Diffusive behavior along mean motion resonances in the Restricted 3 Body Problem

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The Restricted Planar 3 Body Problem (RP3BP)

- Restricted three body problem: three bodies of masses $m_1, m_2 > 0$ and $m_3 = 0$ under the effect of the Newtonian gravitational force.
- The bodies with mass (primaries) are not influenced by the zero mass one.
- They form a two body problem.
- Assume they move on ellipses of eccentricity $e_0 \in (0,1)$ (RPE3BP).
- In this talk:
  - Mass ratio of the primaries $\mu = m_2/m_1 = 10^{-3}$: Realistic value for Sun–Jupiter.
  - $0 < e_0 \ll 1$: RPE3BP as a perturbation RPC3BP.

The RPE3BP

- Hamiltonian of $2\frac{1}{2}$ d.o.f

$$H(q,p,t) = \frac{\|p\|^2}{2} - \frac{1-\mu}{\|q - q_S(t)\|} - \frac{\mu}{\|q - q_J(t)\|}, \quad q, p \in \mathbb{R}^2$$

- In rotating coordinates

$$H_{\mathrm{rot}}(q,p,t) = H_{\mathrm{circ}}(q,p;\mu) + e_0 \Delta H_{\mathrm{ell}}(q,p,t;\mu,e_0)$$

- Associated flow: Φ^t.
- For $e_0 = 0$ the energy H_{rot} is conserved (Jacobi constant).
- For $0 \ll e_0 \ll 1$: Can H_{rot} drift?

Mean motion resonances

- Omit the influence of Jupiter ($\mu = 0$): the system $\equiv$ two uncoupled 2 Body Problems (Sun-Jupiter and Sun-Asteroid).
- Assume that the Asteroid is moving along an ellipse of semimajor axis a and its eccentricity $0 < e < 1$.
- Mean motion resonance:
  (period of the Asteroid)/(period of Jupiter) $\in \mathbb{Q}$.
- After normalizing, mean motion resonance appears when $a^{3/2} \in \mathbb{Q}$.
- Influence of Jupiter ($\mu = 10^{-3}$) on the shape of the Asteroid ellipse when at mean motion resonance?
- We have focused on $3:1$ the mean motion resonance.

Arnold diffusion along the 3 : 1 resonance

Theorem (Féjoz-G.-Kaloshin-Roldan 2016)

Consider the RPE3BP with $\mu = 10^{-3}$ and $0 < e_0 \ll 1$. Assume certain Ansatz. Then, there exist $T > 0$ and a point z such that the (osculating) semimajor axis a and energy H_{rot} satisfy that

$$\left| a(\Phi^t(z)) - 3^{-2/3} \right| \leq 0.149 \quad \text{for all } t \in [0, T] \qquad \text{and}$$

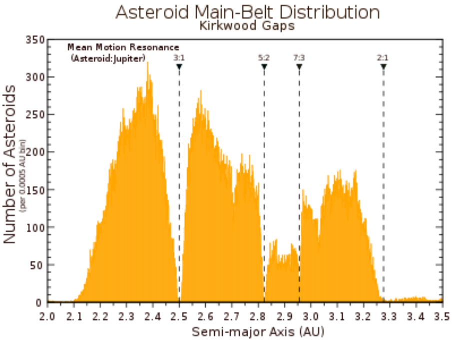
$$H_{\mathrm{rot}}(z) < -1.6 \quad \text{and} \quad H_{\mathrm{rot}}(\Phi^T(z)) > -1.36.$$

- For $e_0 = 0$, H_{rot} is constant.
- For $0 < e_0 \ll 1$: an increase in energy independent of e_0. energy.
- Drift in H_{rot} $\Rightarrow$ Drift in osculating eccentricity e:

$$e(z) < 0.59 \quad \text{and} \quad e(\Phi^T(z)) > 0.91.$$

- Ansatz verified numerically.

The Kirkwood gaps



- The Asteroid Belt is the region of the Solar System located roughly between the orbits of the planets Mars and Jupiter.

- At mean motion resonances of small order $3:1$, $2:1$, $5:2$, $7:3$, there are visible gaps in the distribution of the Asteroids, called Kirkwood gaps.

- This diffusing mechanism could give a justification of its existence.
- Neishtadt-Sidorenko (2004): Different mechanism of instability in the 3 : 1 Kirkwood gap.
- The eccentricity of Jupiter is $e_0 \sim 1/20$ and we need e_0 arbitrarily small.
- Chirikov (70's): Arnold diffusion orbits should have stochastic diffusive behavior.

Main Theorem (G.–Martín–Kaloshin–Roldan)

Assume certain Ansatz. Consider an interval $[H_-, H_+]$ (with a certain property) and a Poincaré map $\mathcal{P}$ associated to the flow Φ^t of the RPE3BP.

Then there are smooth functions $b(H)$ and $\sigma(H)$, $H \in [H_-, H_+]$ such that: for each $H^* \in (H_-, H_+)$, there exists a probability measure ν_{e_0} with the properties

- $\mathrm{dist}\,(H_{\mathrm{rot}}(z), H^*) \lesssim e_0$ for all $z \in \mathrm{supp}\nu_{e_0}$,
- It is supported inside the 3 : 1 mean motion resonance (Kirkwood gap), i.e.

$$\mathrm{dist}\left(a(z), 3^{-2/3}\right) \leq 0.149 \qquad \text{for all } z \in \mathrm{supp}\nu_{e_0}$$

such that...

Stochastic Arnold diffusion

such that:

Fix any $s > 0$. Then, the H_{rot}-distribution of the pushforward measure $\mathcal{P}_*^n \nu_{e_0}$ in the time scale

$$n_{e_0}(s) = \lfloor s\, e_0^{-2} \rfloor \qquad \text{(stopped if hits the boundary of } [H_-, H_+]),$$

converges weakly, as $e_0 \to 0$, to the distribution of $\mathcal{H}_s$, where $\mathcal{H}_\bullet$ is the diffusion process with drift $b(H)$ and variance $\sigma(H)$, i. e.

$$d\mathcal{H}_s = b(\mathcal{H})ds + \sigma(\mathcal{H})dB_s$$

(where B_s is the Brownian motion) starting at $\mathcal{H}_0 = H^*$.

Remarks

- The drift and the variance are "essentially" given by Melnikov-like integrals.
- The support of ν_{e_0} has zero Lebesgue measure.
- $[H_-, H_+] \subset [-1.6, -1.36]$
- Example: $[H_-, H_+] = [-1.591, -1.475]$
- Related results:
  - Arnold diffusion through NHIL: De la Llave (2005), Gelfreich–Turaev (2008).
  - Kaloshin–Zhang–Zhang (2015) (related works by G.-Kaloshin-Zhang and Castejon-G.-Kaloshin): Stochastic behavior for Arnold diffusion for generalized (a priori unstable) Arnold models.
  - Capinski–Gidea (2018): Stochastic behavior for Arnold diffusion for the RP3BP.

Key ingredients of the proof

- Study the RPElliptic3BP as a perturbation of RPCircular3BP for $0 < e_0 \ll 1$.
- Arnold diffusion for a priori chaotic Hamiltonian systems.
- Ansatz 1: RPC3BP has at each $H_{\mathrm{rot}} \in [-1.591, -1.475]$ level a periodic orbit with at least two transverse homoclinic points.
- Choose a subinterval $[H_-, H_+]$ such that these two transverse homoclinic points depend smoothly on H ($+$ another condition).
- Transversality of some of the invariant manifolds at some homoclinic point may fail at some discrete values of H_{rot}.
- RPE3BP $0 < e_0 \ll 1$ has a normally hyperbolic invariant cylinder with "transverse homoclinic channels"

Key ingredients of the proof

- Treschev Separatrix map to study the dynamics close to the homoclinic channels.
- Normally (weakly) hyperbolic invariant lamination localized at small neighborhoods of the homoclinic channels. Homeomorphic to

$$\text{Smale horseshoe} \times \mathbb{T} \times [H_-, H_+]$$

- Dynamics on the lamination:

$$\begin{aligned} \mathcal{F}:\ \Sigma \times \mathbb{T} \times [H_-, H_+] &\longrightarrow \Sigma \times \mathbb{T} \times \mathbb{R} \\ (\omega, \theta, H) &\mapsto (\sigma\omega, \mathcal{F}_\omega(\theta, H)) \end{aligned}$$

- e_0–expansion of $\mathcal{F}_\omega$ (up to order 2) through Melnikov-like integrals.
- They give the drift and the variance.
- Ansatz 2: Certain functions of those Melnikov-like integrals $\neq 0$

Key ingredients of the proof

- Stochastic diffusive behavior for the lamination map: Analysis of the associated martingale problem.
- Key problem: Analysis of circle extensions of hyperbolic maps

$$\begin{aligned} f:\ \Sigma \times \mathbb{T} &\longrightarrow \Sigma \times \mathbb{T} \\ (\omega, \theta) &\mapsto (\sigma\omega, \theta + \beta(\omega)) \,. \end{aligned}$$

- Exponential decay of correlations and CLT for equivariant observables (Field-Melbourne-Torok 2003).

The energy interval

- $[H_-, H_+] \subset [-1.6, -1.36]$ such that there are two "nice" homoclinic channels for the Circular Problem (no tangencies).

- Work in progress: To obtain the theorem for $[H_-, H_+] = [-1.6, -1.36]$

- One has to "join" the result in the different (overlapping) intervals using different transverse homoclinic channels.