problem.

5.3. **The one-dimensional cubic NLS equation.**

In this section we recall some known facts on the dynamics of the one-dimensional cubic defocusing NLS equation with periodic boundary conditions. The interested reader can find more detailed explanations and proofs in [GKK14] [Mol14].

Consider the cubic defocusing NLS equation

$$i\psi_\tau = -\partial^2_{y_1}\psi + 2|\psi|^2\psi, \ \ y_1 \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}). \tag{155}$$

Eq. (155) is a PDE admitting a Hamiltonian structure: indeed, we can set $\mathcal{H}^{\rho,s} = \ell^2_{\rho,s} \times \ell^2_{\rho,s}$ as the phase space with elements denoted by $\phi = (\phi_1, \phi_2)$, while the associated Poisson bracket and the Hamiltonian are given by

$$\{F, G\} := -i\int_{\mathbb{T}} (\partial_{\phi_1}F\,\partial_{\phi_2}G - \partial_{\phi_1}G\,\partial_{\phi_2}F)\,\mathrm{d}y_1, \tag{156}$$

$$H_{NLS}(\phi_1,\phi_2) := \int_{\mathbb{T}} \partial_{y_1}\phi_1\,\partial_{y_1}\phi_2 + \phi_1^2\phi_2^2\,\mathrm{d}y_1. \tag{157}$$

The defocusing NLS equation (155) is obtained by restricting (157) to the invariant subspace of states of real type,

$$\mathcal{H}^{\rho,s}_r := \{\phi \in \mathcal{H}^{\rho,s} : \phi_2 = \bar{\phi}_1\}. \tag{158}$$

The above Hamiltonian (157) is well-defined on $\mathcal{H}^{\rho,s}$ with $s \geq 1$ and $\rho \geq 0$, while the initial value problem for the NLS equation (155) is well-posed on $\mathcal{H}^{0,0} = \ell^2 \times \ell^2$.

It is well known from the work by Zakharov and Shabat that the NLS equation (155) has a Lax pair, and that it admits infinitely many constants of motion in involution. More precisely, for any $\phi \in \mathcal{H}^{0,0}$ consider the Zakharov-Shabat operator

$$L(\phi) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\partial_{y_1} + \begin{pmatrix} 0 & \phi_1 \\ \phi_2 & 0 \end{pmatrix}, \tag{159}$$

where we call ϕ the potential of the operator $L(\phi)$. The spectrum of $L(\phi)$ on the interval $[0,2]$ with periodic boundary conditions is pure point, and it consists of the following sequence of periodic eigenvalues

$$\cdots < \lambda^-_{-1} \leq \lambda^+_{-1} < \lambda^-_0 \leq \lambda^+_0 < \lambda^-_1 \leq \lambda^+_1 < \cdots, \tag{160}$$

where the quantities $\gamma_m := \lambda^+_m - \lambda^-_m$ ($m \in \mathbb{Z}$) are called gap lengths. It has been proved that the squared spectral lengths $(\gamma^2_m)_{m\in\mathbb{Z}}$ form a complete set of analytic constants of motion for (155).

Grêbert, Kappeler and Mityagin proved the following relation between the sequence of the squared spectral gaps and the regularity of the corresponding potential (see Theorem in [GKM98]).

Theorem 5.5. *Let* $\rho \geq 0$ *and* $s > 0$, *then for any bounded subset* $\mathcal{B} \subset \ell^2_{\rho,s} \times \ell^2_{\rho,s}$ *there exists* $n_0 \geq 1$ *and* $M \geq 1$ *such that for any* $|k| \geq n_0$ *and any* $(\phi_1,\phi_2) \in \mathcal{B}$, *the following estimate holds*

$$\sum_{|k|\geq n_0} (1+|k|)^{2s} e^{2\rho|k|}|\gamma_m|^2 \leq M. \tag{161}$$

Moreover, Grébert and Kappeler constructed the following global Birkhoff coordinates (see Theorem 20.1 - Theorem 20.3 in [GKK14])

Theorem 5.6. *There exists a diffeomorphism* $\Omega : L^2_r \to \mathcal{H}^{0,0}_r$ *such that:*

- Ω *is bianalytic and canonical;*
- *for each* $s \geq 0$, *the restriction of* Ω *to* $\mathcal{H}^{0,s}_r$, *namely the map*
  $$\Omega : \mathcal{H}^{0,s}_r \to \mathcal{H}^{0,s}_r$$
  is again bianalytic and canonical;
- *the coordinates* $(x, y) \in \mathcal{H}^{0,1}_r$ *are Birkhoff coordinates for the NLS equation, namely they form a set of canonically conjugated coordinates in which the Hamiltonian of the NLS equation* (155) *depends only on the action* $I_m := \frac{x_m^2 + y_m^2}{2}$ $(m \in \mathbb{Z})$.

The dynamics of the NLS equation (155) in terms of the variables (x, y) is trivial: it can be immediately seen that any solution is periodic, quasiperiodic or almost periodic, depending on the number of spectral gaps (equivalently, depending on the number of actions) initially different from zero.

6. Approximation results

In this section we show how to use the normal form equations in order to construct approximate solutions of (8) and (27), and we estimate the difference with respect to the true solutions with corresponding initial data.

The approach is the same for all the regimes (79), (117) and (133). First we have to point out a relation between the energy of normal mode E_k (defined in (10) for (8), and in (10) for (27)) , $k \in \mathbb{Z}^2_{2N+1}$, and the Fourier coefficients of the solutions of the normal form equations. Then we have to prove that the approximate solutions approximate the energy of the true normal mode E_k up to the time-scale in which the continuous approximation is valid, and finally we can deduce the result about the dynamics of the lattice.

6.1. The KdV regime.

Let $I = [-1, 1]^2$ be as in (80), we define the Fourier coefficients of the function $q : I \to \mathbb{R}$ by

$$\hat{q}(j) := \frac{1}{2} \int_I q(y_1, y_2) e^{-i\pi(j_1 y_1 + j_2 y_2)} \mathrm{d}y_1 \, \mathrm{d}y_2, \tag{162}$$

and similarly for the Fourier coefficients of the function p.

Lemma 6.1. *Consider the lattice* (6) *in the regime (KdV) and with interpolanting function* (79). *Then for a state corresponding to* (q, p) *one has*

$$\mathcal{E}_\kappa = \frac{\mu^4}{2} \sum_{L=(L_1,L_2)\in\mathbb{Z}^2 : \mu L_1, \mu^\sigma L_2 \in 2\mathbb{Z}} |\hat{q}_{K+L}|^2 + \omega_k^2 \left| \frac{\hat{p}_{K+L}}{\mu} \right|^2, \quad \forall k : \kappa(k) = (\mu K_1, \mu^\sigma K_2) \tag{163}$$

(where the ω_k *are defined as in* (11)*), and* $\mathcal{E}_\kappa = 0$ *otherwise.*

Proof. First we introduce a $(2N_1 + 1)(2N_2 + 1)$-periodic interpolating function for Q_j, namely a smooth function $Q : (t, x) \mapsto Q(t, x)$ such that

$$Q_j(t) = Q(t, j), \quad \forall t, j,$$
$$Q(t, x_1 + 2N_1 + 1, x_2 + 2N_2 + 1) = Q(t, x), \quad \forall t, x,$$

and similarly for P_j. We denote by

(164)

$$\hat{Q}(j) := \frac{1}{(2N_1 + 1)^{1/2}(2N_2 + 1)^{1/2}} \int_{[-(N_1+\frac{1}{2}),(N_1+\frac{1}{2})]\times[-(N_2+\frac{1}{2}),(N_2+\frac{1}{2})]} Q(x) e^{-i\frac{j\cdot x \, 2\pi}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}}} \mathrm{d}x,$$

so that by the interpolation property we obtain

$$\begin{aligned}Q_j(t) = Q(t,j) \; &= \; \frac{1}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}} \sum_{k\in\mathbb{Z}^2} \hat{Q}(j) e^{i\frac{j\cdot k\, 2\pi}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}}} \\ &= \frac{1}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}} \\ &\times \sum_{k=(k_1,k_2)\in\mathbb{Z}^2_{2N+1}} \left[\sum_{h=(h_1,h_2)\in\mathbb{Z}^2} \hat{Q}(k_1+(2N_1+1)h_1, k_2+(2N_2+1)h_2) \right] e^{i\frac{j\cdot k\, 2\pi}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}}},\end{aligned}$$

hence

$$\hat{Q}_k = \sum_{h\in\mathbb{Z}^2} \hat{Q}(k_1+(2N_1+1)h_1, k_2+(2N_2+1)h_2). \tag{165}$$

The relation between $\hat{Q}(k)$ and $\hat{q}_k$ can be deduced from (79),

$$\begin{aligned} Q(j) &= \mu^2 q(\mu j_1, \mu^\sigma j_2); \\ \hat{Q}_k &= \frac{1}{2}\mu^{(\sigma+1)/2} \int_{\left[-\frac{1}{\mu},\frac{1}{\mu}\right]\times\left[-\frac{1}{\mu^\sigma},\frac{1}{\mu^\sigma}\right]} Q(x_1,x_2) e^{-i\pi(k_1x_1\mu+k_2x_2\mu^\sigma)} \mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \frac{1}{2}\mu^{(\sigma+1)/2} \int_{\left[-\frac{1}{\mu},\frac{1}{\mu}\right]\times\left[-\frac{1}{\mu^\sigma},\frac{1}{\mu^\sigma}\right]} \mu^2\, q\left(\mu x_1, \mu^\sigma x_2\right) e^{-i\pi(k_1x_1\mu+k_2x_2\mu^\sigma)} \mathrm{d}x_1\,\mathrm{d}x_2 \\ &\overset{(79)}{=} \frac{1}{2}\mu^{(3-\sigma)/2} \int_I q(y) e^{-i\pi(k_1y_1+k_2y_2)} \mathrm{d}y \\ &= \mu^{(3-\sigma)/2}\hat{q}_k, \end{aligned} \tag{166}$$

and similarly

$$\hat{P}_k = \mu^{(1-\sigma)/2}\hat{p}_k. \tag{167}$$

By using (10), (13) and (165)-(167) we have

$$\begin{aligned}\mathcal{E}_\kappa &\overset{(13)}{=} \mu^{\sigma+1}\frac{1}{2} \sum_{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}} |\hat{Q}_{K+L}|^2 + \omega_k^2|\hat{P}_{K+L}|^2 \\ &\overset{(166),(166)}{=} \mu^{\sigma+1}\mu^{3-\sigma}\frac{1}{2} \sum_{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}} |\hat{q}_{K+L}|^2 + \omega_k^2\left|\frac{\hat{p}_{K+L}}{\mu}\right|^2\end{aligned}$$

for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$, and this leads to (163). ☐

Proposition 6.2. *Fix $\rho > 0$ and $0 < \delta \ll 1$. Consider the normal form system* (110)*, and define the Fourier coefficients of (ξ,η) through the following formula*

$$\xi(y) = \frac{1}{2}\sum_{h\in\mathbb{Z}^2} \hat{\xi}_h e^{ih\cdot y\pi}, \tag{168}$$

$$\eta(y) = \frac{1}{2}\sum_{h\in\mathbb{Z}^2} \hat{\eta}_h e^{ih\cdot y\pi},. \tag{169}$$

Consider $(\xi,\eta)\in\mathcal{H}^{\rho,0}$, and denote by $\mathcal{E}_\kappa$ the specific energy of the normal mode with index κ as defined in (12)-(13)*. Then for any positive μ sufficiently small*

$$\left|\mathcal{E}_\kappa - \mu^4\frac{|\hat{\xi}_K|^2+|\hat{\eta}_K|^2}{2}\right| \le C\mu^{4+\frac{6}{5}}\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \tag{170}$$

for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ and $|K_1|+|K_2| \le \frac{(2+\delta)|\log\mu|}{\rho}$. Moreover,

$$|\mathcal{E}_\kappa| \le C\,\mu^8\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \tag{171}$$

for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ and $|K_1|+|K_2| > \frac{(2+\delta)|\log\mu|}{\rho}$, and $\mathcal{E}_\kappa = 0$ otherwise.

We defer the proof of the above Proposition to Appendix B.

Now, consider the following system of uncoupled KdV equations

$$\xi_\tau = -\frac{1}{24}\partial^3_{y_1}\xi - \frac{\alpha}{2\sqrt{2}}\partial_{y_1}(\xi^2), \tag{172}$$

$$\eta_\tau = \frac{1}{24}\partial^3_{y_1}\eta + \frac{\alpha}{2\sqrt{2}}\partial_{y_1}(\eta^2), \tag{173}$$

and consider a solution $(\tau, y) \mapsto (\tilde{\xi}_a(\tau, y), \tilde{\eta}_a(\tau, y))$ such that it belongs to $\mathcal{H}^{\rho,m}$, for some $m > 0$.

We consider the approximate solutions (Q_a, P_a) of the FPU model (76)

$$Q_a(\tau, y) := \frac{\mu^2}{\sqrt{2}}\left[\tilde{\xi}_a(\mu^2\tau, y_1 - \tau, y_2) + \tilde{\eta}_a(\mu^2\tau, y_1 + \tau, y_2)\right] \tag{174}$$

$$\partial_{y_1}P_a(\tau, y) := \frac{\mu}{\sqrt{2}}\left[\tilde{\xi}_a(\mu^2\tau, y_1 - \tau, y_2) - \tilde{\eta}_a(\mu^2\tau, y_1 + \tau, y_2)\right], \tag{175}$$

We need to compare the difference between the approximate solution (174)-(175) and the true solution of (8). Let consider an initial datum (Q_0, P_0) with corresponding Fourier coefficients $(\hat{Q}_{0,k}, \hat{P}_{0,k})$ given by (9), where

$$Q_{0,k} \neq 0 \text{ only if } \kappa(k) = (\mu K_1, \mu^\sigma K_2). \tag{176}$$

We also assume that there exist C, $\rho > 0$ such that

$$\frac{|\hat{Q}_{0,k}|^2 + \omega_k^2|\hat{P}_{0,k}|^2}{N} \leq Ce^{-2\rho|(\kappa_1(k)/\mu, \kappa_2(k)/\mu^\sigma)|}. \tag{177}$$

Moreover, we define an interpolating function for the initial datum (Q_0, P_0) by

$$Q_0(y) = \frac{1}{(2N_1+1)(2N_2+1)} \sum_{K:\left(\mu^2|K_1|^2+\mu^{2\sigma}|K_2|^2\right)^{1/2}=|\kappa(k)|\leq 1} \hat{Q}_{0,k}e^{i\pi(\mu K_1 y_1 + \mu^\sigma K_2 y_2)},$$

and similarly for $y \mapsto P_0(y)$.

Proposition 6.3. *Consider* (8) *with* $\sigma > 2$ *and* $\gamma \geq 1$ *such that* $\sigma + 2\gamma < 7$. *Let us assume that the initial datum satisfies* (176)-(177), *and denote by* $(Q(t), P(t))$ *the corresponding solution. Consider the approximate solution* $(\tilde{\xi}_a(t,x), \tilde{\eta}_a(t,x))$ *with the corresponding initial datum. Assume that* $(\tilde{\xi}_a, \tilde{\eta}_a) \in \mathcal{H}^{\rho,m}$ *for some* $\rho > 0$ *and for some* $m \geq 0$ *for all times, and fix* $T > 0$ *and* $0 < \delta \ll 1$.

Then there exists $\mu_0 = \mu_0(T, \|(\tilde{\xi}_a(0), \tilde{\eta}_a(0))\|_{\mathcal{H}^{\rho,m}})$ *such that, if* $\mu < \mu_0$, *we have that there exists* $C > 0$ *such that*

$$\sup_j |Q_j(t) - Q_a(t,j)| + |P_j(t) - P_a(t,j)| \leq C\mu^\gamma, \quad |t| \leq \frac{T}{\mu^3}, \tag{178}$$

where (Q_a, P_a) *are given by* (174)-(175). *Moreover,*

$$\left|\mathcal{E}_\kappa - \mu^4\frac{|\hat{\xi}_K|^2 + |\hat{\eta}_K|^2}{2}\right| \leq C\,\mu^{4+\gamma} \tag{179}$$

for all k *such that* $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ *and* $|K_1| + |K_2| \leq \frac{(2+\delta)|\log\mu|}{\rho}$. *Moreover,*

$$|\mathcal{E}_\kappa| \leq \mu^{4+\gamma} \tag{180}$$

for all k *such that* $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ *and* $|K_1| + |K_2| > \frac{(2+\delta)|\log\mu|}{\rho}$, *and* $\mathcal{E}_\kappa = 0$ *otherwise.*

Proof. The argument follows along the lines of Appendix C in [BP06].

Exploiting the canonical transformation found in Theorem 3.3, we also define

$$\zeta_a := (\xi_a, \eta_a) = \mathcal{T}^{(1)}_{\mu^2}(\tilde{\xi}_a, \tilde{\eta}_a) = \tilde{\zeta}_a + \psi_a(\tilde{\zeta}_a), \tag{181}$$

where $\psi_a(\tilde{\zeta}_a) := (\psi_\xi(\tilde{\zeta}_a), \psi_\eta(\tilde{\zeta}_a))$; by (59) we have

$$\sup_{\zeta\in\mathcal{B}_{\rho,m}(R)} \|\psi_a(\zeta)\|_{\mathcal{H}^{\rho,m}} \leq C'_m\mu^2 R. \tag{182}$$

For convenience we define

$$q_a(\tau,y) := \frac{1}{\sqrt{2}}\left[\xi_a(\mu^2\tau, y_1-\tau, y_2) + \eta_a(\mu^2\tau, y_1+\tau, y_2)\right] \tag{183}$$

$$\partial_{y_1} p_a(\tau,y) := \frac{1}{\sqrt{2}}\left[\xi_a(\mu^2\tau, y_1-\tau, y_2) - \eta_a(\mu^2\tau, y_1+\tau, y_2)\right], \tag{184}$$

We observe that the pair (q_a, p_a) satisfies

$$\mu^2 (q_a)_t = -\Delta_1\, \mu p_a + \mu^6 \mathcal{R}_q \tag{185}$$

$$\mu (p_a)_t = -\mu^2 q_a - \mu^4\, \alpha\, \pi_0 q_a^2 + \mu^5 \mathcal{R}_p, \tag{186}$$

where the operator Δ_1 acts on the variable x, π_0 is the projector on the space of the functions with zero average, and the remainders are functions of the rescaled variables τ and y which satisfy

$$\sup_{\mathcal{B}_{\rho,m}(R)} \|\mathcal{R}_q\|_{\ell^2_{\rho,0}} \le C,$$

$$\sup_{\mathcal{B}_{\rho,m}(R)} \|\mathcal{R}_p\|_{\ell^2_{\rho,1}} \le C.$$

We now restrict the space variables to integer values; keeping in mind that q_a and p_a are periodic, we assume that $j \in \mathbb{Z}^2_{N,N^\sigma}$.

For a finite sequence $Q = (Q_j)_{j\in\mathbb{Z}^2_{N,N^\sigma}}$ we define the norm

$$\|Q\|^2_{\ell^2_{N,N^\sigma}} := \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |Q_j|^2. \tag{187}$$

Now we consider the discrete model (8): we rewrite in the following form,

$$\dot{Q}_j = -(\Delta_1 P)_j \tag{188}$$

$$\dot{P}_j = -Q_j - \alpha\, \pi_0 Q_j^2 \tag{189}$$

and we want to show that there exist two sequences $E = (E_j)_{j\in\mathbb{Z}^2_{N,N^\sigma}}$ and $F = (F_j)_{j\in\mathbb{Z}^2_{N,N^\sigma}}$ such that

$$Q = \mu^2\, q_a + \mu^{2+\gamma} E, \;\; P = \mu p_a + \mu^{2+\gamma} F$$

fulfills (188)-(189), where $\gamma > 0$ is a parameter we will fix later in the proof. Therefore, we have that

$$\dot{E} = -\Delta_1\, F - \mu^{6-2-\gamma}\mathcal{R}_q \tag{190}$$

$$\dot{F} = -E - \alpha\pi_0\left(\mu^2\, 2 q_a E + \mu^{2+\gamma} E^2\right) - \mu^{5-2-\gamma}\mathcal{R}_p, \tag{191}$$

where we impose initial conditions on (E,F) such that $(\tilde{q},\tilde{p})$ has initial conditions corresponding to the ones of the true initial datum,

$$\mu^2 q_a(0, \mu j_1, \mu^\sigma j_2) + \mu^{2+\gamma} E_{0,j} = Q_{0,j},$$

$$\mu p_a(0, \mu j_1, \mu^\sigma j_2) + \mu^{2+\gamma} F_{0,j} = P_{0,j}.$$

We now define the operator ∂_i, $i = 1, 2$, by $(\partial_i f)_j := f_j - f_{j-e_i}$ for each $f \in \ell^2_{N,N^\sigma}$.

- Claim 1: Let $\sigma > 2$ and $\gamma > 0$, we have

$$\|E_0\|_{\ell^2_{N,N^\sigma}} \le C' \mu^{(3-2\gamma-\sigma)/2},$$

$$\|\partial_1 F_0\|_{\ell^2_{N,N^\sigma}} \le C' \mu^{(3-2\gamma-\sigma)/2},$$

$$\|\partial_2 F_0\|_{\ell^2_{N,N^\sigma}} \le C' \mu^{(1-2\gamma+\sigma)/2}.$$

To prove Claim 1 we observe that

$$E_0 = \mu^2 \frac{\xi_a + \eta_a - (\widetilde{\xi}_a + \widetilde{\eta}_a)}{\sqrt{2}\mu^{2+\gamma}} \;=\; \mu^{-\gamma}\, \frac{\psi_\xi + \psi_\eta}{\sqrt{2}},$$

$$F_0 = \mu \frac{\partial_{y_1}^{-1}[\xi_a - \eta_a - (\widetilde{\xi}_a - \widetilde{\eta}_a)]}{\sqrt{2}\mu^{2+\gamma}} \;=\; \mu^{-1-\gamma}\, \frac{\partial_{y_1}^{-1}(\psi_\xi - \psi_\eta)}{\sqrt{2}},$$

from which we can deduce

$$\|E_0\|^2_{\ell^2_{N,N^\sigma}} \le \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |E_{0,j}|^2 \;\le\; C\,4N^{\sigma+1}\,(\mu^{2-\gamma})^2 \;=\; C\,\mu^{3-2\gamma-\sigma},$$

$$\|\partial_1 F_0\|^2_{\ell^2_{N,N^\sigma}} \le \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |\partial_1 F_{0,j}|^2 \;\le\; C\,4N^{\sigma+1}\,(\mu^{2-\gamma})^2 \;\le\; C\,\mu^{3-2\gamma-\sigma}$$

$$\|\partial_2 F_0\|^2_{\ell^2_{N,N^\sigma}} \le \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |\partial_2 F_{0,j}|^2 \;\le\; C\,4N^{\sigma+1}\,(\mu^{1+\sigma-\gamma})^2 \;=\; C\,\mu^{1-2\gamma+\sigma}$$

and this leads to the thesis.

- Claim 2: Fix $m \ge 0$, $T > 0$ and $K_* > 0$, then for any $\mu < \mu_s$ and for any $\sigma > 2$ and $\gamma \ge 1$ such that $\sigma + 2\gamma < 7$ we have

$$\|E\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_1 F\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_2 F\|^2_{\ell^2_{N,N^\sigma}} \le K_*,\;\; |t| < \frac{T}{\mu^3}. \tag{192}$$

To prove the claim, we define

$$\mathcal{F}(E,F) := \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} \frac{E_j^2 + F_j(-\Delta_1 F)_j}{2} + \frac{2\mu^2\alpha q_a E_j^2}{2}, \tag{193}$$

and we remark that

$$\frac{1}{2}\mathcal{F}(E,F) \;\le\; \|E\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_1 F_0\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_2 F_0\|^2_{\ell^2_{N,N^\sigma}} \;\le\; 2\mathcal{F}(E,F).$$

Now we compute the time derivative of $\mathcal{F}$. Exploiting (190)-(191)

$$\dot{\mathcal{F}} = \sum_j E_j \left[-(\Delta_1 F)_j - \mu^{4-\gamma}(\mathcal{R}_q)_j\right] \tag{194}$$

$$+ \sum_j (-\Delta_1 F)_j \left[-E_j - \alpha(\mu^2 2\tilde{q}_j E_j + \mu^{2+\gamma} E_j^2) - \mu^{3-\gamma}(\mathcal{R}_p)_j\right] \tag{195}$$

$$+ \sum_j 2\mu^2\,\alpha\,\tilde{q}_j E_j \left[-(\Delta_1 F)_j - \mu^{4-\gamma}(\mathcal{R}_q)_j\right] \tag{196}$$

$$+ \sum_j \mu^2 \alpha E_j^2\, \mu\, \frac{\partial \tilde{q}_j}{\partial \tau} \tag{197}$$

$$= \sum_j -E_j\, \mu^{4-\gamma}(\mathcal{R}_q)_j + \sum_j (-\Delta_1 F)_j \left[-\alpha\,\mu^{2+\gamma} E_j^2 - \mu^{3-\gamma}(\mathcal{R}_p)_j\right] \tag{198}$$

$$- \sum_j 2\mu^2\,\alpha\,\tilde{q}_j E_j\, \mu^{4-\gamma}(\mathcal{R}_q)_j + \sum_j \mu^2\alpha E_j^2\,\mu\,\frac{\partial \tilde{q}_j}{\partial \tau} \tag{199}$$

In order to estimate (198)-(199), we notice that

$$\sup_j |(\Delta_1 F)_j| \le 2\sup_j |(\partial_1 F)_j| + |(\partial_2 F)_j| \le 4\sqrt{\mathcal{F}},$$

$$\|\mathcal{R}_q\|^2_{\ell^2_{N,N^\sigma}} \;\le\; \sum_j |(\mathcal{R}_q)_j|^2 \le 4N^{\sigma+1} \sup_y |\mathcal{R}_q(y)|^2 \;\le\; C\mu^{-1-\sigma},$$

and that $|(\partial_i \mathcal{R}_p)_j| \leq \mu \sup_y \left|\frac{\partial \mathcal{R}_p}{\partial y}(y)\right|$, which implies

$$\|\partial_i \mathcal{R}_p\|^2_{\ell^2_{N,N^\sigma}} \leq C\mu^{1-\sigma}.$$

Now, the first sum in (198) is estimated by $C\mathcal{F}^{1/2}\mu^{(7-2\gamma-\sigma)/2}$; the second sum can be bounded by

$$C(\mu^{2+\gamma}\mathcal{F}^{3/2} + \mu^{(7-2\gamma-\sigma)/2}\mathcal{F}^{1/2}).$$

Recalling that $\tilde{q}_j$ is bounded, the first sum in (199) can be bounded by $C\mathcal{F}^{1/2}\mu^{(11-2\gamma-\sigma)/2}$, while the second one is estimated by $C\mu^3\mathcal{F}$. Hence, as long as $\mathcal{F} < 2K_*$ we have

$$\left|\dot{\mathcal{F}}\right| \leq C\left|\mathcal{F}^{1/2}\mu^{(7-2\gamma-\sigma)/2} + \mu^{2+\gamma}\mathcal{F}^{3/2} + \mu^{(7-2\gamma-\sigma)/2}\mathcal{F}^{1/2} + \mathcal{F}^{1/2}\mu^{(11-2\gamma-\sigma)/2} + \mu^3\mathcal{F}\right|$$

$$\leq C(\mu^{2+\gamma}\sqrt{2}\,K_*^{1/2} + \mu^3)\mathcal{F} + C(2\mu^{(7-2\gamma-\sigma)/2} + \mu^{(11-2\gamma-\sigma)/2})\sqrt{2}\,K_*^{1/2}, \tag{200}$$

$$\overset{\gamma\geq 1}{\leq} C\,\mu^3\,2\sqrt{2}\,K_*^{1/2}\mathcal{F} + C\,3\mu^{(7-2\gamma-\sigma)/2}\,\sqrt{2}\,K_*^{1/2}, \tag{201}$$

and by applying Gronwall's lemma we get

$$\mathcal{F}(t) \leq \mathcal{F}(0)e^{C\,2\sqrt{2}\,K_*^{1/2}\,\mu^3 t} + e^{C\,2\sqrt{2}\,K_*^{1/2}\,\mu^3 t}\,C\,2\sqrt{2}\,K_*^{1/2}\,\mu^3 t\,C\,3\mu^{(7-2\gamma-\sigma)/2}\,\sqrt{2}\,K_*^{1/2}, \tag{202}$$

from which we can deduce the thesis. □

Proof of Theorem (2.1). First we prove (17).

We consider an initial datum as in (16); when passing to the continuous approximation (76), this initial datum corresponds to an initial data $(\xi_0, \eta_0) \in \mathcal{H}^{\rho_0,m}$. By Theorem 5.1 the corresponding sequence of gaps belongs to $\mathcal{H}^{\rho_0,m}$, and that the solution $(\xi(\tau), \eta(\tau))$ is analytic in a complex strip of width $\rho(t)$. Taking the minimum of such quantities one gets the coefficient ρ appearing in the statement of Theorem (2.1). Applying Proposition 6.3, we can deduce the corresponding result for the discrete model (8) and the specific quantities (13).

Next, we prove (19). In order to do so, we exploit the Birkhoff coordinates (x, y) introduced in Theorem 5.2; indeed, by rewriting the normal form system (110) in Birkhoff coordinates we get that every solution is almost-periodic in time. Now, let us introduce the quantities

$$E_K^{(1)} := |\xi_K|^2\,,$$
$$E_K^{(2)} := |\eta_K|^2\,,$$

then $\tau \mapsto E_K^{(1)}(x(\tau), y(\tau))$ and $\tau \mapsto E_K^{(2)}(x(\tau), y(\tau))$ are almost-periodic. If we set $E_K := \frac{1}{2}\left(E_K^{(1)} + E_K^{(2)}\right)$, we can exploit (179) of Proposition 6.3 to translate the results in terms of the specific quantities $\mathcal{E}_\kappa$, and we get the thesis. □

6.2. **The KP regime.** Similarly to Lemma 6.1, Proposition 6.2 we can prove the following results

Lemma 6.4. *Consider the lattice* (6) *in the regime (KP) and with interpolanting function* (117)*. Then for a state corresponding to* (q, p) *one has*

$$\mathcal{E}_\kappa = \frac{\mu^4}{2}\sum_{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^2 L_2\in 2\mathbb{Z}} |\hat{q}_{K+L}|^2 + \omega_k^2\left|\frac{\hat{p}_{K+L}}{\mu}\right|^2,\ \ \forall k : \kappa(k) = (\mu K_1, \mu^2 K_2) \tag{203}$$

(where the ω_k *are defined as in* (11)*), and* $\mathcal{E}_\kappa = 0$ *otherwise.*

Proposition 6.5. *Fix* $\rho > 0$ *and* $0 < \delta \ll 1$*. Consider the normal form system* (130)*, and define the Fourier coefficients of* (ξ, η) *through the following formula*

$$\xi(y) = \frac{1}{2}\sum_{h\in\mathbb{Z}^2} \hat{\xi}_h e^{ih\cdot y\pi}, \tag{204}$$

$$\eta(y) = \frac{1}{2}\sum_{h\in\mathbb{Z}^2} \hat{\eta}_h e^{ih\cdot y\pi},. \tag{205}$$

Consider $(\xi,\eta)\in\mathcal{H}^{\rho,0}$, *and denote by* $\mathcal{E}_\kappa$ *the specific energy of the normal mode with index* κ *as defined in* (12)-(13). *Then for any positive* μ *sufficiently small*

$$\left|\mathcal{E}_\kappa-\mu^4\frac{|\hat{\xi}_K|^2+|\hat{\eta}_K|^2}{2}\right|\le C\mu^{4+\frac{6}{5}}\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \tag{206}$$

for all k *such that* $\kappa(k)=(\mu K_1,\mu^2K_2)$ *and* $|K_1|+|K_2|\le\frac{(2+\delta)|\log\mu|}{\rho}$. *Moreover,*

$$|\mathcal{E}_\kappa|\le C\,\mu^8\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \tag{207}$$

for all k *such that* $\kappa(k)=(\mu K_1,\mu^2K_2)$ *and* $|K_1^2+K_2^2|^{1/2}>\frac{(2+\delta)|\log\mu|}{\rho}$, *and* $\mathcal{E}_\kappa=0$ *otherwise.*

Now, consider the following systems of uncoupled KP equations

$$\xi_\tau=-\frac{1}{24}\partial^3_{y_1}\xi-\frac{1}{2}\partial^{-1}_{y_1}\partial^2_{y_2}\xi-\frac{\alpha}{2\sqrt{2}}\partial_{y_1}(\xi^2), \tag{208}$$

$$\eta_\tau=\frac{1}{2}\partial^{-1}_{y_1}\partial^2_{y_2}\eta+\frac{1}{24}\partial^3_{y_1}\eta+\frac{\alpha}{2\sqrt{2}}\partial_{y_1}(\eta^2). \tag{209}$$

and consider a solution $(\tau,y)\mapsto(\widetilde{\xi_a}(\tau,y),\widetilde{\eta_a}(\tau,y))$ such that it belongs to $\mathcal{H}^{\rho,m}$, for some $m>0$.

We consider the approximate solutions (Q_a,P_a) of the FPU model (76)

$$Q_a(\tau,y):=\frac{\mu^2}{\sqrt{2}}\left[\widetilde{\xi_a}(\mu^2\tau,y_1-\tau,y_2)+\widetilde{\eta_a}(\mu^2\tau,y_1+\tau,y_2)\right] \tag{210}$$

$$\partial_{y_1}P_a(\tau,y):=\frac{\mu}{\sqrt{2}}\left[\widetilde{\xi_a}(\mu^2\tau,y_1-\tau,y_2)-\widetilde{\eta_a}(\mu^2\tau,y_1+\tau,y_2)\right], \tag{211}$$

We need to compare the difference between the approximate solution (210)-(211) and the true solution of (8). Let consider an initial datum (Q_0,P_0) with corresponding Fourier coefficients $(\hat{Q}_{0,k},\hat{P}_{0,k})$ given by (9), where

$$Q_{0,k}\neq0\ \text{ only if }\ \kappa(k)=(\mu K_1,\mu^2K_2). \tag{212}$$

We also assume that there exist C, $\rho>0$ such that

$$\frac{|\hat{Q}_{0,k}|^2+\omega_k^2|\hat{P}_{0,k}|^2}{N}\le Ce^{-2\rho|(\kappa_1(k)/\mu,\kappa_2(k)/\mu^2)|}. \tag{213}$$

Moreover, we define an interpolating function for the initial datum (Q_0,P_0) by

$$Q_0(y)=\frac{1}{(2N_1+1)(2N_2+1)}\sum_{K:\left(\mu^2|K_1|^2+\mu^4|K_2|^2\right)^{1/2}=|\kappa(k)|\le1}\hat{Q}_{0,k}e^{i\pi(\mu K_1y_1+\mu^2K_2y_2)},$$

and similarly for $y\mapsto P_0(y)$.

Arguing as for Proposition 6.3, we obtain

Proposition 6.6. *Consider* (8) *with* $\sigma=2$, *and fix* $1\le\gamma\le\frac{5}{2}$. *Let us assume that the initial datum for* (8) *satisfying* (212)-(213), *and denote by* $(Q(t),P(t))$ *the corresponding solution. Consider the approximate solution* $(\widetilde{\xi_a},\widetilde{\eta_a})$ *with the corresponding initial datum. Assume that* $(\widetilde{\xi_a},\widetilde{\eta_a})\in\mathcal{H}^{\rho,m}$ *for some* $\rho>0$ *and for some* $m\ge0$ *for all times, and fix* $T>0$ *and* $0<\delta\ll1$.

Then there exists $\mu_0=\mu_0(T,\|(\widetilde{\xi_a}(0),\widetilde{\eta_a}(0))\|_{\mathcal{H}^{\rho,m}})$ *such that, if* $\mu<\mu_0$, *we have that there exists* $C>0$ *such that*

$$\sup_j|Q_j(t)-Q_a(t,j)|+|P_j(t)-P_a(t,j)|\le C\mu^\gamma,\ \ |t|\le\frac{T}{\mu^3}, \tag{214}$$

where (Q_a,P_a) *are given by* (174)-(175). *Moreover,*

$$\left|\mathcal{E}_\kappa-\mu^4\frac{|\hat{\xi}_K|^2+|\hat{\eta}_K|^2}{2}\right|\le C\mu^{4+\gamma} \tag{215}$$

for all k *such that* $\kappa(k)=(\mu K_1,\mu^2K_2)$ *and* $|K_1|+|K_2|\le\frac{(2+\delta)|\log\mu|}{\rho}$. *Moreover,*

$$|\mathcal{E}_\kappa|\le\mu^{4+\gamma} \tag{216}$$

for all k such that $\kappa(k) = (\mu K_1, \mu^2 K_2)$ and $|K_1| + |K_2| > \frac{(2+\delta)|\log \mu|}{\rho}$, and $\mathcal{E}_\kappa = 0$ otherwise.

Proof of Theorem (2.2)*.* First we prove (21).

We consider an initial datum as in (20); when passing to the continuous approximation (76), this initial datum corresponds to an initial data $(\xi_0, \eta_0) \in \mathcal{H}^{\rho_0,m}$. By Theorem 5.3 the corresponding solution $(\xi(\tau), \eta(\tau))$ is analytic in a complex strip of width $\rho(t)$. Taking the minimum of such quantities one gets the coefficient ρ appearing in the statement of Theorem (2.2). Applying Proposition 6.6, we can deduce the corresponding result for the discrete model (8) and the specific quantities (13). ☐

6.3. **The one-dimensional NLS regime.** Let I be as in (80), we define the Fourier coefficients of the function $q : I \to \mathbb{R}$ by

$$\hat{q}(j) := \frac{1}{2} \int_I q(y_1, y_2) e^{-i\pi(j_1 y_1 + j_2 y_2)} \mathrm{d}y_1 \, \mathrm{d}y_2, \tag{217}$$

and similarly for the Fourier coefficients of the function p.

Lemma 6.7. *Consider the lattice* (24) *in the regime (1D NLS) and with interpolanting function* (133)*. Then for a state corresponding to* (q, p) *one has*

$$\mathcal{E}_\kappa = \frac{\mu^2}{2} \sum_{L=(L_1,L_2)\in\mathbb{Z}^2 : \mu L_1, \mu^\sigma L_2 \in 2\mathbb{Z}} |\hat{p}_{K+L}|^2 + \omega_k^2 |\hat{q}_{K+L}|^2, \quad \forall k : \kappa(k) = (\mu K_1, \mu^\sigma K_2) \tag{218}$$

(where the ω_k are defined as in (30)*), and $\mathcal{E}_\kappa = 0$ otherwise.*

Proof. First we introduce a $(2N_1+1)(2N_2+1)$-periodic interpolating function for Q_j, namely a smooth function $Q : (t, x) \mapsto Q(t, x)$ such that

$$Q_j(t) = Q(t, j), \quad \forall t, j,$$
$$Q(t, x_1 + 2N_1 + 1, x_2 + 2N_2 + 1) = Q(t, x), \quad \forall t, x,$$

and similarly for P_j. We denote by

$$\hat{Q}(j) := \frac{1}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}} \int_{[-(N_1+\frac{1}{2}),(N_1+\frac{1}{2})]\times[-(N_2+\frac{1}{2}),(N_2+\frac{1}{2})]} Q(x) e^{-i\frac{j\cdot x\, 2\pi}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}}} \mathrm{d}x, \tag{219}$$

so that by the interpolation property we obtain

$$\begin{aligned} Q_j(t) = Q(t, j) \;&=\; \frac{1}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}} \sum_{k\in\mathbb{Z}^2} \hat{Q}(j) e^{i\frac{j\cdot k\, 2\pi}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}}} \\ &= \frac{1}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}} \\ &\quad \times \sum_{k=(k_1,k_2)\in\mathbb{Z}^2_{2N+1}} \left[\sum_{h=(h_1,h_2)\in\mathbb{Z}^2} \hat{Q}(k_1 + (2N_1+1)h_1, k_2 + (2N_2+1)h_2) \right] e^{i\frac{j\cdot k\, 2\pi}{(2N_1+1)^{1/2}(2N_2+1)^{1/2}}}, \end{aligned}$$

hence

$$\hat{Q}_k = \sum_{h\in\mathbb{Z}^2} \hat{Q}(k_1 + (2N_1+1)h_1, k_2 + (2N_2+1)h_2). \tag{220}$$

The relation between $\hat{Q}(k)$ and $\hat{q}_k$ can be deduced from (133),

$$\begin{aligned}
Q(j) &= \mu q(\mu j_1, \mu^\sigma j_2);\\
\hat{Q}_k &= \frac{1}{2}\mu^{(\sigma+1)/2} \int_{\left[-\frac{1}{\mu},\frac{1}{\mu}\right]\times\left[-\frac{1}{\mu^\sigma},\frac{1}{\mu^\sigma}\right]} Q(x_1,x_2)e^{-i\pi(k_1x_1\mu+k_2x_2\mu^\sigma)}\mathrm{d}x_1\,\mathrm{d}x_2\\
&\overset{(133)}{=} \frac{1}{2}\mu^{(\sigma+1)/2} \int_{\left[-\frac{1}{\mu},\frac{1}{\mu}\right]\times\left[-\frac{1}{\mu^\sigma},\frac{1}{\mu^\sigma}\right]} \mu\, q\left(\mu x_1, \mu^\sigma x_2\right) e^{-i\pi(k_1x_1\mu+k_2x_2\mu^\sigma)}\mathrm{d}x_1\,\mathrm{d}x_2\\
&= \frac{1}{2}\mu^{(1-\sigma)/2}\int_I q(y)e^{-i\pi(k_1y_1+k_2y_2)}\mathrm{d}y\\
&= \mu^{(1-\sigma)/2}\hat{q}_k, \qquad (221)
\end{aligned}$$

and similarly

$$\hat{P}_k = \mu^{(1-\sigma)/2}\hat{p}_k. \qquad (222)$$

By using (29), (13) and (220)-(222) we have

$$\begin{aligned}
\mathcal{E}_\kappa &\overset{(13)}{=} \mu^{\sigma+1}\frac{1}{2}\sum_{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}} |\hat{P}_{K+L}|^2+\omega_k^2|\hat{Q}_{K+L}|^2\\
&\overset{(221),(221)}{=} \mu^{\sigma+1}\,\mu^{1-\sigma}\frac{1}{2}\sum_{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}} |\hat{p}_{K+L}|^2+\omega_k^2|\hat{q}_{K+L}|^2
\end{aligned}$$

for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$, and this leads to (218). ☐

Proposition 6.8. *Fix $\rho > 0$ and $0 < \delta \ll 1$. Consider the normal form equation* (148)*, and define the Fourier coefficients of $(\psi,\bar{\psi})$ through the following formula*

$$\psi(y) = \frac{1}{2}\sum_{h\in\mathbb{Z}^2}\hat{\psi}_h e^{ih\cdot y\pi}, \qquad (223)$$

(224)

Consider $(\psi,\bar{\psi}) \in \mathcal{H}^{\rho,0}$, and denote by $\mathcal{E}_\kappa$ the specific energy of the normal mode with index κ as defined in (12)-(13)*. Then for any positive μ sufficiently small*

$$\left|\mathcal{E}_\kappa - \mu^2\frac{|\hat{\psi}_K|^2}{2}\right| \le C\mu^{2+\frac{6}{5}}\|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,0}} \qquad (225)$$

for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ and $|K_1| + |K_2| \le \frac{(2+\delta)|\log\mu|}{\rho}$. Moreover,

$$|\mathcal{E}_\kappa| \le C\,\mu^8\|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,1}} \qquad (226)$$

for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ and $|K_1| + |K_2| > \frac{(2+\delta)|\log\mu|}{\rho}$, and $\mathcal{E}_\kappa = 0$ otherwise.

We defer the proof of the above Proposition to Appendix C.

Now, consider the normal form equation, namely the following cubic defocusing one-dimensional NLS

$$-i\psi_t = -\,\partial^2_{y_1}\psi + \frac{3\beta}{4}\,|\psi|^2\psi. \qquad (227)$$

and consider a solution $(\widetilde{\psi_a}, \widetilde{\bar{\psi}_a})$ such that it belongs to $\mathcal{H}^{\rho,n}$, for some $n > 0$.

We consider the approximate solutions (Q_a, P_a) of the KG lattice (24) (in the following $\tau = \mu^2 t$)

$$Q_a(\tau,y) := \frac{\mu}{\sqrt{2}}\left[e^{i\tau}\widetilde{\psi_a}(\tau,y_1,y_2) + e^{-i\tau}\widetilde{\bar{\psi}_a}(\tau,y_1,y_2)\right] \qquad (228)$$

$$P_a(\tau,y) := \frac{\mu}{\sqrt{2}i}\left[e^{i\tau}\widetilde{\psi_a}(\tau,y_1,y_2) + e^{-i\tau}\widetilde{\bar{\psi}_a}(\tau,y_1,y_2)\right] \qquad (229)$$

(230)

We need to compare the difference between the approximate solution (174)-(175) and the true solution of (24). Let consider an initial datum (Q_0, P_0) with corresponding Fourier coefficients $(\hat{Q}_{0,k}, \hat{P}_{0,k})$ given by (9), where

$$Q_{0,k} \neq 0 \ \text{ only if } \ \kappa(k) = (\mu K_1, \mu^\sigma K_2). \tag{231}$$

We also assume that there exist C, $\rho > 0$ such that

$$\frac{|\hat{P}_{0,k}|^2 + \omega_k^2 |\hat{Q}_{0,k}|^2}{N} \leq C e^{-2\rho |(\kappa_1(k)/\mu, \kappa_2(k)/\mu^\sigma)|}. \tag{232}$$

Moreover, we define an interpolating function for the initial datum (Q_0, P_0) by

$$Q_0(y) = \frac{1}{(2N_1+1)(2N_2+1)} \sum_{K : \left(\mu^2 |K_1|^2 + \mu^{2\sigma} |K_2|^2\right)^{1/2} = |\kappa(k)| \leq 1} \hat{Q}_{0,k} e^{i\pi(\mu K_1 y_1 + \mu^\sigma K_2 y_2)},$$

and similarly for $y \mapsto P_0(y)$.

Proposition 6.9. *Consider* (24) *with* $\sigma > 1$ *and* $\gamma > 0$ *such that* $\sigma + 2\gamma < 7$. *Let us assume that the initial datum satisfies* (231)-(232), *and denote by* $(Q(t), P(t))$ *the corresponding solution. Consider the approximate solution* $(\widetilde{\psi_a}(t,x), \widetilde{\bar{\psi}_a}(t,x))$ *with the corresponding initial datum. Assume that* $(\widetilde{\psi_a}, \widetilde{\bar{\psi}_a}) \in \mathcal{H}^{\rho,n}$ *for some* $\rho > 0$ *and for some* $n \geq 0$ *for all times, and fix* $T > 0$ *and* $0 < \delta \ll 1$.

Then there exists $\mu_0 = \mu_0(T, \|(\widetilde{\psi_a}(0), \widetilde{\bar{\psi}_a}(0))\|_{\mathcal{H}^{\rho,n}})$ *such that, if* $\mu < \mu_0$, *we have that there exists* $C > 0$ *such that*

$$\sup_j |Q_j(t) - Q_a(t,j)| + |P_j(t) - P_a(t,j)| \leq C\mu^\gamma, \ \ |t| \leq \frac{T}{\mu^2}, \tag{233}$$

where (Q_a, P_a) *are given by* (228)-(229). *Moreover,*

$$\left| \mathcal{E}_\kappa - \mu^2 \frac{|\hat{\xi}_K|^2 + |\hat{\eta}_K|^2}{2} \right| \leq C\, \mu^{2+\gamma} \tag{234}$$

for all k *such that* $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ *and* $|K_1| + |K_2| \leq \frac{(2+\delta)|\log \mu|}{\rho}$. *Moreover,*

$$|\mathcal{E}_\kappa| \leq \mu^{2+\gamma} \tag{235}$$

for all k *such that* $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ *and* $|K_1| + |K_2| > \frac{(2+\delta)|\log \mu|}{\rho}$, *and* $\mathcal{E}_\kappa = 0$ *otherwise.*

Proof. The argument follows along the lines of Appendix C in [BP06].

Exploiting the canonical transformation found in Theorem 3.3, we also define

$$\zeta_a := (\psi_a, \bar{\psi}_a) = \mathcal{T}^{(1)}_{\mu^2}(\widetilde{\psi_a}, \widetilde{\bar{\psi}_a}) = \tilde{\zeta}_a + \phi_a(\tilde{\zeta}_a), \tag{236}$$

where $\phi_a(\tilde{\zeta}_a) := (\phi_\xi(\tilde{\zeta}_a), \phi_\eta(\tilde{\zeta}_a))$; by (59) we have

$$\sup_{\zeta \in \mathcal{B}_{\rho,n}(R)} \|\phi_a(\zeta)\|_{\mathcal{H}^{\rho,n}} \leq C'_n \mu^2\, R. \tag{237}$$

For convenience we define

$$q_a(\tau, y) := \frac{1}{\sqrt{2}} \left[e^{i\tau} \widetilde{\psi_a}(\tau, y_1, y_2) + e^{-i\tau} \widetilde{\bar{\psi}_a}(\tau, y_1, y_2) \right] \tag{238}$$

$$p_a(\tau, y) := \frac{1}{\sqrt{2}i} \left[e^{i\tau} \widetilde{\psi_a}(\tau, y_1, y_2) - e^{-i\tau} \widetilde{\bar{\psi}_a}(\tau, y_1, y_2) \right], \tag{239}$$

We observe that the pair (q_a, p_a) satisfies

$$\mu (q_a)_t = \mu p_a + \mu^5 \mathcal{R}_q \tag{240}$$

$$\mu (p_a)_t = -\mu q_a + \mu \Delta_1 q_a - \mu^3\, \beta\, \pi_0 q_a^3 + \mu^5 \mathcal{R}_p, \tag{241}$$

where the operator Δ_1 acts on the variable x, π_0 is the projector on the space of the functions with zero average, and the remainders are functions of the rescaled variables τ and y which satisfy

$$\sup_{\mathcal{B}_{\rho,n}(R)} \|\mathcal{R}_q\|_{\ell^2_{\rho,0}} \leq C,$$
$$\sup_{\mathcal{B}_{\rho,n}(R)} \|\mathcal{R}_p\|_{\ell^2_{\rho,1}} \leq C.$$

We now restrict the space variables to integer values; keeping in mind that q_a and p_a are periodic, we assume that $j \in \mathbb{Z}^2_{N,N^\sigma}$.

For a finite sequence $Q = (Q_j)_{j\in\mathbb{Z}^2_{N,N^\sigma}}$ we define the norm

$$\|Q\|^2_{\ell^2_{N,N^\sigma}} := \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |Q_j|^2. \tag{242}$$

Now we consider the discrete model (8): we rewrite in the following form,

$$\dot{Q}_j = P_j \tag{243}$$
$$\dot{P}_j = -Q_j + (\Delta_1 Q)_j - \beta\,\pi_0 Q_j^3 \tag{244}$$

and we want to show that there exist two sequences $E = (E_j)_{j\in\mathbb{Z}^2_{N,N^\sigma}}$ and $F = (F_j)_{j\in\mathbb{Z}^2_{N,N^\sigma}}$ such that

$$Q \,=\, \mu\, q_a + \mu^{1+\gamma} E,\;\; P \,=\, \mu p_a + \mu^{1+\gamma} F$$

fulfills (243)-(244), where $\gamma > 0$ is a parameter we will fix later in the proof. Therefore, we have that

$$\dot{E} = F - \mu^{5-1-\gamma}\mathcal{R}_q \tag{245}$$
$$\dot{F} = -E + \Delta_1 E - \beta\pi_0\,(3\mu^{3+\gamma-1-\gamma}\, q_a^2 E + 3\mu^{1+2+2\gamma-1-\gamma}\, q_a E^2 + \mu^{3+3\gamma-1-\gamma} E^3) - \mu^{5-1-\gamma}\mathcal{R}_p, \tag{246}$$

where we impose initial conditions on (E, F) such that $(\tilde{q}, \tilde{p})$ has initial conditions corresponding to the ones of the true initial datum,

$$\mu q_a(0, \mu j_1, \mu^\sigma j_2) + \mu^{1+\gamma} E_{0,j} = Q_{0,j},$$
$$\mu p_a(0, \mu j_1, \mu^\sigma j_2) + \mu^{1+\gamma} F_{0,j} = P_{0,j}.$$

We now define the operator ∂_i, $i = 1, 2$, by $(\partial_i f)_j := f_j - f_{j-e_i}$ for each $f \in \ell^2_{N,N^\sigma}$.

- Claim 1: Let $\sigma > 1$ and $\gamma > 0$, we have

$$\|E_0\|_{\ell^2_{N,N^\sigma}} \leq C'\mu^{(3-2\gamma-\sigma)/2},$$
$$\|F_0\|_{\ell^2_{N,N^\sigma}} \leq C'\mu^{(3-2\gamma-\sigma)/2},$$
$$\|\partial_1 E_0\|_{\ell^2_{N,N^\sigma}} \leq C'\mu^{(5-2\gamma-\sigma)/2},$$
$$\|\partial_2 E_0\|_{\ell^2_{N,N^\sigma}} \leq C'\mu^{(3-2\gamma+\sigma)/2},$$
$$\|\partial_1 F_0\|_{\ell^2_{N,N^\sigma}} \leq C'\mu^{(5-2\gamma-\sigma)/2},$$
$$\|\partial_2 F_0\|_{\ell^2_{N,N^\sigma}} \leq C'\mu^{(3-2\gamma+\sigma)/2}.$$

To prove Claim 1 we observe that

$$E_0 = \mu\frac{\psi_a + \bar{\psi}_a - (\widetilde{\psi}_a + \bar{\widetilde{\psi}}_a)}{\sqrt{2}\mu^{1+\gamma}} \;=\; \mu^{-\gamma}\,\frac{\phi_\xi + \phi_\eta}{\sqrt{2}},$$
$$F_0 = \mu\frac{\psi_a - \bar{\psi}_a - (\widetilde{\psi}_a - \bar{\widetilde{\psi}}_a)]}{\sqrt{2}i\,\mu^{1+\gamma}} \;=\; \mu^{-\gamma}\,\frac{\phi_\xi - \phi_\eta}{\sqrt{2}i},$$

from which we can deduce

$$
\begin{aligned}
\|E_0\|^2_{\ell^2_{N,N^\sigma}} &\le \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |E_{0,j}|^2 \,\le\, C\,4N^{\sigma+1}\,(\mu^{2-\gamma})^2 \,=\, C\,\mu^{3-2\gamma-\sigma},\\
\|F_0\|^2_{\ell^2_{N,N^\sigma}} &\le \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |F_{0,j}|^2 \,\le\, C\,4N^{\sigma+1}\,(\mu^{2-\gamma})^2 \,=\, C\,\mu^{3-2\gamma-\sigma},\\
\|\partial_1 E_0\|^2_{\ell^2_{N,N^\sigma}} &\le \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |\partial_1 E_{0,j}|^2 \,\le\, C\,4N^{\sigma+1}\,(\mu^{2+1-\gamma})^2 \,\le\, C\,\mu^{5-2\gamma-\sigma},\\
\|\partial_2 E_0\|^2_{\ell^2_{N,N^\sigma}} &\le \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |\partial_2 E_{0,j}|^2 \,\le\, C\,4N^{\sigma+1}\,(\mu^{2+\sigma-\gamma})^2 \,=\, C\,\mu^{3-2\gamma+\sigma},\\
\|\partial_1 F_0\|^2_{\ell^2_{N,N^\sigma}} &\le \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |\partial_1 F_{0,j}|^2 \,\le\, C\,4N^{\sigma+1}\,(\mu^{2+1-\gamma})^2 \,\le\, C\,\mu^{5-2\gamma-\sigma},\\
\|\partial_2 F_0\|^2_{\ell^2_{N,N^\sigma}} &\le \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} |\partial_2 F_{0,j}|^2 \,\le\, C\,4N^{\sigma+1}\,(\mu^{2+\sigma-\gamma})^2 \,=\, C\,\mu^{3-2\gamma+\sigma},
\end{aligned}
$$

and this leads to the thesis.

- Claim 2: Fix $n \ge 0$, $T > 0$ and $K_* > 0$, then for any $\mu < \mu_s$ and for any $\sigma > 1$ and $\gamma > 0$ such that $\sigma + 2\gamma < 7$ we have

$$
\|E\|^2_{\ell^2_{N,N^\sigma}} + \|F\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_1 E_0\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_2 E_0\|^2_{\ell^2_{N,N^\sigma}} \le K_*,\;\; |t| < \frac{T}{\mu^2}. \tag{247}
$$

To prove the claim, we define

$$
\mathcal{F}(E,F) := \sum_{j\in\mathbb{Z}^2_{N,N^\sigma}} \frac{F_j^2 + E_j^2 + E_j(-\Delta_1 E)_j}{2} + \frac{3\mu^2\beta q_a^2 E_j^2 + 3\mu^{2+\gamma}\beta q_a E_j^3}{2}, \tag{248}
$$

and we remark that

$$
\frac{1}{2}\mathcal{F}(E,F) \,\le\, \|E\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_1 F_0\|^2_{\ell^2_{N,N^\sigma}} + \|\partial_2 F_0\|^2_{\ell^2_{N,N^\sigma}} \,\le\, 2\mathcal{F}(E,F).
$$

Now we compute the time derivative of $\mathcal{F}$. Exploiting (190)-(191)

$$
\begin{aligned}
\dot{\mathcal{F}} &= \sum_j F_j\left[-E_j + (\Delta_1 E)_j - \beta\pi_0\,(3\mu^2\,q_a^2 E_j + 3\mu^{2+\gamma}\,q_a E_j^2 + \mu^{2+2\gamma} E_j^3) - \mu^{4-\gamma}(\mathcal{R}_p)_j\right] && (249)\\
&\quad + \sum_j (E_j - (\Delta_1 E)_j)\left[F_j - \mu^{4-\gamma}(\mathcal{R}_q)_j\right] && (250)\\
&\quad + \sum_j 3\mu^2\,\beta\, q_a^2 E_j\left[F_j - \mu^{4-\gamma}(\mathcal{R}_q)_j\right] + 3\mu^2\beta E_j^2 q_a\,\mu\frac{\partial q_a}{\partial\tau} && (251)\\
&\quad + \sum_j \frac{9}{2}\mu^{2+\gamma}\beta E_j^2\left[F_j - \mu^{4-\gamma}(\mathcal{R}_q)_j\right] + \frac{3}{2}\mu^{2+\gamma}\beta E_j^3\,\mu\frac{\partial q_a}{\partial\tau} && (252)\\
&= \sum_j F_j\left[-\beta\pi_0\,(3\mu^{2+\gamma}\,q_a E_j^2 + \mu^{2+2\gamma}E_j^3) - \mu^{4-\gamma}(\mathcal{R}_p)_j\right] && (253)\\
&\quad + \sum_j E_j\left[-\mu^{4-\gamma}(\mathcal{R}_q)_j\right] - (\Delta_1 E)_j\left[-\mu^{4-\gamma}(\mathcal{R}_q)_j\right] && (254)\\
&\quad + \sum_j 3\mu^2\,\beta\, q_a^2 E_j\left[-\mu^{4-\gamma}(\mathcal{R}_q)_j\right] + 3\mu^2\beta E_j^2 q_a\,\mu\frac{\partial q_a}{\partial\tau} && (255)\\
&\quad + \sum_j \frac{9}{2}\mu^{2+\gamma}\beta E_j^2\left[F_j - \mu^{4-\gamma}(\mathcal{R}_q)_j\right] + \frac{3}{2}\mu^{2+\gamma}\beta E_j^3\,\mu\frac{\partial q_a}{\partial\tau} && (256)
\end{aligned}
$$

In order to estimate (253)-(256), we notice that

$$\sup_j |(\Delta_1 E)_j| \leq 2 \sup_j |(\partial_1 E)_j| + |(\partial_2 E)_j| \leq 4\sqrt{\mathcal{F}},$$
$$\|\mathcal{R}_q\|^2_{\ell^2_{N,N^\sigma}} \ \leq \ \sum_j |(\mathcal{R}_q)_j|^2 \leq 4N^{\sigma+1} \sup_y |\mathcal{R}_q(y)|^2 \ \leq \ C\mu^{-1-\sigma},$$
$$\|\mathcal{R}_p\|^2_{\ell^2_{N,N^\sigma}} \ \leq \ C\mu^{-1-\sigma},$$

and that $|(\partial_i \mathcal{R}_q)_j| \leq \mu \sup_y \left| \frac{\partial \mathcal{R}_q}{\partial y}(y) \right|$, which implies

$$\|\partial_i \mathcal{R}_q\|^2_{\ell^2_{N,N^\sigma}} \leq C\mu^{1-\sigma}.$$

Now, we can estimate (253) by

$$C \left(\mu^{2+\gamma} \mathcal{F}^{3/2} + \mu^{2+2\gamma} \mathcal{F}^2 + \mu^{4-\gamma} \mu^{-(1+\sigma)/2} \mathcal{F}^{1/2} \right). \tag{257}$$

Then, (254) can be bounded by

$$C \left(\mu^{4-\gamma-(1+\sigma)/2} \mathcal{F}^{1/2} + \mu^{4-\gamma+(1-\sigma)/2} \mathcal{F}^{1/2} \right); \tag{258}$$

next, we can estimate (255) by

$$C \left(\mu^{6-\gamma-(1+\sigma)/2} \mathcal{F}^{1/2} + \mu^3 \mathcal{F} \right), \tag{259}$$

while (256) can be bounded by

$$C \left(\mu^{2+\gamma} \mathcal{F}^{3/2} + \mu^{6-(1+\sigma)/2} \mathcal{F} + \mu^{2+\gamma} \mathcal{F}^{3/2} \right). \tag{260}$$

Hence, as long as $\mathcal{F} < 2K_*$ we have

$$\left| \dot{\mathcal{F}} \right| \leq C \left[\mu^{2+\gamma} K_*^{1/2} + \mu^{2+2\gamma} K_* + \mu^3 + \mu^{2+\gamma} K_*^{1/2} + \mu^{6-(1+\sigma)/2} + \mu^{2+\gamma} K_*^{1/2} \right] \mathcal{F} \tag{261}$$
$$+ C \left[\mu^{4-\gamma} \mu^{-(1+\sigma)/2} + \mu^{4-\gamma-(1+\sigma)/2} + \mu^{4-\gamma+(1-\sigma)/2} + \mu^{6-\gamma-(1+\sigma)/2} \right] K_*^{1/2} \tag{262}$$
$$\overset{\sigma+2\gamma<7}{\leq} C \mu^2 (1 + K_*^{1/2}) \mathcal{F} + C \mu^{(7-2\gamma-\sigma)/2} K_*^{1/2} \tag{263}$$

and by applying Gronwall's lemma we get

$$\mathcal{F}(t) \leq \mathcal{F}(0) e^{C (1+K_*^{1/2}) \mu^2 t} + e^{C (1+K_*^{1/2}) \mu^2 t} C (1 + K_*^{1/2}) \mu^2 t \, C \mu^{(7-2\gamma-\sigma)/2} K_*^{1/2}, \tag{264}$$

from which we can deduce the thesis. □

Proof of Theorem (2.4). First we prove (32).

We consider an initial datum as in (31); when passing to the continuous approximation (132), this initial datum corresponds to an initial data $(\xi_0, \eta_0) \in \mathcal{H}^{\rho_0, n}$. By Theorem 5.5 the corresponding sequence of gaps belongs to $\mathcal{H}^{\rho_0, n}$, and that the solution $(\xi(\tau), \eta(\tau))$ is analytic in a complex strip of width $\rho(t)$. Taking the minimum of such quantities one gets the coefficient ρ appearing in the statement of Theorem (2.4). Applying Proposition 6.9, we can deduce the corresponding result for the discrete model (27) and the specific quantities (13).

Next, we prove (34). In order to do so, we exploit the Birkhoff coordinates (x, y) introduced in Theorem 5.6; indeed, by rewriting the normal form system (148) in Birkhoff coordinates we get that every solution is almost-periodic in time. Now, let us introduce the quantities

$$E_K^{(1)} := |\xi_K|^2,$$
$$E_K^{(2)} := |\eta_K|^2,$$

then $\tau \mapsto E_K^{(1)}(x(\tau), y(\tau))$ and $\tau \mapsto E_K^{(2)}(x(\tau), y(\tau))$ are almost-periodic. If we set $E_K := \frac{1}{2}\left(E_K^{(1)} + E_K^{(2)} \right)$, we can exploit (234) of Proposition 6.9 to translate the results in terms of the specific quantities $\mathcal{E}_\kappa$, and we get the thesis. □

APPENDIX A. PROOF OF LEMMA 3.7

In order to normalize system (61), we used an adaptation of Theorem 4.4 in [Bam99]. The result is based on the method of Lie transform, that we will recall in the following.

Let $s \geq s_1$ and $\rho \geq 0$ be fixed.
Given an auxiliary function χ analytic on $\mathcal{H}^{\rho,s}$, we consider the auxiliary differential equation

$$\dot{\zeta} = X_\chi(\zeta) \tag{265}$$

and denote by Φ^t_χ its time-t flow. A simple application of Cauchy inequality gives

Lemma A.1. *Let χ and its symplectic gradient be analytic in $\mathcal{B}_{\rho,s}(R)$. Fix $\delta < R$, and assume that*

$$\sup_{\mathcal{B}_{\rho,s}(R-\delta)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq \delta.$$

Then, if we consider the time-t flow Φ^t_χ of X_χ we have that for $|t| \leq 1$

$$\sup_{\mathcal{B}_{\rho,s}(R-\delta)} \|\Phi^t_\chi(\zeta) - \zeta\|_{\mathcal{H}^{\rho,s}} \leq \sup_{\mathcal{B}_{\rho,s}(R-\delta)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}}.$$

Definition A.2. *The map $\Phi := \Phi^1_\chi$ will be called the* Lie transform *generated by χ.*

Remark A.3. *Given G analytic on $\mathcal{H}^{\rho,s}$, let us consider the differential equation*

$$\dot{\zeta} = X_G(\zeta), \tag{266}$$

where by X_G we denote the vector field of G. Now define

$$\Phi^* G(\tilde{\zeta}) := G \circ \Phi(\zeta).$$

In the new variable $\tilde{\zeta}$ defined by $\zeta = \Phi(\tilde{\zeta})$ equation (266) is equivalent to

$$\dot{\tilde{\zeta}} = X_{\Phi^* G}(\zeta). \tag{267}$$

Using the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi^t_\chi)^* G = (\Phi^t_\chi)^* \{\chi, G\},$$

we formally get

$$\Phi^* G = \sum_{l=0}^{\infty} G_l, \tag{268}$$

$$G_0 := G, \tag{269}$$

$$G_l := \frac{1}{l}\{\chi, G_{l-1}\}, \quad l \geq 1. \tag{270}$$

In order to estimate the terms appearing in (268) we exploit the following results

Lemma A.4. *Let $R > 0$, and assume that χ, G are analytic on $\mathcal{B}_{\rho,s}(R)$.
Then, for any $d \in (0, R)$ we have that $\{\chi, G\}$ is analytic on $\mathcal{B}_{\rho,s}(R-d)$, and*

$$\sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{\{\chi,G\}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \lesssim \frac{2}{d}. \tag{271}$$

Lemma A.5. *Let $R > 0$, and assume that χ, G are analytic on $\mathcal{B}_{\rho,s}(R)$. Let $l \geq 1$, and consider G_l as defined in (268); for any $d \in (0, R)$ we have that G_l is analytic on $\mathcal{B}_{\rho,s}(R-d)$, and*

$$\sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{G_l}(\zeta)\|_{\mathcal{H}^{\rho,s}} \lesssim \left(\frac{2e}{d}\right)^l. \tag{272}$$

Proof. Fix l, and denote $\delta := d/l$. We look for a sequence $C_m^{(l)}$ such that

$$\sup_{\mathcal{B}_{\rho,s}(R-m\delta)} \|X_{G_m}(\zeta)\|_{\mathcal{H}^{\rho,s}} \lesssim C_m^{(l)}, \quad \forall m \leq l.$$

By (271) we can define the sequence

$$\begin{aligned} C_0^{(l)} &:= \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}}, \\ C_m^{(l)} &= \frac{2}{\delta m} C_{m-1}^{(l)} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \\ &= \frac{2l}{dm} C_{m-1}^{(l)} \sup_{B_{k,p}(R)} \|X_\chi(\psi,\bar{\psi})\|_{k,p}. \end{aligned}$$

One has

$$C_l^{(l)} = \frac{1}{l!}\left(\frac{2l}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{k,p}\right)^l \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}},$$

and by using the inequality $l^l < l! e^l$ we can conclude. ☐

Remark A.6. *Let $s \geq s_1$, and assume that χ, F are analytic on $\mathcal{B}_{\rho,s}(R)$. Fix $d \in (0,R)$, and assume also that*

$$\sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq d/3,$$

Then for $|t| \leq 1$

$$\begin{aligned} \sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{(\Phi_\chi^t)^* F - F}(\zeta)\|_{\mathcal{H}^{\rho,s}} &= \sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{F\circ\Phi_\chi^t - F}(\zeta)\|_{\mathcal{H}^{\rho,s}} \qquad (273) \\ &\overset{(271)}{\leq} \frac{5}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_F(\zeta)\|_{\mathcal{H}^{\rho,s}}. \qquad (274) \end{aligned}$$

Lemma A.7. *Let $s \geq s_1$. Assume that G is analytic on $\mathcal{B}_{\rho,s}(R)$, and that h_0 satisfies PER. Then there exists χ analytic on $\mathcal{B}_{\rho,s}(R)$ and Z analytic on $\mathcal{B}_{\rho,s}(R)$ with Z in normal form, namely $\{h_0, Z\} = 0$, such that*

$$\{h_0, \chi\} \; + \; G \; = \; Z. \tag{275}$$

Furthermore, we have the following estimates on the vector fields

$$\sup_{\mathcal{B}_{\rho,s}(R)} \|X_Z(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}}, \tag{276}$$

$$\sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}} \lesssim \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}}. \tag{277}$$

Proof. One can check that the solution of (275) is

$$\chi(\zeta) = \frac{1}{T}\int_0^T t\,\left[G(\Phi^t(\zeta)) - Z(\Phi^t(\zeta))\right] \mathrm{d}t,$$

with $T = 2\pi$. Indeed,

$$\begin{aligned} \{h_0, \chi\}(\zeta) &= \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} \chi(\Phi^s(\zeta)) \\ &= \frac{1}{2\pi}\int_0^{2\pi} t \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} \left[G(\Phi^{t+s}(\zeta)) - Z(\Phi^{t+s}(\zeta))\right] \mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} t \frac{\mathrm{d}}{\mathrm{d}t} \left[G(\Phi^t(\zeta)) - Z(\Phi^t(\zeta))\right] \mathrm{d}t \\ &= \frac{1}{2\pi}\left[tG(\Phi^t(\zeta)) - tZ(\Phi^t(\zeta))\right]_{t=0}^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} \left[G(\Phi^t(\zeta)) - Z(\Phi^t(\zeta))\right] \mathrm{d}t \\ &= G(\zeta) - Z(\zeta). \end{aligned}$$

Finally, (276) follows from the fact that

$$X_\chi(\zeta) = \frac{1}{T}\int_0^T t\,\Phi^{-t} \circ X_{G-Z}(\Phi^t(\zeta)\mathrm{d}t$$

by applying property (52). □

Lemma A.8. *Let $s \geq s_1$. Assume that G is analytic on $\mathcal{B}_{\rho,s}(R)$, and that h_0 satisfies PER. Let χ be analytic on $\mathcal{B}_{\rho,s}(R)$, and assume that it solves* (275). *For any $l \geq 1$ denote by $h_{0,l}$ the functions defined recursively as in* (268) *from h_0. Then for any $d \in (0, R)$ one has that $h_{0,l}$ is analytic on $\mathcal{B}_{\rho,s}(R-d)$, and*

$$\sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{h_{0,l}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq 2 \sup_{\mathcal{B}_{\rho,s}(R)} \|X_G(\zeta)\|_{\mathcal{H}^{\rho,s}} \left(\frac{5}{d} \sup_{\mathcal{B}_{\rho,s}(R)} \|X_\chi(\zeta)\|_{\mathcal{H}^{\rho,s}}\right)^l . \tag{278}$$

Proof. By using (275) one gets that $h_{0,1} = Z - G$ is analytic on $\mathcal{B}_{\rho,s}(R)$. Then by exploiting (274) one gets the result. □

Lemma A.9. *Let $s \geq s_1 \gg 1$, $R > 0$, $m \geq 0$, and consider the Hamiltonian*

$$H^{(m)}(\zeta) = h_0(\zeta) + \delta Z^{(m)}(\zeta) + \delta^{m+1} F^{(m)}(\zeta). \tag{279}$$

Assume that h_0 satisfies PER and INV, and that

$$\sup_{\mathcal{B}_{\rho,s}(R)} \|X_{F^{(0)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq F.$$

Fix $d < R/(m+1)$, and assume also that $Z^{(m)}$ are analytic on $\mathcal{B}_{\rho,s}(R-md)$, and that

$$\sup_{\mathcal{B}_{\rho,s}(R-md)} \|X_{Z^{(0)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} = 0,$$

$$\sup_{\mathcal{B}_{\rho,s}(R-md)} \|X_{Z^{(m)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq F \sum_{i=0}^{m-1} \delta^i K_s^i, \quad m \geq 1,$$

$$\sup_{\mathcal{B}_{\rho,s}(R-md)} \|X_{F^{(m)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq F\, K_s^m, \quad m \geq 1, \tag{280}$$

with $K_s := \frac{2\pi}{d}\, 18F$.

Then, if $\delta K_s < 1/2$ there exists a canonical transformation $\mathcal{T}_\delta^{(m)}$ analytic on $\mathcal{B}_{\rho,s}(R-(m+1)d)$ such that

$$\sup_{\mathcal{B}_{\rho,s}(R-md)} \|\mathcal{T}_\delta^{(m)}(\zeta) - \zeta\|_{\mathcal{H}^{\rho,s}} \leq 2\pi\delta^{m+1}F, \tag{281}$$

$H^{(m+1)} := H^{(m)} \circ \mathcal{T}^{(m)}$ has the form (279) *and satisfies* (280) *with m replaced by $m+1$.*

Proof. The key point of the lemma is to look for $\mathcal{T}_\delta^{(m)}$ as the time-one map of the Hamiltonian vector field of an analytic function $\delta^{m+1}\chi_m$. Hence, consider the differential equation

$$\dot{\zeta} = X_{\delta^{m+1}\chi_m}(\zeta). \tag{282}$$

By standard theory we have that, if $\|X_{\delta^{m+1}\chi_m}\|_{\mathcal{B}_{\rho,s}(R-md)}$ is sufficiently small and $\zeta_0 \in \mathcal{B}_{\rho,s}(R-(m+1)d)$, then the solution of (282) exists for $|t| \leq 1$.

Therefore we can define $\mathcal{T}_{m,\delta}^t : \mathcal{B}_{\rho,s}(R-(m+1)d) \to \mathcal{B}_{\rho,s}(R-md)$, and in particular the corresponding time-one map $\mathcal{T}_\delta^{(m)} := \mathcal{T}_{m,\delta}^1$, which is an analytic canonical transformation, δ^{m+1}-close to the identity. We have

$$\begin{aligned}
(\mathcal{T}_\delta^{(m+1)})^* \,(h_0 + \delta Z^{(m)} + \delta^{m+1} F^{(m)}) &= h_0 + \delta Z^{(m)} \\
&+ \delta^{m+1}\left[\{\chi_m, h_0\} + F^{(m)}\right] + \\
&+ \left(h_0 \circ \mathcal{T}^{(m+1)} - h_0 - \delta^{m+1}\{\chi_m, h_0\}\right) + \delta\left(Z^{(m)} \circ \mathcal{T}^{(m+1)} - Z^{(m)}\right) && (283)\\
&+ \delta^{m+1}\left(F^{(m)} \circ \mathcal{T}^{(m+1)} - F^{(m)}\right). && (284)
\end{aligned}$$

It is easy to see that the first two terms are already normalized, that the term in the second line is the non-normalized part of order m+1 that will vanish through the choice of a suitable χ_m, and that the last lines contains all the terms of order higher than m+1.

Now we want to determine χ_m in order to solve the so-called "homological equation"

$$\{\chi_m, h_0\} + F^{(m)} \; = \; Z_{m+1},$$

with Z_{m+1} in normal form. The existence of χ_m and Z_{m+1} is ensured by Lemma A.7, and by applying (276) and the inductive hypothesis we get

$$\sup_{\mathcal{B}_{\rho,s}(R-md)} \|X_{\chi_m}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq 2\pi F, \tag{285}$$

$$\sup_{\mathcal{B}_{\rho,s}(R-md)} \|X_{Z_{m+1}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq 2\pi F. \tag{286}$$

Now define $Z^{(m+1)} := Z^{(m)} + \delta^m \, Z_{m+1}$, and notice that by Lemma A.1 we can deduce the estimate of $X_{Z^{(m+1)}}$ on $\mathcal{B}_{\rho,s}(R-(m+1)d)$ and (281) at level $m+1$. Next, set $\delta^{m+2} F^{(m+1)} :=$ (283) + (284). Then we can use (274) and (278), in order to get

$$\begin{aligned} &\sup_{\mathcal{B}_{\rho,s}(R-(m+1)d)} \|X_{\delta^{m+2} F^{(m+1)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \qquad (287)\\ &\leq \left(\frac{10}{d} \delta^m K_s^m \, \delta F + \frac{5}{d} \delta F \sum_{i=0}^{m-1} \delta^i K_s^i + \frac{5}{d} \delta F \, \delta^m K_s^m \right) \delta^{m+1} \sup_{\mathcal{B}_{\rho,s}(R-md)} \|X_{\chi_m}(\zeta)\|_{\mathcal{H}^{\rho,s}} \\ &= \delta^{m+2} \left(\frac{10}{d} \delta^m K_s^m \, F + \frac{5}{d} F \sum_{i=0}^{m-1} \delta^i K_s^i + \frac{5}{d} F \, \delta^m K_s^m \right) \sup_{\mathcal{B}_{\rho,s}(R-md)} \|X_{\chi_m}(\zeta)\|_{\mathcal{H}^{\rho,s}}. \qquad (288) \end{aligned}$$

If $m = 0$, then the second term is not present, and (288) reads

$$\sup_{\mathcal{B}_{\rho,s}(R-d)} \|X_{\delta^2 F^{(1)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} \leq \delta^2 \, \frac{15}{d} \, F \, 2\pi F < \delta^2 K_s F.$$

If $m \geq 1$, we exploit the smallness condition $\delta K_s < 1/2$, and (288) reads

$$\sup_{\mathcal{B}_{\rho,s}(R-(m+1)d)} \|X_{\delta^{m+2} F^{(m+1)}}(\zeta)\|_{\mathcal{H}^{\rho,s}} < \frac{18}{d} \delta F \, 2\pi \, \delta F \, \delta^m K_s^m = \delta^{m+2} \, F K_s^{m+1}.$$

□

Now fix $R > 0$.

Proof of Lemma 3.7. The Hamiltonian (61) satisfies the assumptions of Lemma A.9 with $m = 0$, $F_{1,M}$ in place of $F^{(0)}$, $F = K_{1,s}^{(F)} \, M^2$. So we apply Lemma A.9 with $d = R/4$, provided that

$$\frac{8\pi}{R} \, 18 F \delta < \frac{1}{2},$$

which is true due to (71). Hence there exists an analytic canonical transformation $\mathcal{T}_{\delta,M}^{(1)} : \mathcal{B}_{\rho,s}(3R/4) \to \mathcal{B}_{\rho,s}(R)$ with

$$\sup_{\mathcal{B}_{\rho,s}(3R/4)} \|\mathcal{T}_{\delta,M}^{(1)}(\zeta) - (\zeta)\|_{\mathcal{H}^{\rho,s}} \leq 2\pi F \, \delta,$$

such that

$$
\begin{align}
& H_{1,M} \circ \mathcal{T}^{(1)}_{\delta,M} = h_0 + \delta Z^{(1)}_M + \delta^2 \mathcal{R}^{(1)}_M , \tag{289}\\
& Z^{(1)}_M := \langle F_{1,M} \rangle , \tag{290}\\
& \delta^2 \mathcal{R}^{(1)}_M := \delta^2 F^{(1)} \nonumber\\
& = \left(h_0 \circ \mathcal{T}^{(1)}_{\delta,M} - h_0 - \delta \{ \chi_1 , h_0 \} \right) + \delta \left(Z^{(1)}_M \circ \mathcal{T}^{(1)}_{\delta,M} - Z^{(1)}_M \right) \nonumber\\
& \quad + \delta^2 \left(F_{1,M} \circ \mathcal{T}^{(1)}_{\delta,M} - F_{1,M} \right) , \tag{291}\\
& \sup_{\mathcal{B}_{\rho,s}(3R/4)} \| X_{Z^{(1)}_M} (\zeta) \|_{\mathcal{H}^{\rho,s}} \leq F, \tag{292}\\
& \sup_{\mathcal{B}_{\rho,s}(3R/4)} \| X_{\mathcal{R}^{(1)}_N} (\zeta) \|_{\mathcal{H}^{\rho,s}} \leq \frac{8\pi}{R} \, 18 F^2 . \tag{293}
\end{align}
$$

□

Appendix B. Proof of Proposition 6.2

In order to prove Proposition 6.2 we first discuss the specific energies associated to the high modes, and then the ones associated to the low modes.

First we remark that for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ we have

$$
\begin{align}
\left| \frac{\omega_k^2}{\mu^2} \right| & \overset{(11)}{=} \frac{4}{\mu^2} \left[\sin^2 \left(\frac{k_1 \pi}{2N+1} \right) + \sin^2 \left(\frac{k_2 \pi}{2N+1} \right) \right] \nonumber\\
& = \frac{4}{\mu^2} \left[\sin^2 \left(\frac{\mu K_1 \pi}{2} \right) + \sin^2 \left(\frac{\mu^\sigma K_2 \pi}{2} \right) \right] \nonumber\\
& \leq \pi^2 (K_1^2 + \mu^{2(\sigma-1)} K_2^2); \tag{294}
\end{align}
$$

moreover, for $K_1 \neq 0$

$$
\begin{align}
\frac{|\hat{q}_K|^2 + \pi^2 (K_1^2 + \mu^{2(\sigma-1)} K_2^2) |\hat{p}_K|^2}{2} & \leq \pi^2 \, e^{-2\rho|K|} \, \frac{|\hat{q}_K|^2 + (K_1^2 + \mu^{2(\sigma-1)} K_2^2) |\hat{p}_K|^2}{2} e^{2\rho|K|} \nonumber\\
& \leq \pi^2 \, e^{-2\rho|K|} \, \left(1 + \mu^{2(\sigma-1)} \frac{K_2^2}{K_1^2} \right) \, \| (\xi, \eta) \|^2_{\mathcal{H}^{\rho,0}} , \tag{295}
\end{align}
$$

while for $|K_2| \leq |K_1|$

$$
\frac{|\hat{q}_K|^2 + \pi^2 (K_1^2 + \mu^{2(\sigma-1)} K_2^2) |\hat{p}_K|^2}{2} \overset{|K_2| \leq |K_1|}{\leq} 2\pi^2 \, e^{-2\rho|K|} \, \| (\xi, \eta) \|^2_{\mathcal{H}^{\rho,0}} . \tag{296}
$$

Hence, by (163) we obtain that for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ and $|K_1| + |K_2| > \frac{(2+\delta)|\log\mu|}{\rho}$

$$
\begin{aligned}
&\frac{\mathcal{E}_\kappa}{\mu^4} \\
&= \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|\le|K_1+L_1|}} \left(|\hat{q}_{K+L}|^2 + \omega_k^2\left|\frac{\hat{p}_{K+L}}{\mu}\right|^2\right) + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|}} \left(|\hat{q}_{K+L}|^2 + \omega_k^2\left|\frac{\hat{p}_{K+L}}{\mu}\right|^2\right) \\
&\overset{(294),(296),(94)}{\le} \pi^2 \left\|(\xi,\eta)\right\|^2_{\mathcal{H}^{\rho,0}}\, 2 \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|\le|K_1+L_1|}} e^{-2\rho|K+L|} \\
&\quad + \pi^2 \left\|(\xi,\eta)\right\|^2_{\mathcal{H}^{\rho,0}} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0}} e^{-2\rho|K+L|}\left(1+\mu^{2(\sigma-1)}\frac{(K_2+L_2)^2}{(K_1+L_1)^2}\right) \\
&\quad + \pi^2 \left\|(\xi,\eta)\right\|^2_{\mathcal{H}^{\rho,0}} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1=0}} e^{-2\rho|K_2+L_2|}.
\end{aligned}
$$

Now,

$$
\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|} \tag{297}
$$

$$
\begin{aligned}
&\le e^{-2\rho|K|} + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|} + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1\neq 0,L_2=0}} e^{-2\rho|K+L|} \\
&\quad + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|}.
\end{aligned} \tag{298}
$$

We now estimate the last sum in (298); we point out that for $L_1, L_2 \neq 0$ we have

$$
|L| \ge \frac{2}{\mu} + \frac{2}{\mu^\sigma},
$$

hence

$$
2|K| \le |L|. \tag{299}
$$

Therefore, for any k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ and $|K_1| + |K_2| \geq \frac{(2+\delta)|\log \mu|}{\rho}$

$$
\begin{aligned}
\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|} &\leq \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{-2\rho\,|\,|K|-|L|\,|} \\
&\leq \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{2\rho|K|} e^{-2\rho|L|} \\
&\leq e^{2\rho|K|}\, 2\pi \int_{2|K|}^{+\infty} R e^{-2\rho R} \mathrm{d}R \\
&= 2\pi\, e^{2\rho|K|} \left(-\frac{1}{2}\right) \frac{\mathrm{d}}{\mathrm{d}\rho} \left[\int_{2|K|}^{+\infty} e^{-2\rho R} \mathrm{d}R\right] \\
&= -\pi\, e^{2\rho|K|} \frac{\mathrm{d}}{\mathrm{d}\rho} \left(\frac{e^{-4\rho|K|}}{2\rho}\right) \\
&= -\pi e^{2\rho|K|} \left(-\frac{1}{2\rho^2} e^{-4\rho|K|} - 2|K|\, e^{-4\rho|K|}\right) \\
&= \pi \left(\frac{1}{2\rho^2} + 2|K|\right) e^{-2\rho|K|}.
\end{aligned}
\tag{300}
$$

Next we estimate the second sum in (298); we have

$$
\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1\neq 0, L_2=0}} e^{-2\rho|K+L|} \leq e^{-2\rho\,(|K_1|+|K_2|)} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu},
\tag{301}
$$

which is exponentially small with respect to μ. Similarly,

$$
\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0, L_2\neq 0}} e^{-2\rho|K+L|} \leq e^{-2\rho\,(|K_1|+|K_2|)} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu^\sigma}.
\tag{302}
$$

Then,

$$
\begin{aligned}
&\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0}} e^{-2\rho|K+L|}\,\frac{(K_2+L_2)^2}{(K_1+L_1)^2}\\
&\le e^{-2\rho|K|}\left(\frac{K_2}{K_1}\right)^2\\
&\quad+\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0\\ L_1\neq 0,L_2=0}} e^{-2\rho|K+L|}\,\frac{(K_2+L_2)^2}{(K_1+L_1)^2}+\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|}\,\frac{(K_2+L_2)^2}{(K_1+L_1)^2}\\
&\quad+\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|}\,\frac{(K_2+L_2)^2}{(K_1+L_1)^2}.
\end{aligned}
\tag{303}
$$

First we estimate the last term in (303): we have that $|L+K|\ge|K|$, hence

$$
\begin{aligned}
&\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|}\,\frac{(K_2+L_2)^2}{(K_1+L_1)^2}\\
&=\int_{|K|}^{+\infty}\int_0^{\pi/4} e^{-2\rho\,\xi}\,\xi\tan^2\phi\,\mathrm{d}\phi\,\mathrm{d}\xi\\
&=\left(1-\frac{\pi}{4}\right)\,e^{-2\rho|K|}\,\frac{1+2\rho|K|}{4\rho^2}\\
&\le\left(1-\frac{\pi}{4}\right)\,\mu^4\,e^{-2\rho\left[|K|-\frac{2|\log\mu|}{\rho}-\frac{1}{2\rho}\log(2\rho|K|)\right]}\\
&\overset{\delta<1-1/e}{\le}\left(1-\frac{\pi}{4}\right)\,\mu^4\,e^{-2\rho\,\left[\delta|K|-\frac{2|\log\mu|}{\rho}\right]}
\end{aligned}
\tag{304}
$$

Now we bound the other two nontrivial terms in (303); on the one hand, we notice that

$$
\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0\\ L_1\neq 0,L_2=0}} e^{-2\rho|K+L|}L_2^2
\tag{305}
$$

vanishes, while on the other hand

$$\begin{aligned}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ |K_2+L_2|>|K_1+L_1|\\ K_1+L_1\neq 0\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|}L_2^2 &\leq e^{-2\rho|K|}\sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu^\sigma}\,\frac{\ell^2}{\mu^{2\sigma}}\\ &\leq 2e^{-2\rho|K|}\int_1^{+\infty} e^{-4\rho|\ell|/\mu^\sigma}\,\frac{\ell^2}{\mu^{2\sigma}}\,\mathrm{d}\ell, \qquad (306)\end{aligned}$$

where the last integral is exponentially small with respect to μ.

On the other hand, for any k such that $\kappa(k)=(\mu K_1,\mu^\sigma K_2)$ and $|K_1|+|K_2|\leq\frac{(2+\delta)|\log\mu|}{\rho}$

$$\begin{aligned}&\left|\frac{\mathcal{E}_\kappa}{\mu^4}-\frac{|\hat\xi_K|^2+|\hat\eta_K|^2}{2}\right|\\ &\leq\left|\frac{\omega_k^2-\mu^2\,\pi^2K_1^2}{2\mu^2}\right||\hat p_K|^2+\frac12\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}|\hat q_{K+L}|^2+\omega_k^2\left|\frac{\hat p_{K+L}}{\mu}\right|^2,\\ &\overset{(294)}{\leq}(\mu^2\pi^4K_1^4+\pi^2\mu^{2(\sigma-1)}K_2^2)|\hat p_K|^2\\ &\quad+\frac12\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}|\hat q_{K+L}|^2+\pi^2[(K_1+L_1)^2+\mu^{2(\sigma-1)}(K_2+L_2)^2]|\hat p_{K+L}|^2,\\ &\leq\left(\pi^4\,\mu^2K_1^4+\pi^2\mu^{2(\sigma-1)}\,\frac{9|\log\mu|^2}{\rho^2}\right)|\hat p_K|^2\\ &\quad+\frac12\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}|\hat q_{K+L}|^2+\pi^2[(K_1+L_1)^2+\mu^{2(\sigma-1)}(K_2+L_2)^2]|\hat p_{K+L}|^2,\\ &\leq\left(\pi^4\,\mu^2+\pi^2\mu^{2(\sigma-1)}\right)\frac{9|\log\mu|^2}{\rho^2}\,2\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \qquad (307)\\ &\quad+\frac{\pi^2}{2}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}} e^{2\rho|K+L|}\,(|\hat\xi_{K+L}|^2+|\hat\eta_{K+L}|^2)\left(1+2\mu^{2(\sigma-1)}\frac{K_2^2+L_2^2}{(K_1+L_1)^2}\right)e^{-2\rho|K+L|}, \qquad (308)\end{aligned}$$

and we can conclude by estimating (307) by exploiting the fact that $|\log\mu|\leq\mu^{-2/5}$, while we can estimate (308) by

$$\begin{aligned}&\frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}\left(1+2\mu^{2(\sigma-1)}\frac{K_2^2+L_2^2}{(K_1+L_1)^2}\right)e^{-2\rho|K+L|}\\ &\leq\frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}\left(1+2\,\mu^{2(\sigma-1)}\,K_2^2+2\,\mu^{2(\sigma-1)}\,L_2^2\right)e^{-2\rho|K+L|}\\ &\leq\frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\left[(1+2\,\mu^{2(\sigma-1)}\,K_2^2)2\pi\int_{2/\mu}^{+\infty}e^{-2\rho\ell}\,\ell\mathrm{d}\ell+4\pi\int_{2/\mu}^{+\infty}e^{-2\rho\ell}\,\ell^3\mathrm{d}\ell\right]\\ &=\frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\times\\ &\quad\left[2\pi\left(1+2\,\mu^{2(\sigma-1)}\,\frac{9|\log\mu|^2}{\rho^2}\right)e^{-4\rho/\mu}\,\frac{\mu+4\rho}{4\mu\rho^2}+4\pi\,e^{-4\rho/\mu}\,\frac{3\mu^3+12\rho\mu^2+24\rho^2\mu+32\rho^3}{8\mu^3\rho^4}\right]. \qquad (309)\end{aligned}$$

Appendix C. Proof of Proposition 6.8

We argue as in the proof of Proposition (6.2).

First we remark that for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ we have

$$
\begin{aligned}
|\omega_k^2| &\overset{(30)}{=} 1 + 4\left[\sin^2\left(\frac{k_1\pi}{2N+1}\right) + \sin^2\left(\frac{k_2\pi}{2N+1}\right)\right] \\
&= 1 + 4\left[\sin^2\left(\frac{\mu K_1\pi}{2}\right) + \sin^2\left(\frac{\mu^\sigma K_2\pi}{2}\right)\right] \\
&\le 1 + \pi^2(\mu^2\, K_1^2 + \mu^{2\sigma}\, K_2^2), \\
&\le \pi^2(1 + \mu^2\, K_1^2 + \mu^{2\sigma}\, K_2^2), \qquad (310)
\end{aligned}
$$

hence

$$
\begin{aligned}
\frac{|\hat{p}_K|^2 + \pi^2(1+\mu^2K_1^2+\mu^{2\sigma}K_2^2)|\hat{q}_K|^2}{2} &\le \pi^2\, e^{-2\rho|K|}\, \frac{|\hat{p}_K|^2 + (1+\mu^2K_1^2+\mu^{2\sigma}K_2^2)|\hat{q}_K|^2}{2} e^{2\rho|K|} \\
&\le \pi^2\, e^{-2\rho|K|}\, \left(1+\mu^2\, K_1^2+\mu^{2\sigma}\, K_2^2\right)\, \|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,0}}. \qquad (311)
\end{aligned}
$$

Hence, by (218) we obtain that for all k such that $\kappa(k) = (\mu K_1, \mu^\sigma K_2)$ and $|K_1| + |K_2| > \frac{(2+\delta)|\log\mu|}{\rho}$

$$
\begin{aligned}
&\frac{\mathcal{E}_\kappa}{\mu^2} \\
&\le \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} \left(|\hat{p}_{K+L}|^2 + \omega_k^2\,|\hat{q}_{K+L}|^2\right) \qquad (312)\\
&\overset{(310),(311)}{\le} \pi^2\, \|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,0}}\, 2 \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|}\left[1+\mu^2\,(K_1+L_1)^2+\mu^{2\sigma}\,(K_2+L_2)^2\right],
\end{aligned}
$$

where the sum in (312) can be rewritten as follows,

$$
\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|} \qquad (313)
$$

$$
+\,\mu^2 \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|}(K_1+L_1)^2 \qquad (314)
$$

$$
+\,\mu^{2\sigma} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|}(K_2+L_2)^2. \qquad (315)
$$

Now,

$$
\begin{aligned}
&\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|} \\
&\le e^{-2\rho|K|} + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|} + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1\neq 0,L_2=0}} e^{-2\rho|K+L|} \\
&\quad + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|}, \qquad (316)
\end{aligned}
$$

and we can estimate the above terms as for (298) in Proposition 6.2; indeed, by (300), (301) and (302) we have that (316) is bounded by

$$\begin{aligned} & e^{-2\rho|K|} + \pi\left(\frac{1}{2\rho^2} + 2|K|\right) e^{-2\rho|K|} + e^{-2\rho\,(|K_1|+|K_2|)} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu} \\ & + e^{-2\rho\,(|K_1|+|K_2|)} \sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu^\sigma}. \end{aligned} \tag{317}$$

Now we estimate (314). We have

$$\begin{aligned} & \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}}} e^{-2\rho|K+L|}\,(K_1+L_1)^2 \\ & \le e^{-2\rho|K|}\,K_1^2 \\ & \quad + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1\neq 0, L_2=0}} e^{-2\rho|K+L|}\,(K_1+L_1)^2 + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0, L_2\neq 0}} e^{-2\rho|K+L|}\,K_1^2 \\ & \quad + \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|}\,(K_1+L_1)^2. \end{aligned} \tag{318}$$

First we estimate the last term in (318): we have that $|L+K|\ge|K|$, hence

$$\begin{aligned} & \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{(2+\delta)|\log\mu|}{\rho}\\ L_1,L_2\neq 0}} e^{-2\rho|K+L|}\,(K_1+L_1)^2 \\ & = \int_{|K|}^{+\infty}\int_0^{2\pi} e^{-2\rho\,\xi}\,\xi\cos^2\phi\,\mathrm{d}\phi\,\mathrm{d}\xi \\ & = \pi\,e^{-2\rho|K|}\,\frac{1+2\rho|K|}{4\rho^2} \\ & \le \pi\,\mu^4\,e^{-2\rho\left[|K|-\frac{2|\log\mu|}{\rho}-\frac{1}{2\rho}\log(2\rho|K|)\right]} \\ & \overset{\delta<1-1/e}{\le} \pi\,\mu^4\,e^{-2\rho\,\left[\delta|K|-\frac{2|\log\mu|}{\rho}\right]} \end{aligned} \tag{319}$$

Now we bound the other two nontrivial terms in (318); on the one hand, we notice that

$$\begin{aligned} & \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0, L_2\neq 0}} e^{-2\rho|K+L|}(K_1+L_1)^2 \\ & \le 2 \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0, L_2\neq 0}} e^{-2\rho|K+L|}\,K_1^2 \\ & + 2 \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0, L_2\neq 0}} e^{-2\rho|K+L|}\,L_1^2, \end{aligned} \tag{320}$$

where the first sum can be bounded as the second term in (316), while

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|}L_1^2 \leq e^{-2\rho|K|}\sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu}\,\frac{\ell^2}{\mu^2}$$

$$\leq 2e^{-2\rho|K|}\int_1^{+\infty} e^{-4\rho|\ell|/\mu}\,\frac{\ell^2}{\mu^2}\,\mathrm{d}l, \tag{321}$$

where the last integral is exponentially small with respect to μ.

Similarly,

$$\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2:\mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}\\ |K_1|+|K_2|>\frac{2|\log\mu|}{\rho}\\ L_1=0,L_2\neq 0}} e^{-2\rho|K+L|}L_2^2 \leq e^{-2\rho|K|}\sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-4\rho|\ell|/\mu^\sigma}\,\frac{\ell^2}{\mu^{2\sigma}}$$

$$\leq 2e^{-2\rho|K|}\int_1^{+\infty} e^{-4\rho|\ell|/\mu^\sigma}\,\frac{\ell^2}{\mu^{2\sigma}}\,\mathrm{d}\ell, \tag{322}$$

where the last integral is exponentially small with respect to μ.

On the other hand, for any k such that $\kappa(k)=(\mu K_1,\mu^\sigma K_2)$ and $|K_1|+|K_2|\leq\frac{(2+\delta)|\log\mu|}{\rho}$

$$\left|\frac{\mathcal{E}_\kappa}{\mu^2}-\frac{|\hat{\psi}_K|^2}{2}\right|$$

$$\leq\left|\omega_k^2-1\right|\left|\hat{q}_K\right|^2+\frac{1}{2}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}\left|\hat{p}_{K+L}\right|^2+\omega_k^2\left|\hat{q}_{K+L}\right|^2,$$

$$\overset{(310)}{\leq}(\mu^2\pi^2K_1^2+\pi^2\mu^{2\sigma}K_2^2)|\hat{p}_K|^2$$

$$+\frac{1}{2}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}}|\hat{p}_{K+L}|^2+|\hat{q}_{K+L}|^2+\pi^2[\mu^2(K_1+L_1)^2+\mu^{2\sigma}(K_2+L_2)^2]|\hat{q}_{K+L}|^2,$$

$$\leq\left(\pi^2\mu^2K_1^2+\pi^2\mu^{2\sigma}K_2^2\right)|\hat{p}_K|^2$$

$$+\|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,0}}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}} e^{-2\rho|K+L|}[1+\pi^2\mu^2(K_1+L_1)^2+\pi^2\mu^{2\sigma}(K_2+L_2)^2]$$

$$\leq\pi^2\mu^2\left(1+\mu^{2(\sigma-1)}\right)\frac{9|\log\mu|^2}{\rho^2}\|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,0}} \tag{323}$$

$$+\|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,0}}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}} e^{-2\rho|K+L|} \tag{324}$$

$$+\pi^2\mu^2\|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,0}}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}} e^{-2\rho|K+L|}(K_1+L_1)^2 \tag{325}$$

$$+\pi^2\mu^{2\sigma}\|(\psi,\bar{\psi})\|^2_{\mathcal{H}^{\rho,0}}\sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^\sigma L_2\in 2\mathbb{Z}}} e^{-2\rho|K+L|}(K_2+L_2)^2 \tag{326}$$

(327)

and we can conclude by estimating (323) by exploiting the fact that $|\log\mu| \le \mu^{-2/5}$, while we can bound (324)-(325) by

$$\begin{aligned}
&\frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^{\sigma} L_2\in 2\mathbb{Z}}} [1+\mu^2(K_1+L_1)^2]\,e^{-2\rho|K+L|}\\
&\frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^{\sigma} L_2\in 2\mathbb{Z}}} (1+2\,\mu^2\,K_1^2+2\,\mu^2\,L_1^2)\,e^{-2\rho|K+L|}\\
&\le \frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}} \left[(1+2\,\mu^2\,K_1^2)\,2\pi\int_{2/\mu}^{+\infty} e^{-2\rho\ell}\,\ell\mathrm{d}\ell + 4\pi\,\mu^2\int_{2/\mu}^{+\infty} e^{-2\rho\ell}\,\ell^3\mathrm{d}\ell\right]\\
&= \frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\times\\
&\quad\left[2\pi\left(1+2\mu^2\frac{9|\log\mu|^2}{\rho^2}\right) e^{-4\rho/\mu}\,\frac{\mu+4\rho}{4\mu\rho^2} + 4\pi\,\mu^2 e^{-4\rho/\mu}\,\frac{3\mu^3+12\rho\mu^2+24\rho^2\mu+32\rho^3}{8\mu^3\rho^4}\right],
\end{aligned}\tag{328}$$

and we can estimate (326) by

$$\begin{aligned}
&\frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\,\mu^{2(\sigma-1)} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^{\sigma} L_2\in 2\mathbb{Z}}} (K_2+L_2)^2\,e^{-2\rho|K+L|}\\
&\frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\,\mu^{2(\sigma-1)} \sum_{\substack{L=(L_1,L_2)\in\mathbb{Z}^2\setminus\{0\}\\ \mu L_1,\mu^{\sigma} L_2\in 2\mathbb{Z}}} (2K_2^2+2L_2^2)\,e^{-2\rho|K+L|}\\
&\le \frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\,\mu^{2(\sigma-1)} \left[2K_1^2\,2\pi\int_{2/\mu^{\sigma}}^{+\infty} e^{-2\rho\ell}\,\ell\mathrm{d}\ell + 4\pi\int_{2/\mu^{\sigma}}^{+\infty} e^{-2\rho\ell}\,\ell^3\mathrm{d}\ell\right]\\
&= \frac{\pi^2}{2}\,\|(\xi,\eta)\|^2_{\mathcal{H}^{\rho,0}}\,\mu^{2(\sigma-1)}\times\\
&\quad\left[2\pi\,2\frac{9|\log\mu|^2}{\rho^2}\,e^{-4\rho/\mu^{\sigma}}\,\frac{\mu^{\sigma}+4\rho}{4\mu^{\sigma}\rho^2} + 4\pi\,e^{-4\rho/\mu^{\sigma}}\,\frac{3\mu^{3\sigma}+12\rho\mu^{2\sigma}+24\rho^2\mu^{\sigma}+32\rho^3}{8\mu^{3\sigma}\rho^4}\right].
\end{aligned}\tag{329}$$

APPENDIX D. MORE REGIMES FOR TWO-DIMENSIONAL LATTICES

D.1. **The mKdV regime.** One can also study the β model (namely, (76) with $\alpha = 0$ and $\beta \neq 0$) in the following regime,

(mKdV) the β model in the very weakly transverse regime,

$$Q(t,x) = \mu\, q(\mu^3 t, \mu x_1, \mu^{\sigma} x_2)\tag{330}$$

with $\mu \ll 1$, $2 < \sigma$.

The computations for the mKdV regime (330) follow closely the ones of Sec. 4.1 for the KdV regime. We introduce again the rescaled variables $\tau = \mu t$, $y_1 = \mu x_1$, $y_2 = \mu^{\sigma} x_2$, and the domain I as in (80).

If we plug the following ansatz

$$Q(t,x) = \mu q(\mu t, \mu x_1, \mu^{\sigma} x_2),\tag{331}$$

into (76), we get

$$q_{\tau\tau} = \frac{\Delta_{\mu,y_1,\sigma}}{\mu^2}\left(q + \mu^2\beta q^3\right),\tag{332}$$

where $\Delta_{\mu,y_1,\sigma}$ is the operator introduced in (82). Eq. (332) is a Hamiltonian PDE with the following corresponding Hamiltonian,

$$K_2(q,p) = \int_I \frac{-p\,\Delta_{\mu,y_1,\sigma}p}{2\mu^2} + \frac{q^2}{2} + \beta\mu^2\frac{q^4}{4}\mathrm{d}y,\tag{333}$$

where p is the variable canonically conjugated to q.

Recalling that (94) holds true, one can exploit the non-canonical change of coordinates $(q,p) \mapsto (\xi,\eta)$ introduced in (90)-(91) and the Poisson tensor (92), thus obtaining that

$$K_2(\xi,\eta) \sim h_0(\xi,\eta) + \mu^2 F_1(\xi,\eta) + \mu^4 \mathcal{R}(\xi,\eta), \tag{334}$$

where h_0 is the same as in (97), while

$$F_1(\xi,\eta) = \int_I -\frac{[\partial_{y_1}(\xi-\eta)]^2}{48} + \beta\frac{(\xi+\eta)^4}{2^4}\mathrm{d}y. \tag{335}$$

Therefore we can apply the averaging Theorem 3.3 to the Hamiltonian (334), with $\delta = \mu^2$. Along the lines of the proof of Proposition (100) we get

$$\langle F_1 \rangle (\xi,\eta) = -\int_I \frac{(\partial_{y_1}\xi)^2 + (\partial_{y_1}\eta)^2}{48}\mathrm{d}y + \frac{\beta}{16}([\xi^4] + 6[\xi^2][\eta^2] + [\eta^4]), \tag{336}$$

where we denote by $[f^j]$ the average $\int_I f^j(y)\frac{\mathrm{d}y}{4}$.

Therefore, the equations of motion associated to $h_0(\xi,\eta) + \mu^2 \langle F_1 \rangle (\xi,\eta)$ are given by

$$\begin{cases} \xi_\tau &= -\left(1+\frac{3}{4}[\eta^2]\right)\partial_{y_1}\xi - \frac{\mu^2}{24}\partial^3_{y_1}\xi - \frac{\mu^2\beta}{4}\partial_{y_1}(\xi^3) \\ \eta_\tau &= \left(1+\frac{3}{4}[\xi^2]\right)\partial_{y_1}\eta + \frac{\mu^2}{24}\partial^3_{y_1}\eta + \frac{\mu^2\beta}{4}\partial_{y_1}(\eta^3) \end{cases}. \tag{337}$$

which is a system of two uncoupled mKdV equations in translating frames with respect to the direction y_1.

The integrability properties of the mKdV equation and the existence of Birkhoff coordinates for this model have been proved in [KST08].

D.2. **The two-dimensional NLS regime.** We consider the Hamiltonian (24) in the following regime,

(2-D NLS) the scalar model (24) with $m = 1$, $p = 1$ and

$$Q(t,x) = \mu\, q(\mu^2 t, \mu x) \tag{338}$$

with $\mu \ll 1$ and $\sigma = 1$.

We introduce the rescaled variable $y = \mu x$, and we define I as in (80). The Hamiltonian (24) in the rescaled variable is given by

$$K_5(q,p) = \int_I \frac{p^2}{2} + \frac{q^2}{2} - \frac{q\,\Delta_\mu q}{2} + \beta\mu^2\frac{q^4}{4}\mathrm{d}y, \tag{339}$$

$$\Delta_\mu := 4\sinh^2\left(\frac{\mu\partial_{y_1}}{2}\right) + 4\sinh^2\left(\mu\frac{\partial_{y_2}}{2}\right), \tag{340}$$

and p is the variable canonically conjugated to q. The corresponding equation of motion are given by

$$q_{tt} = -q + \Delta_\mu q - \beta\mu^2 q^3. \tag{341}$$

Observe that the the operator Δ_μ admits the following asymptotic expansion up to terms of order $\mathcal{O}(\mu^4)$,

$$\frac{\Delta_\mu}{\mu^2} \sim \Delta + \mathcal{O}(\mu^2), \tag{342}$$

Therefore the Hamiltonian (339) admits the following asymptotic expansion

$$K_5(q,p) \sim \hat{h}_0(q,p) + \mu^2\hat{F}_1(q,p) + \mu^4\hat{\mathcal{R}}(q,p), \tag{343}$$

$$\hat{h}_0(q,p) = \int_I \frac{p^2+q^2}{2}\mathrm{d}y, \tag{344}$$

$$\hat{F}_1(q,p) = \int_I -\frac{q\,\Delta q}{2} + \beta\frac{q^4}{4}\mathrm{d}y, \tag{345}$$

and the equation of motion associated to $h_0 + F_1$ is given by the following cubic nonlinear Klein-Gordon (NLKG) equation,

$$q_{tt} = -(q - \mu^2\Delta q) - \mu^2\beta q^3. \tag{346}$$

We now exploit again the change of coordinates $(q,p) \mapsto (\psi,\bar\psi)$ given by (140), while the Poisson tensor is given by $-i\mathrm{d}\psi \wedge \mathrm{d}\bar\psi$. With this change of variables the Hamiltonian takes the form

$$K_5(\psi,\bar\psi) \sim h_0(\psi,\bar\psi) + \mu^2 F_1(\psi,\bar\psi) + \mu^4 \mathcal{R}(\psi,\bar\psi), \tag{347}$$

$$h_0(\psi,\bar\psi) = \int_I \psi\,\bar\psi \mathrm{d}y, \tag{348}$$

$$F_1(\psi,\bar\psi) = \int_I -\frac{(\psi+\bar\psi)\,[-\Delta(\psi+\bar\psi)]}{4} + \beta\frac{(\psi+\bar\psi)^4}{16}\mathrm{d}y. \tag{349}$$

Now we apply the averaging Theorem 3.3 to the Hamiltonian (143), with $\delta = \mu^2$. Recall that by (146) h_0 generates a periodic flow.

Proposition D.1. *Consider the Hamiltonian* (143)*. Then the average of* F_1 *with respect to the flow of* h_0 *is given by*

$$\langle F_1\rangle(\psi,\bar\psi) = \int_I \frac{\bar\psi\,(-\Delta\psi)}{2}\mathrm{d}y + \frac{3\beta}{8}\int_I |\psi|^4 \mathrm{d}y. \tag{350}$$

Corollary D.2. *The equation of motion associated to* $h_0(\psi,\bar\psi) + \mu^2\langle F_1\rangle(\psi,\bar\psi)$ *is given by the cubic nonlinear Schrödinger (NLS) equation*

$$-i\psi_t = \psi - \mu^2\,\Delta\psi + \mu^2\frac{3\beta}{4}\,|\psi|^2\psi. \tag{351}$$

The cubic NLS equation (351) on the two-dimensional torus has been extensively studied. The local well-posedness in the Sobolev space $H^s(\mathbb{T}^2)$, $s>0$, has been discussed by Bourgain in [Bou93a]; along with the conversation laws, this implies the global existence in the defocusing case ($\beta > 0$), and the global existence for small solution in the focusing case ($\beta < 0$).

The long time dynamics of the NLS equation has also been studied, especially in relation with the growth of the Sobolev norms of its solution. Indeed, since the pioneering papers by Bourgain in the 1990s, there has been much interest in the problem of growth of Sobolev norms for nonlinear Hamiltonian PDEs in dimension $d \geq 2$ [CKS$^+$10] [Han14] [GK15] [GHP16].

We also mention [CF12], in which the authors exhibit orbits of the cubic of the NLS equation which spread energy along the modes; this phenomenon, even though it describes a sort of energy cascade between the modes, does not lead to growth of Sobolev norms.

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