

Asymptotic density of collision orbits in the Restricted Planar Circular 3 Body Problem

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The 3 body problem

- Consider three bodies q_1 , q_2 and q_3 with masses $m_1, m_2, m_3 > 0$,

$$\frac{d^2 q_i}{dt^2} = \sum_{j=1, j \neq i}^3 m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}, \quad q_i \in \mathbb{R}^3$$

- Long term behavior?
- Chazy (1922): Final motions – Behavior of the bodies $q_k(t)$ as $t \rightarrow \pm\infty$.

Chazy classification

- Types of final motions:
 - \mathcal{H}^+ : $|r_k| \rightarrow \infty$, $|\dot{r}_k| \rightarrow c_k \neq 0$ as $t \rightarrow +\infty$;
 - \mathcal{HP}_k^+ : $|r_k| \rightarrow \infty$, $|\dot{r}_k| \rightarrow 0$, $|\dot{r}_i| \rightarrow c_i > 0$ ($i \neq k$);
 - \mathcal{HE}_k^+ : $|r_k| \rightarrow \infty$, $|\dot{r}_i| \rightarrow c_i > 0$ ($i \neq k$), $\sup_{t \geq 0} |r_k| < \infty$;
 - \mathcal{PE}_k^+ : $|r_k| \rightarrow \infty$, $|\dot{r}_i| \rightarrow 0$ ($i \neq k$), $\sup_{t \geq 0} |r_k| < \infty$;
 - \mathcal{P}_+ : $|r_k| \rightarrow \infty$, $|\dot{r}_k| \rightarrow 0$;
 - \mathcal{B}^+ : $\sup_{t \geq 0} |r_k| < \infty$;
 - \mathcal{OS}^+ : $\limsup_{t \rightarrow \infty} \max_k |r_k| = \infty$, $\liminf_{t \rightarrow \infty} \max_k |r_k| < \infty$.
- Classification for trajectories defined for all time.
- Some orbits are not defined for all time: orbits hitting collisions.

Collision orbits

$$\frac{d^2 q_i}{dt^2} = \sum_{j=1, j \neq i}^3 m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}, \quad q_i \in \mathbb{R}^2$$

- **Collision set:** $\mathcal{C} = \{q_1 = q_2\} \cup \{q_1 = q_3\} \cup \{q_2 = q_3\}$
- **Collision orbit:** orbit which hits a collision at some time $t = t^*$.

Herman conjecture

- Fix the center of mass at the origin.
- Reparameterize the flow so that it takes infinite time to get to collision.
- **Non-wandering set:** Consider a dynamical system $\phi : X \rightarrow X$, $x \in X$ is non-wandering if for every open neighborhood U of x and any N satisfies $\phi^n(U) \cap U \neq \emptyset$ for some $n > N$.
- **Herman question:** Is the non-wandering set nowhere dense in all energy levels?
- In particular: Is the set of bounded orbits nowhere dense?

How abundant/rare are collision orbits?

- Saari 1970's (also Fleischer & Knauf 2018): The set of collision orbits has measure zero.
- Alexeev conjecture (1981): Is there an open set \mathcal{U} in phase space possessing a dense subset $\mathcal{D} \subset \mathcal{U}$ whose points lead to collision?
- This conjecture goes back to Siegel.
- If Alexeev conjecture is true, would imply a (spurious) dense set of “bounded orbits”.
- Could Alexeev conjecture (if true) lead to a negative answer to Herman conjecture?
- To understand Alexeev conjecture: consider the case $m_2 = m_3 = 0$.

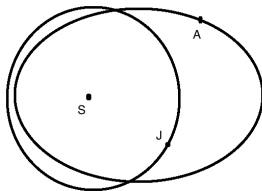
The case: $m_2 = m_3 = 0$

- Body 1 does not move.
- Body 2 and 3

$$\frac{d^2 q_i}{dt^2} = m_1 \frac{q_1 - q_i}{\|q_1 - q_i\|^3}$$

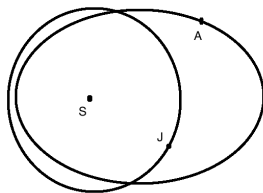
form two uncoupled 2BPs.

- Place them on ellipses.
- Take ellipses that intersect transversally.
- They form an open set in phase space foliated by 2-tori.
- All solutions are either periodic or quasi-periodic.
- Each torus contains two (invisible) collisions.



The case: $m_2 = m_3 = 0$

- If periods of q_2 and q_3 are incommensurable, collision orbits are dense in this \mathbb{T}^2 .
- Periods is $2\pi a^{3/2}$ where a is the semimajor axis of the ellipse.
- For a dense set of a 's the periods are incommensurable.
- Tori with dense collision orbits are dense in an open set.
- Alexeev conjecture is true when $m_2 = m_3 = 0$.



General case: $m_2, m_3 > 0$

- Alexeev: Does density of collision orbits still hold?
- For $m_2, m_3 > 0$ small, this is not a regular perturbation problem.
- The system blows up in a small neighborhood of collisions.
- We consider a simpler model: The Restricted Planar Circular 3 Body problem.
- In rotating coordinates

$$H(q, p) = \frac{\|p\|^2}{2} - (p_2 q_1 - p_1 q_2) - \frac{1 - \mu}{\|q + \mu\|} - \frac{\mu}{\|q - (1 - \mu)\|}$$

Collisions are asymptotically dense

Theorem (M. G. – V. Kaloshin – J. Zhang)

Consider the RPC3BP. There exists an open set $\mathcal{U} \subset \mathbb{R}^4$ and $\tau > 0$, independent of μ , such that, for μ small enough, there is a μ^τ -dense set $\mathcal{D} \subset \mathcal{U}$ whose points lead to collision.

- μ^τ dense $\equiv \mu^\tau$ neighborhoods of all points in \mathcal{D} cover \mathcal{U} .
- τ can be taken $\tau = \frac{1}{17 + \sigma}$ for any $\sigma > 0$.
- \mathcal{U} gives open sets in the energy level $H = h$ for energies

$$h \in \left(-\frac{3}{2}, \sqrt{2} \right).$$

The set \mathcal{U}

- The set \mathcal{U} can be easily characterized in terms of Delaunay coordinates.
- Then \mathcal{U} is the interior of any compact set contained in

$$\mathcal{V} = \left\{ (l, g, L, G) \in \mathbb{T}^2 \times (0, +\infty) \times (-L, 0) \cup (0, L) : \right. \\ \left. \frac{G^2}{1+e} < 1 < \frac{G^2}{1-e}, \quad H(l, g, L, G) \in \left(-\frac{3}{2}, \sqrt{2} \right) \right\}.$$

where $e = \sqrt{1 - \frac{G^2}{L^2}}$.

The set \mathcal{U}

- \mathcal{U} corresponds to where in the unperturbed case ($\mu = 0$) the ellipses of the two bodies intersect transversally.
- In particular we only consider collisions with the small primary at $(1 - \mu, 0)$.
- The same set were the existence of second species periodic solutions are looked for (Niedermaier, Marco, Bolotin, McKay,...)
- Collisions with the massive primary – Punctured tori: Chenciner, Llibre, Féjóz, Zhao.

Other almost density results: punctured tori

- Punctured tori also give an almost density result of collision orbits for the RP3BP and the full 3BP.
- However, for the RP3BP the corresponding set \mathcal{U} is very small (either its measure goes to zero as $\mu \rightarrow 0$ or two bodies are arbitrarily close).
- For the full 3BP almost density in “big” sets if one places one of the bodies very far away.

A disproof of a weak version of Herman conjecture

- **Herman question:** Is the non-wandering set nowhere dense in all energy levels?
- Consider a dynamical system $\{\phi_t\}_{t \in \mathbb{R}}$ defined on a topological space X . Then, a point $x \in X$ is called δ -non-wandering, if for any neighborhood V of it containing the δ -ball $B_\delta(x)$, there exists $T > 1$ such that $\phi_T(V) \cap V \neq \emptyset$.

Theorem (M. G. – V. Kaloshin – J. Zhang)

Any point belonging to the open set \mathcal{U} is $\mathcal{O}(\mu^\tau)$ -non wandering under the flow ϕ_t of the RPC3BP.

More concretely, for any $z \in \mathcal{U}$, we can find a $\mathcal{O}(\mu^\tau)$ -neighborhood V_μ of it and times $0 < T'_\mu < T_\mu$ such that $\phi_{T'_\mu}(V_\mu)$ is $\mathcal{O}(\mu^\tau)$ -close to a collision and $\phi_{T_\mu}(V_\mu) \cap V_\mu \neq \emptyset$.

Summarizing

- For the RPC3BP, at μ^τ scales for $\tau = \frac{1}{17 + \sigma}$, $\sigma > 0$:
 - Alexeev conjecture is correct
 - Herman conjecture is not.
- What happens for the true conjectures for the full 3BP?
- Are they incompatible?

Some ideas of the proof of almost density of collisions

- Take any point $P \in \mathcal{U}$: we want to find Q μ^τ -close to it hitting a collision.
- **Case $\mu = 0$:**
 - \mathcal{U} foliated by 2 dimensional tori.
 - Choose Q in an orbit in a non-resonant torus hitting collision (they are dense).
- Q may need a very long time to hit collision.
- **Case $\mu > 0$:** Choose a $\mu^{3\tau}$ -long curve μ^τ -close to P and show that a point in this curve hits a collision.

Some ideas of the proof: three regimes

- 1 **Far from collision** (points $\mu^{3\tau}$ away from collision) the zero mass body q (essentially) only notices the main primary: nearly integrable setting.
- 2 **Transition zone:** q notices the two primaries but orbits spend there very short time.
- 3 **Small neighborhood of the collision** ($\mu^{1/2}$ neighborhood of the collision): q (essentially) only notices the small primary – A different nearly integrable setting.

Regime 1: far from collision

- We are in a nearly integrable regime.
- Problem: the point may need a very long time to reach Regime 2.
- We apply **KAM Theory**.
- KAM is global: it cannot be applied directly due to the collisions (the Hamiltonian blows up at collision!)
- Remove the collision by multiplying H by a bump function supported at $\mu^{3\tau}$ -ball centered at the collision.
- The modified Hamiltonian is close to a 2 body problem (in low regularity).

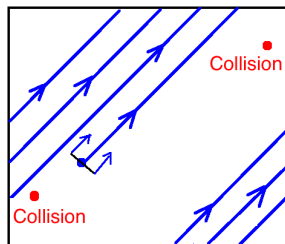
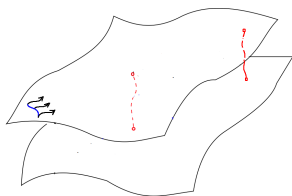
Regime 1: far from collision

- We want to apply KAM with lowest possible regularity: the more regularity, the worse estimate on the Hamiltonian with bump functions.
- Constant type frequencies are γ -dense

$$|q\omega - p| \geq \frac{\gamma}{|q|}.$$

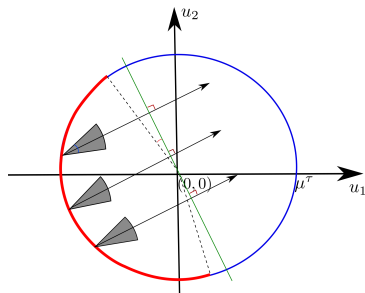
- We apply Herman version of KAM (for $\mathcal{C}^{3+\sigma}$ maps and constant type frequencies): **tori are γ -dense**.
- Each torus has two (removed) collisions.
- Orbits on the tori are true orbits of the RPC3BP as long as do not intersect a $\mu^{3\tau}$ neighborhood of the collisions.

Regime 1: How to reach well Regime 2



- We wanted: any point P has a $\mu^{3\tau}$ -long curve μ^τ -close to it such that a point in this curve hits collision.
- Take a KAM torus μ^τ close to P and $\mu^{3\tau}$ -long curve in this torus

Regime 1: How to reach well Regime 2



- The forward orbit of the small curve has to hit “well” the puncture around one of the collisions so that it can be sent forward to Regimes 2 and 3.

- Well:

- The orbit cannot have intersected before the punctures around collisions (we want a true orbit of RPC3BP!).
- The image of the segment covers the half of the boundary of the neighborhood where the velocity is pointing inwards.

Regime 1: Far from collision

- We want to optimize the density coefficient
- Small γ : gives better density of tori.
- To have the segment hitting well we need to avoid close encounters with collisions before a good hitting.
- We need a strong Diophantine condition $\rightarrow \gamma$ big.
- KAM + Non-homogeneous Dirichlet Theorem leads to

$$\gamma = \mu^\tau \quad \text{with} \quad \tau = \frac{1}{17 + \sigma}, \quad \sigma > 0.$$

Regime 2

- Regime 2: $\mu^{3\tau}$ -close to collision and $\mu^{1/2}$ -far from collision.
- It is a small annulus of width $\mu^{3\tau}$ where the two bodies are “not too close”.
- We use the true RPC3BP.
- Velocity of order ~ 1 (collisions are “far enough” to control it).
- Thus: the flow is almost tubular.
- Conclusion: the propagated segment goes from the outer to the inner boundary with **almost constant velocity**.

Regime 3

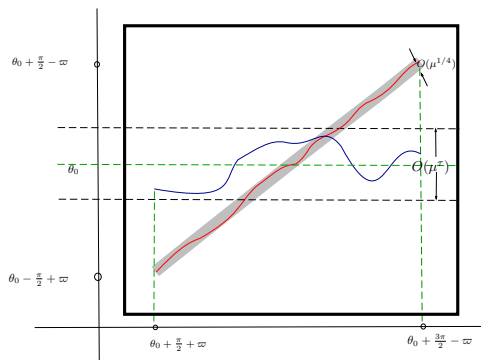
- Small primary is dominant: Flow close to a new 2 body problem (close to collision).
- By Levi-Civita regularization (+ scaling): the RPC3BP becomes

$$K(z, w) = \frac{1}{2}(|w|^2 - |z|^2) + \mu^{1/2}\mathcal{O}_4(z, w)$$

where $z = 0$ is the collision set.

- Run backwards orbits departing from collisions to the boundary between Regimes 2 and 3.
- Restricting to the level of energy, they give a curve at the boundary.

The collision orbit



- Plot the curves in the plane $(\arg(z), \arg(w))$.
- **Blue**: the incoming curve coming from Regime 1 and 2.
- **Red**: backward orbits departing from collision.

They are both \mathcal{C}^0 curves: they must intersect.