Asymptotic density of collision orbits in the Restricted Planar Circular 3 Body Problem

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The 3 body problem

- Consider three bodies q_1, q_2 and q_3 with masses $m_1, m_2, m_3 > 0$,

$$\frac{d^2 q_i}{dt^2} = \sum_{j=1, j\neq i}^{3} m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}, \qquad q_i \in \mathbb{R}^3$$

- Long term behavior?

- Chazy (1922): Final motions – Behavior of the bodies $q_k(t)$ as $t \to \pm\infty$.

Chazy classification

- Types of final motions:
  - $\mathcal{H}^+$: $|r_k| \to \infty, \ |\dot{r}_k| \to c_k \neq 0$ as $t \to +\infty$;
  - $\mathcal{HP}_k^+$: $|r_k| \to \infty, \ |\dot{r}_k| \to 0, \ |\dot{r}_i| \to c_i > 0 \ (i \neq k)$;
  - $\mathcal{HE}_k^+$: $|r_k| \to \infty, \ |\dot{r}_i| \to c_i > 0 \ (i \neq k), \ \sup_{t\geq 0} |r_k| < \infty$;
  - $\mathcal{PE}_k^+$: $|r_k| \to \infty, \ |\dot{r}_i| \to 0 \ (i \neq k), \ \sup_{t\geq 0} |r_k| < \infty$;
  - $\mathcal{P}_+$: $|r_k| \to \infty, \ |\dot{r}_k| \to 0$;
  - $\mathcal{B}^+$: $\sup_{t\geq 0} |r_k| < \infty$;
  - $\mathcal{OS}^+$: $\limsup_{t\to\infty} \max_k |r_k| = \infty, \ \liminf_{t\to\infty} \max_k |r_k| < \infty$.
- Classification for trajectories defined for all time.
- Some orbits are not defined for all time: orbits hitting collisions.

Collision orbits

$$\frac{d^2 q_i}{dt^2} = \sum_{j=1, j\neq i}^{3} m_j \frac{q_j - q_i}{\|q_j - q_i\|^3}, \qquad q_i \in \mathbb{R}^2$$

- Collision set: $\mathcal{C} = \{q_1 = q_2\} \cup \{q_1 = q_3\} \cup \{q_2 = q_3\}$
- Collision orbit: orbit which hits a collision at some time $t = t^*$.

Herman conjecture

- Fix the center of mass at the origin.
- Reparameterize the flow so that it takes infinite time to get to collision.
- Non-wandering set: Consider a dynamical system $\phi : X \to X$, $x \in X$ is non-wandering if for every open neighborhood U of x and any N satisfies $\phi^n(U) \cap U \neq \emptyset$ for some $n > N$.
- Herman question: Is the non-wandering set nowhere dense in all energy levels?
- In particular: Is the set of bounded orbits nowhere dense?

How abundant/rare are collision orbits?

- Saari 1970's (also Fleischer & Knauf 2018): The set of collision orbits has measure zero.
- Alexeev conjecture (1981): Is there an open set $\mathcal{U}$ in phase space posessing a dense subset $\mathcal{D} \subset \mathcal{U}$ whose points lead to collision?
- This conjecture goes back to Siegel.
- If Alexeev conjecture is true, would imply a dense set of "bounded orbits".
- Could Alexeev conjecture (if true) lead to a negative answer to Herman conjecture?
- To understand Alexeev conjecture: consider the case $m_2 = m_3 = 0$.

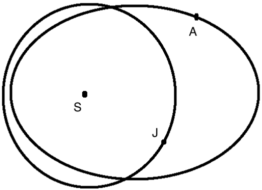
The case: $m_2 = m_3 = 0$

- Body 1 does not move.
- Body 2 and 3

$$\frac{d^2 q_i}{dt^2} = m_1 \frac{q_1 - q_i}{\|q_1 - q_i\|^3}$$

form two uncoupled 2BPs.

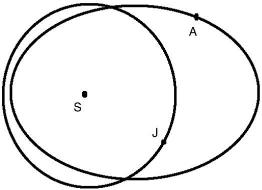
- Place them on ellipses.



- Take ellipses that intersect transversally.
- They form an open set in phase space foliated by 2-tori.
- All solutions are either periodic or quasi-periodic

The case: $m_2 = m_3 = 0$

- If periods of q_2 and q_3 are inconmensurable, collision orbits are dense in this $\mathbb{T}^2$.
- Periods is $2\pi a^{3/2}$ where a is the semimajor axis of the ellipse.
- For a dense set of a's the periods are inconmensurable.



- Tori with dense collision orbits are dense in an open set.

General case: $m_2, m_3 > 0$

- Alexeev: Does density of collision orbits still hold?
- For $m_2, m_3 > 0$ small, this is not a regular perturbation problem.
- The system blows up in a small neighborhood of collisions.
- We consider a simpler model: The Restricted Planar Circular 3 Body problem.

The Restricted Planar Circular 3 Body problem

- Three bodies of masses $1-\mu$, μ and 0 under the effects of the Newtonian gravitational force.
- Primaries q_1 and q_2 orbiting on circles.
- Rotating coordinates:
  - Primaries at $q=(-\mu,0)$ and $q=(1-\mu,0)$
  - Dynamics of the third body q is given by the 2 dof Hamiltonian

$$H(q,p)=\frac{\|p\|^2}{2}-(p_2q_1-p_1q_2)-\frac{1-\mu}{\|q+\mu\|}-\frac{\mu}{\|q-(1-\mu)\|}$$

- Phase space: $\mathbb{R}^4\setminus\{\text{collisions}\}$.

Collisions are asymptotically dense

Theorem (M. G. – V. Kaloshin – J. Zhang)

Consider the RPC3BP. There exists an open set $\mathcal{U} \subset \mathbb{R}^4$ and $\tau > 0$, independent of μ, such that, for μ small enough, there is a μ^τ-dense set $\mathcal{D} \subset \mathcal{U}$ whose points lead to collision.

- μ^τ dense $\equiv$ μ^τ neighborhoods of all points in $\mathcal{D}$ cover $\mathcal{U}$.
- τ can be taken $\tau = \dfrac{1}{17+\sigma}$ for any $\sigma > 0$.
- $\mathcal{U}$ gives open sets in the energy level $H = h$ for energies

$$h \in \left(-\frac{3}{2}, \sqrt{2}\right).$$

The set $\mathcal{U}$

- The set $\mathcal{U}$ can be easily characterized in terms of Delaunay coordinates:
  - L square root of the semimajor axis of the ellipse.
  - G is the angular momentum.
  - ℓ is the mean anomaly.
  - g is the argument of the perihelion with respect the primaries line.
- Then $\mathcal{U}$ is the interior of any compact set contained in

$$\mathcal{V} = \Big\{(\ell, g, L, G) \in \mathbb{T}^2 \times (0, +\infty) \times (-L, 0) \cup (0, L) : \\ \frac{G^2}{1+e} < 1 < \frac{G^2}{1-e}, \quad H(\ell, g, L, G) \in \left(-\frac{3}{2}, \sqrt{2}\right)\Big\}.$$

where $e = \sqrt{1 - \frac{G^2}{L^2}}$.

The set $\mathcal{U}$

- $\mathcal{U}$ corresponds to where in the unperturbed case ($\mu = 0$) the ellipses of the two bodies intersect transversally.
- In particular we only consider collisions with the small primary at $(1 - \mu, 0)$.
- The same set were the existence of second species periodic solutions are looked for (Niederman, Marco, Bolotin, McKay,...)
- Collisions with the massive primary – Punctured tori: Chenciner, Llibre, Féjoz, Zhao.

Other almost density results: punctured tori

- Punctured tori also give an almost density result of collision orbits for the RP3BP and the full 3BP.

- However, for the RP3BP the corresponding set $\mathcal{U}$ is very small (either its measure goes to zero as $\mu \to 0$ or two bodies are arbitrarily close).

- For the full 3BP almost density in “big” sets if one places one of the bodies very far away.

A disproof of a weak version of Herman conjecture

- Herman question: Is the non-wandering set nowhere dense in all energy levels?
- Consider a a dynamical system $\{\phi_t\}_{t\in\mathbb{R}}$ defined on a topological space X. Then, a point $x \in X$ is called δ-non-wandering, if for any neighborhood V of it containing the δ-ball $B_\delta(x)$, there exists $T > 1$ such that $\phi_T(V) \cap V \neq \emptyset$.

Theorem (M. G. – V. Kaloshin – J. Zhang)

Any point belonging to the open set $\mathcal{U}$ is $\mathcal{O}(\mu^\tau)$–non wandering under the flow ϕ_t of the RPC3BP.
More concretely, for any $z \in \mathcal{U}$, we can find a $\mathcal{O}(\mu^\tau)$-neighborhood V_μ of it and times $0 < T'_\mu < T_\mu$ such that $\phi_{T'_\mu}(V_\mu)$ is $\mathcal{O}(\mu^\tau)$–close to a collision and $\phi_{T_\mu}(V_\mu) \cap V_\mu \neq \emptyset$.

Summarizing

- For the RPC3BP, at μ^τ scales for $\tau = \dfrac{1}{17+\sigma}$, $\sigma > 0$:
  - Alexeev conjecture is correct
  - Herman conjecture is not.
- What happens for the true conjectures for the full 3BP?
- Are they incompatible?

Some ideas of the proof of almost density of collisions

- Take any point $P \in \mathcal{U}$: we want to find Q μ^τ-close to it hitting a collision.
- Case $\mu = 0$:
  - $\mathcal{U}$ foliated by 2 dimensional tori.
  - Choose Q in an orbit in a non-resonant torus hitting collision (they are dense).
- Q may need a very long time to hit collision.
- Case $\mu > 0$: Choose a $\mu^{3\tau}$-long curve μ^τ-close to P and show that a point in this curve hits a collision.

Some ideas of the proof: three regimes

1. Far from collision (points $\mu^{3\tau}$ away from collision) the zero mass body q (basically) only notices the main primary: nearly integrable setting.

2. Transition zone: q notices the two primaries but orbits spend there very short time.

3. Small neighborhood of the collision ($\mu^{1/2}$ neighborhood of the collision): q (basically) only notices the small primary – A different nearly integrable setting.

Regime 1: far from collision

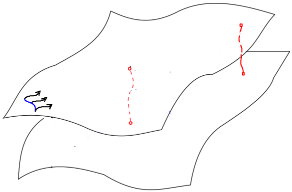
- We are in a nearly integrable regime.
- Problem: the point may need a very long time to reach Regime 2.
- We apply KAM Theory.
- KAM is global: it cannot be applied directly due to the collisions (the Hamiltonian blows up at collision!)
- Remove the collision by multiplying H by a bump function supported at $\mu^{3\tau}$-ball centered at the collision.
- The modified Hamiltonian is close to a 2 body problem (in low regularity).

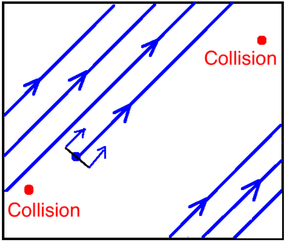
Regime 1: far from collision

- We want to apply KAM with lowest possible regularity: the more regularity, the worse estimate on the Hamiltonian with bump functions.
- Constant type frequencies are γ-dense

$$|q\omega - p| \geq \frac{\gamma}{|q|}.$$

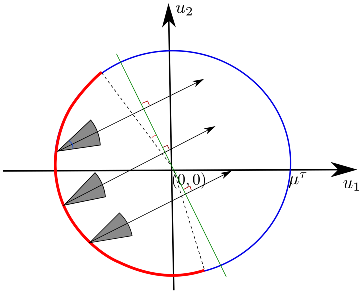
- We apply Herman version of KAM (for $\mathcal{C}^{3+\sigma}$ maps and constant type frequencies): tori are γ-dense.
- Each torus has two (removed) collisions.
- Orbits on the tori are true orbits of the RPC3BP as long as do not intersect a $\mu^{3\tau}$ neighborhood of the collisions.





- We wanted: any point P has a $\mu^{3\tau}$-long curve μ^{τ}-close to it and a point in this curve hits collision.
- Take a KAM torus μ^{τ} close to P and $\mu^{3\tau}$-long curve in this torus

Regime 1: How to reach well Regime 2



- The forward orbit of the small curve has to hit "well" the puncture around one of the collisions so that it can be sent forward to Regimes 2 and 3.

- Well:
  - The orbit cannot have intersected before the punctures around collisions (we want a true orbit of RPC3BP!).
  - The image of the segment covers the half of the boundary of the neighborhood where the velocity is pointing inwards.

Regime 1: Far from collision

- We want to optimize the density coefficient
- Small γ: gives better density of tori.
- To have the segment hitting well we need to avoid close encounters with collisions before a good hitting.
- We need a strong Diophantine condition $\to \gamma$ big.
- KAM + Non-homogeneous Dirichlet Theorem leads to

$$\gamma = \mu^{\tau} \quad \text{with} \quad \tau = \frac{1}{17+\sigma}, \quad \sigma > 0.$$

Regime 2

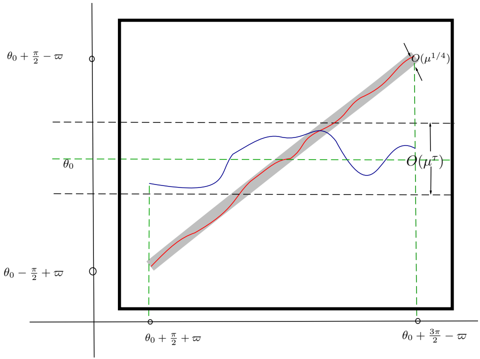
- Regime 2: $\mu^{3\tau}$-close to collision and $\mu^{1/2}$-far from collision with $\gg 1$.
- It is a small annulus of width $\mu^{3\tau}$ where the two bodies are "not too close".
- We use the true RPC3BP.
- Velocity of order ~ 1 (collisions are "far enough" to control it).
- Thus: the flow is almost tubular.
- Conclusion: the propagated segment goes from the outer to the inner boundary with almost constant velocity.

Regime 3

- Small primary is dominant: Flow close to a new 2 body problem (close to collision).
- By Levi-Civita regularization (+ scaling): the RPC3BP becomes

$$K(z, w) = \frac{1}{2}(|w|^2 - |z|^2) + \mu^{1/2}\mathcal{O}_4(z, w)$$

  where $z = 0$ is the collision set.
- Run backwards orbits departing from collisions to the boundary between Regimes 2 and 3.
- Restricting to the level of energy, they give a curve at the boundary.



- Plot the curves in the plane $(\arg(z), \arg(w))$.
- Blue: the incoming curve coming from Regime 1 and 2.
- Red: backward orbits departing from collision.

They are both $\mathcal{C}^0$ curves: they must intersect.