KAM theory for quasi-linear PDEs

Filippo Giuliani

BGSMath-BMS Meeting (Berlin), 27 June 2019

This research project HamInstab has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 757802.

Quasi-periodic solutions of PDEs

We are interested in finding **quasi-periodic in time** solutions of equations of the form

$$u_t = Lu + N(u), \quad u = u(t,x),$$

$$t \text{ time variable}, \; x \text{ space variable},$$

L is a **linear** operator acting on a functional space.
N is a **nonlinear** (smooth) function.

Quasi-linear PDEs

If $L = \partial_x^k$, $N = N(u, u_x, \dots, \partial_x^m u)$
Semilinear PDE: $m < k$, **Quasi-linear PDE**: $m = k$,

Examples

$$\mathrm{i}\, u_t = -u_{xx} + |u|^{2p} u, \quad L = -\partial_{xx}, \qquad x \in \mathbb{R}.$$

$$u_t = -u_{xxx} + 6u\, u_x + u\, u_{xxx}, \quad L = -\partial_{xxx}, \qquad x \in \mathbb{R}.$$

Results of existence of quasi-periodic solutions for quasi-linear PDEs are quite recent.

Quasi-periodic function

We say that a function $u(t)$ with values in some Hilbert space $\mathcal{H}$ is quasi-periodic with frequency $\omega \in \mathbb{R}^d$ if there exists a function $\mathcal{U}\colon \mathbb{T}^d \to \mathcal{H}$ such that

- $u(t) = \mathcal{U}(\omega t)$,
- ω is **irrational**, namely $\omega \cdot k \neq 0$ for all $k \in \mathbb{Z}^d, k \neq 0$.

When $d = 1$ we say that $u(t)$ is periodic.
Example: $\sin(\sqrt{2}t) + \cos(t)$ is a quasi-periodic real-valued function.

The KAM problems change a lot depending on the choice of the phase space $\mathcal{H}$. We consider PDEs under periodic boundary condition and with spatial domain of dimension 1, namely

$$x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z},$$

Sobolev regularity (in space)

$$H^s(\mathbb{T}) := \{u \in L^2(\mathbb{T}) : u(x) = \sum_{j \in \mathbb{Z}} u_j \, e^{\mathrm{i} j\, x} \,, \|u\|_s^2 := \sum_{j \in \mathbb{Z}} |u_j|^2 \, \langle j \rangle^{2s} < \infty\}$$

Quasi-periodic solution

A quasi-periodic solution of a system $u_t = X(u)$ is an embedding $\mathcal{U}\colon \mathbb{T}^d \to \mathcal{H}$ (at least C^1) such that

$$\mathcal{U} \circ \Psi^t_\omega = \Phi^t \circ \mathcal{U}, \quad \Psi^t_\omega(\varphi) := \varphi + \omega\, t$$

Φ^t is the flow of X. Note that the dynamics on the invariant torus is *prescribed*.

Some remarks:

- the flow Φ^t has to be well defined just on $\mathcal{U}(\mathbb{T}^d)$. We can deal also with ill-posed PDEs.
- this is not a Cauchy problem. We prove existence of ω-family of solutions for ω in a *large* set.
- we are looking for finite-dimensional tori in a *infinite dimensional* phase space $\mathcal{H}$.

Invariance equation

Invariance equation

We can reformulate the problem as

$$\mathcal{F}(\omega, \mathcal{U}(\omega)) := \omega \cdot \partial_\varphi \mathcal{U} - X(\mathcal{U}) = 0, \quad \mathcal{U}(\varphi) = \mathcal{U}(\varphi)[x], \quad \varphi = \omega\, t.$$

The aim is to solve this nonlinear functional equation *for some appropriate choice* of the parameter ω.

Invariance equation

We can reformulate the problem as

$$\mathcal{F}(\omega,\mathcal{U}(\omega)) := \omega \cdot \partial_\varphi \mathcal{U} - X(\mathcal{U}) = 0, \quad \mathcal{U}(\varphi) = \mathcal{U}(\varphi)[x], \quad \varphi = \omega\, t.$$

The aim is to solve this nonlinear functional equation *for some appropriate choice* of the parameter ω.

Small divisors problem

$\omega \in \mathbb{R}^d$ irrational $\Rightarrow$ the set $\{\omega \cdot \ell\}_{\ell \in \mathbb{Z}^d}$ accumulates to zero. These quantities will appear at the denominator of some Fourier coefficients.

Diophantine conditions (Kolmogorov)

We shall impose *non-resonance* conditions such as

$$|\omega \cdot \ell| \geq \frac{\gamma}{\ell^\tau}, \quad \ell \in \mathbb{Z}^d, \ell \neq 0.$$

If $\tau > d - 1$ the set of diophantine frequencies has **positive measure**.

Dynamical system approach

Dynamical system idea 1

Look for invariant manifolds (equilibrium points, periodic orbits...), on which the dynamics is simple, in order to study the behavior of the orbits close to them by using **perturbative arguments**;

We will consider quasi-periodic solutions close to *elliptic fixed points*.

$$u_t = L\,u + N(u), \quad u = u(t,x).$$

Assume that $u = 0$ is an elliptic equilibrium (**small amplitude** solutions), hence

$$\sigma(L) = \{\mathrm{i}\,\lambda_j\} \subseteq \mathrm{i}\,\mathbb{R}$$

Suppose that L is diagonal in the basis $\{e^{\mathrm{i}jx}\}_{j\in\mathbb{Z}}$ and $N(0) = N'(0) = 0$.

Infinite dimensional dynamical systems

Close to the origin, the PDE behaves like an *infinite* chain of harmonic oscillators (with frequency $\lambda(j)$) weakly coupled by the nonlinear terms

$$\dot{u}_j = \mathrm{i}\,\lambda_j\,u_j + \Big(N(u)\Big)_j, \qquad j \in \mathbb{Z}$$

Dynamical systems idea 2

Look for *approximately* invariant manifolds and we prove that close to these objects there are truly invariant manifolds

We excite a **finite number** of modes $S := \{\mathtt{j}_1, \dots, \mathtt{j}_N\}$ and we look for quasi-periodic in time solutions of the linear equation of the form

$$\sum_{j \in S} u_j \, e^{\mathrm{i}(\lambda_j t + jx)}$$

$$\lambda_j \text{ linear frequencies of oscillations}, \quad j \mapsto \lambda_j \text{ dispersion relation.}$$

These ones are approximately invariant tori for the system $u_t = X(u)$ and we search finite-dimensional invariant tori close to them $(\omega \sim (\lambda_j)_{j \in S})$.

Dispersion

The dispersion relation plays an important role in the measure estimates of the set of the good parameters ω.

Implicit function theorem fails

We try to solve $\mathcal{F}(u(\varphi, x)) = 0$ by using the IFT. We work on the space

$$H^s(\mathbb{T}^{d+1}) := \{u = u(\varphi, x) \in L^2(\mathbb{T}^{d+1}) : \|u\|_s^2 := \sum_{j\in\mathbb{Z},\ell\in\mathbb{Z}^d} |u_{j\,\ell}|^2 \langle j, \ell\rangle^{2s} < \infty\}$$

$$d_u\mathcal{F}(\omega, 0)[h] = \omega \cdot \partial_\varphi h - d_u X(0)[h] = \Big(\omega \cdot \partial_\varphi - L\Big) h$$

$$g(\varphi, x) = \sum_{\ell\in\mathbb{Z}^d, j\in\mathbb{Z}} g_{j\ell}\, e^{\mathrm{i}(\ell\cdot\varphi + jx)}, \quad \Big(\omega \cdot \partial_\varphi - L\Big) h = g$$

$$h = \Big(\omega \cdot \partial_\varphi - L\Big)^{-1} g = \sum_{\ell\in\mathbb{Z}^d, j\in\mathbb{Z}} \frac{g_{j\ell}}{\mathrm{i}\,(\omega \cdot \ell + \lambda_j)}\, e^{\mathrm{i}(\ell\cdot\varphi + jx)}$$

Since ω is irrational we have a small divisor problem. Then we impose on the parameter ω a non-resonance condition such as

$$|\omega \cdot \ell + \lambda_j| \geq \frac{\gamma}{\ell^\tau}, \quad \ell \in \mathbb{Z}^d, \ell \neq 0, j \in \mathbb{Z}.$$

Implicit function theorem fails

We try to solve $\mathcal{F}(u(\varphi, x)) = 0$ by using the IFT. We work on the space

$$H^s(\mathbb{T}^{d+1}) := \{u = u(\varphi, x) \in L^2(\mathbb{T}^{d+1}) : \|u\|_s^2 := \sum_{j \in \mathbb{Z}, \ell \in \mathbb{Z}^d} |u_{j\,\ell}|^2 \langle j, \ell \rangle^{2s} < \infty\}$$

$$d_u \mathcal{F}(\omega, 0)[h] = \omega \cdot \partial_\varphi h - d_u X(0)[h] = \Big(\omega \cdot \partial_\varphi - L\Big) h$$

$$g(\varphi, x) = \sum_{\ell \in \mathbb{Z}^d, j \in \mathbb{Z}} g_{j\ell}\, e^{\mathrm{i}(\ell \cdot \varphi + jx)}, \quad \Big(\omega \cdot \partial_\varphi - L\Big) h = g$$

$$h = \Big(\omega \cdot \partial_\varphi - L\Big)^{-1} g = \sum_{\ell \in \mathbb{Z}^d, j \in \mathbb{Z}} \frac{g_{j\ell}}{\mathrm{i}\,(\omega \cdot \ell + \lambda_j)}\, e^{\mathrm{i}(\ell \cdot \varphi + jx)}$$

Since ω is irrational we have a small divisor problem. Then we impose on the parameter ω a non-resonance condition such as

$$|\omega \cdot \ell + \lambda_j| \geq \frac{\gamma}{\ell^\tau}, \quad \ell \in \mathbb{Z}^d, \ell \neq 0, j \in \mathbb{Z}.$$

Loss of derivatives $\Big(\omega \cdot \partial_\varphi - L\Big)^{-1} : H^s(\mathbb{T}^{d+1}) \to H^{s-\tau}(\mathbb{T}^{d+1}).$

Nash-Moser scheme

We need a quadratic Newton-type scheme which works on **scale of functional spaces**.

$$\{H^s(\mathbb{T}^{d+1})\}_{s=s_0}^{S}$$

Craig-Wayne ('93) and Bourgain ('94) used a Nash-Moser algorithm to find, respectively, periodic solutions of the nonlinear wave eq. and quasi-periodic solutions of PDEs in higher dimension.

Nash-Moser iteration

$$u_{n+1} = u_n - [d_u\mathcal{F}(\omega, u_n(\omega))]^{-1}\,\mathcal{F}(\omega, u_n(\omega))$$

$\{u_n\}_n$ is a sequence of *approximate solutions* ($\mathcal{F}(u_n) \sim 0$) such that $u_n \to u$ where $\mathcal{F}(u) = 0$.

We have to invert the linearized operator $\mathcal{L}_\omega := d_u\mathcal{F}(\omega, u_n(\omega))$ in *all neighborhood of the origin* (not just at a point).

We have to provide *quantitative estimates* on $\mathcal{L}_\omega^{-1}$ in order to prove the convergence of the scheme.

Tame estimates

We have to prove bounds like

$$\|(\mathcal{L}_\omega(u_n))^{-1}h\|_s \leq C(s)\Big(\|h\|_{s+\mu} + \|u_n\|_{s+\mu}\|h\|_{s_0+\mu}\Big)$$

for $s \in [s_0, S]$, with S large. The loss of derivatives μ is the same at each step of the Nash-Moser scheme.

The **super-exponetial convergence** of the scheme + the **tame estimates** kill the loss of derivatives.

The equation $\mathcal{L}_\omega h := d_u \mathcal{F}(\omega, u_n(\omega))[h] = g$ is a quasi-periodically forced linear PDE.

KdV example

Recall $\varphi = \omega\, t$.

$$\mathcal{L}_\omega = \omega \cdot \partial_\varphi + a_3(\varphi, x)\partial_{xxx} + a_2(\varphi, x)\partial_{xx} + a_1(\varphi, x)\partial_x + a_0(\varphi, x) + \mathcal{R}(\varphi)$$

a_2, a_1, a_0 are small.
semilinear case: $a_3 = 1$, **quasi-linear** case $a_3 - 1 \ll 1$.

Reducibility

In dimension 1 it is convenient to **reduce** the operator $\mathcal{L}_\omega$, namely find bounded changes of coordinates $\Phi(\omega t) \colon H^s(\mathbb{T}) \to H^s(\mathbb{T})$ such that

$$\Phi(\omega t)\, \mathcal{L}_\omega\, \Phi(\omega t)^{-1} = \omega \cdot \partial_\varphi - \mathcal{D}, \quad \mathcal{D} := \mathrm{diag}_{j \in \mathbb{Z}}(\mathrm{i}\, d_j).$$

where d_j is close to λ_j. The problem is to construct such changes of variables and provide tame estimates for them.
Remark: this allows to prove also the **linear stability** of the solutions.

Second Melnikov conditions

The diagonalization of $\mathcal{L}_\omega$ is related to non-resonance conditions involving the difference of the eigenvalues

$$|\omega \cdot \ell + d_j - d_k| \geq \frac{\gamma}{\langle \ell \rangle^\tau}, \quad \ell \in \mathbb{Z}^d,\, j, k \in \mathbb{Z},\, (\ell, j, k) \neq (0, j, j).$$

where $d_j - \lambda_j \ll 1$.

The most difficult part is to prove that the bad set

$$\bigcup_{\ell \in \mathbb{Z}^d,\, j,k \in \mathbb{Z}} R_{\ell\, j\, k}, \qquad R_{\ell\, j\, k} := \{\omega : |\omega \cdot \ell + d_j - d_k| \leq \frac{\gamma}{\langle \ell \rangle^\tau}\}$$

has small measure.

- Estimate for the single bad set $R_{\ell\, j\, k}$;
- Summability of the series.

What about the starting points u_0 of the Nash-Moser scheme?
Usually one tries to bifurcates from the linear solutions

$$\sum_{j\in S} v_j\, e^{\mathrm{i}(\lambda(j)t+j\,x)}, \quad S\subset\mathbb{Z},\ \#S<\infty.$$

Resonant cases

The linear frequencies of oscillations $\lambda(j)$ are rational, then all the linear solutions are periodic.
We have to find other approximate solutions.

External parameters

Since we have to impose infinitely many non-resonance conditions we need some **parameters** that allow us to tune the frequencies ω.

$$\mathrm{i}u_t = u_{xx} + V\star u + f(|u|^2)u, \quad \lambda(j) = j^2 + V_j,$$

$$u_{tt} = u_{xx} + m\,u + u^3, \quad \lambda(j) = \sqrt{j^2+m}.$$

If external parameters are not present in the equation we have to extract them directly from the equation by using *Birkhoff normal form techniques.*

Literature

KAM for PDE's (semilinear perturbations)

- (**space 1d**): Kuksin ('87-'98), Wayne ('90), Craig-Wayne ('93), Pöschel ('96), Chierchia-You ('00), Berti-Biasco ('11).
  Kappeler-Pöschel ('03), Zhang-Gao-Yuan ('11), Berti-Biasco-Procesi ('13).

- (**higher space dimension**): Bourgain ('96-'98-'05), Eliasson-Kuksin ('10), Berti-Bolle ('13), Geng-Xu-You ('11), Procesi-Procesi ('12-'15), Eliasson-Grebert-Kuksin ('16).

KAM for PDE's (quasi-linear perturbations, 1d space)

- (time-**periodic**): Iooss-Plotnikov ('05), Iooss-Plotnikov-Toland ('05), Plotnikov-Toland ('01), Baldi ('13), Alazard-Baldi ('15).

- (time-**quasi-periodic**): Baldi-Berti-Montalto [*KdV-mKdV* ('14-'15)] , Feola-Procesi [*NLS* ('15)] , Berti-Montalto [*capillary WW* ('16)] ,
  G. [*Generalized quasi-linear KdV* ('17)],
  Baldi-Berti-Haus-Montalto [*gravity WW* ('17)], Feola-G.-Procesi [*DP* ('18)].

Degasperis-Procesi equation

The Degasperis-Procesi equation

$$u_t = (1 - \partial_{xx})^{-1}(4 - \partial_{xx})\partial_x \left(u - \frac{u^2}{2} + f(u) \right)$$

is a model for nonlinear shallow water phenomena. It is an *integrable* quasi-linear Hamiltonian PDE

$$u_t = J\nabla_{L^2} H(u), \quad H(u) = \int_{\mathbb{T}} u^2 + u^3 \, dx, \quad J := (1 - \partial_{xx})^{-1}(4 - \partial_{xx})\partial_x.$$

The linear frequencies of oscillations are

$$\lambda_j = \frac{j(4 + j^2)}{1 + j^2} = j + \frac{3j}{1 + j^2} \in \mathbb{Q}.$$

Twist condition

By using Birkhoff normal form techniques we find approximate solutions of the form

$$\sum_{j\in S} \sqrt{\xi_j}\, e^{\mathrm{i}(\omega(j)t+jx)}, \quad \omega(j) - \lambda_j = O(\sqrt{|\xi|})$$

where ξ_j are the **amplitudes**, $\omega := (\omega(j))_{j\in S}$ is the frequency vector (close to the linear frequencies), which depends on the ξ_j's.
The set S is called **tangential set**.

We want to use the amplitudes as parameters to modulate the frequency ω. Usually we have that

$$\omega = \lambda + \mathbb{A}\xi, \quad \lambda = (\lambda_j)_{j\in S},\ \mathbb{A} \in \mathbb{R}^{d\times d}$$

Twist condition

We impose that $\det \mathbb{A} \neq 0$. In this way $\xi \mapsto \omega(\xi)$ is a local diffeomorphism.

Unfortunately this condition does not hold for all choices of the set S.

Theorem (Feola-G.-Procesi (2018), Arxiv)

Let $f \in C^\infty$, $f(u) = O(u^9)$, fix $d \in \mathbb{N}, d \geq 2$. There exist a set $A_d \subset \mathbb{N}^d$ such that if the tangential set $S \subseteq A_d$ there exists ε_0 such that: for all $\varepsilon \in (0, \varepsilon_0)$, there exists a Cantor set

$$\mathcal{C}_\varepsilon \subset \varepsilon \left[\frac{1}{2}, \frac{3}{2}\right]^d, \quad \frac{|\mathcal{C}_\varepsilon|}{|\varepsilon \left[\frac{1}{2}, \frac{3}{2}\right]^d|} \to 1 \quad as \quad \varepsilon \to 0,$$

such that for all $\xi \in \mathcal{C}_\varepsilon$ the equation

$$u_t = (1 - \partial_{xx})^{-1}(4 - \partial_{xx})\partial_x \left(u - \frac{u^2}{2} + f(u)\right)$$

has a quasi-periodic solution with frequency ω:

$$u(t, x) = 2 \sum_{j \in S} \sqrt{\xi_j} \cos(\omega(j)t + jx) + r(\omega t, x), \quad \omega(j) = \lambda(j) + O(\varepsilon)$$

where $r(\omega t, x) = o(\sqrt{|\xi|})$ is small in some $H^s(\mathbb{T}^{d+1})$-norm for some large s. These quasi-periodic solutions are **linearly stable**.

Thanks for your attention!