Growth of Sobolev norms in the nonlinear Schrödinger equation

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The defocusing cubic NLS

- Consider the equation

$$-iu_t + \Delta u = |u|^2 u$$

where $x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $t \in \mathbb{R}$ and $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$.

- NLS defines a complete flow in $H^s(\mathbb{T}^2)$, $s \geq 1$.

- Conserved quantities: the Hamiltonian and the mass

$$E[u] = \int_{\mathbb{T}^2} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4 \right) \frac{dx}{(2\pi)^2} dx$$

$$\mathcal{M}[u] = \int_{\mathbb{T}^2} |u|^2 dx.$$

Two very related problems

- Transfer of energy.

- Lyapunov instability of invariant objects of the dynamical system.

Transfer of energy

- Fourier series: $u(x,t) = \sum_{n\in\mathbb{Z}^2} a_n(t)e^{inx}$.
- Can we have transfer of energy to high modes as $t \to +\infty$?
- Can a solution u start oscillating only on scales comparable to the spatial period and eventually oscillate on arbitrarily small scale?
- Zakharov-Shabat eq: cubic defocusing NLS for $x \in \mathbb{T}$ is integrable.
- No transfer of energy is possible.
- What about $\mathbb{T}^D$ with $D \geq 2$? How we measure the transfer?

Sobolev norms

$$\|u(t)\|_{H^s(\mathbb{T}^2)} = \|u(t,\cdot)\|_{H^s(\mathbb{T}^2)} = \left(\sum_{n\in\mathbb{Z}^2}(1+|n|^2)^s|a_n(t)|^2\right)^{1/2}$$

- Mass and energy conservation $\Rightarrow L^2$ norm is conserved and

$$\|u(t)\|_{H^1(\mathbb{T}^2)} \leq C\|u(0)\|_{H^1(\mathbb{T}^2)} \quad \text{for all} \quad t \geq 0.$$

- Energy transfer measured by the growth of H^s norms, $s > 0$, $s \neq 1$.
- Simultaneous forward and backward cascade.

Bourgain conjecture

$$-iu_t + \Delta u = |u|^2 u, \quad x \in \mathbb{T}^2$$

Theorem (Bourgain (1996))

Fix $\delta > 0$. Then any solution u of the cubic defocusing NLS on $\mathbb{T}^2$ satisfies

$$\|u(t)\|_{H^s} \leq t^{2(s-1)+\delta} \quad \textit{as} \quad t \to +\infty.$$

- Question by Bourgain (2000): Are there solutions u such that for $s > 1$,

$$\|u(t)\|_{H^s} \to +\infty \qquad \text{as } t \to +\infty?$$

The I-team result: Finite large growth

$$-iu_t + \Delta u = |u|^2 u$$

Theorem (Colliander, Keel, Staffilani, Takaoka, Tao (2010))

Fix $s > 1$, $K \gg 1$ and $\delta \ll 1$. Then there exists a global solution u of NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(0)\|_{H^s} \leq \delta, \qquad \|u(T)\|_{H^s} \geq K.$$

- The result also applies to $s \in (0, 1)$.

More results

- M. G. and V. Kaloshin: $T \sim e^{\left(\frac{K}{\delta}\right)^{\beta}}$ for some $\beta > 1$.
- E. Haus and M. Procesi: generalized the result to the quintic NLS

$$-iu_t + \Delta u = |u|^4 u.$$

- M. G, E. Haus and M. Procesi: generalization to

$$-iu_t + \Delta u = |u|^{2p} u \qquad \text{with } p \geq 3.$$

- Z. Hani, B. Pausader, N. Tzvetkov, N. Visciglia: unbounded growth (Bourgain conjecture) for the cubic NLS in $\mathbb{R} \times \mathbb{T}^2$.
- It remains open in all compact manifolds.

Dynamical systems point of view

$$-iu_t + \Delta u = |u|^2 u$$

- $u = 0$ is an elliptic critical point: linearly stable in any H^s topology.
- It is Lyapunov stable or unstable? It depends on the topology.
- Stability in L^2 and H^1 topology.
- I-team result $\equiv$ Instability in the H^s topology, $s \neq 1$.
- Similar to Arnold diffusion.

Instability of others invariant objects?

- What about the Lyapunov stability/instability of other invariant objects of the cubic NLS?
- For which time ranges?
- Can we attain Sobolev norm explosion starting arbitrarily close to an invariant object?

  (Arnold diffusion like orbits close to an invariant object)

Stability/Instability of invariant objects: plane waves

- Plane waves (periodic orbits): $u(t,x) = Ae^{i(mx-\omega t)}$, $\omega = m^2 + A^2$.
- Stability result (Faou, Gauckler, Lubich 2013): Fix $N > 1$. There exists $s_0 > 0$ such that "many" plane waves satisfy the following:

  For any $s \geq s_0$, $\delta \ll 1$ and any initial condition $u_0(x)$ satisfying $\|u_0(x) - Ae^{imx}\|_{H^s} \leq \delta$, its orbit satisfies

  $$\|u(t,x) - Ae^{i(mx-\omega t)}\|_{H^s} \lesssim \delta \quad \text{for} \quad t \lesssim \delta^{-N}.$$

- Many: For any m and a full measure set of A

Stability/Instability of invariant objects: plane waves

$$u(t,x) = Ae^{i(mx-\omega t)}, \qquad \omega = m^2 + A^2$$

- Instability result (Hani 2011): Fix $s \in (0,1)$, $K \gg 1$ and $\delta \ll 1$. Then there exists a global solution u of NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(0) - Ae^{imx}\|_{H^s} \le \delta, \qquad \|u(T)\|_{H^s} \ge K.$$

- Remark: Instabilities are only proven for $s \in (0,1)$.

Transfer of energy close to invariant tori

- Goal: Transfer of energy close to invariant (quasiperiodic) tori.
- We look at the "simplest" tori: 1D tori (finite gap solutions).
- 1D Cubic NLS $i\partial_t q = -\partial_{xx} q + |q|^2 q$, $x \in \mathbb{T}$, is integrable.
- It admits global Birkhoff coordinates

$$\begin{aligned} \Phi : L^2(\mathbb{T}) &\longrightarrow \ \ell^2 \times \ell^2 \\ q \quad &\longmapsto \ (z_m, \overline{z}_m)_{m\in\mathbb{Z}}, \end{aligned}$$

  such that $i\dot{z}_m = \alpha_m(I) z_m$ where $I_m = |z_m|^2$ are the actions ($\dot{I}_m = 0$).

  (i.e ∞ dimensional nonlinear oscillator).
- All solutions are periodic/quasi-periodic/almost-periodic.

1D invariant tori

- Consider one of the $\mathtt{d}$-dimensional quasiperiodic tori:
  - Excited "modes" (oscillators) $\mathcal{S}_0 = (\mathtt{m}_1, \dots \mathtt{m}_\mathtt{d}) \subset \mathbb{Z}$.
  - Actions vector $I_\mathtt{m} = (I_{\mathtt{m}_1}, \dots, I_{\mathtt{m}_\mathtt{d}}) \in \mathbb{R}^\mathtt{d}_+$.
- Define

$$\mathtt{T}^\mathtt{d} = \left\{ (z_m)_{m \in \mathbb{Z}} : |z_{\mathtt{m}_i}|^2 = I_{\mathtt{m}_i}, i = 1, \dots, \mathtt{d},\ z_m = 0 \text{ if } m \notin \mathcal{S}_0 \right\}.$$

- Lyapunov stable as invariant objects of 1D cubic NLS.
- Consider them as invariant objects for the 2D equation.
- For which time scales is Lyapunov stable/unstable?
- H^s norm explosion from a small neighborhood of it?

Stability of the finite gaps solution

Theorem (A. Maspero – M. Procesi)

Fix $s > 0$, $s \neq 1$. For a generic choice of support sites $\mathcal{S}_0$ there exists ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a Cantor set $\mathcal{I} \subset (0, \varepsilon)^{\mathtt{d}}$ such that the following holds true for any torus $\mathtt{T}^{\mathtt{d}} = \mathtt{T}^{\mathtt{d}}(\mathcal{S}_0, I_{\mathtt{m}})$ with $I_{\mathtt{m}} \in \mathcal{I}$:

Fix any $\delta \ll 1$. Then, any solution of cubic defocusing NLS $u(t)$ such that

$$\mathrm{dist}_{H^s}\left(u(0), \mathtt{T}^{\mathtt{d}}\right) \leq \delta$$

satisfies

$$\mathrm{dist}_{H^s}\left(u(t), \mathtt{T}^{\mathtt{d}}\right) \lesssim \delta \qquad \text{for all } |t| \lesssim \delta^{-2}.$$

Some comments

- These tori are Diophantine.
- ε is small $\Rightarrow$ the tori are small.
- Generic choice of $\mathcal{S}_0$: The modes in $\mathcal{S}_0$ cannot satisfy a finite number of relations (number of relations only depending on $\mathtt{d}$).
- The set of "good" actions $\mathcal{I} \subset (0, \varepsilon)^{\mathtt{d}}$ has positive relative measure.

Main result

Theorem (M.G.– Z. Hani – E. Haus – A. Maspero – M. Procesi)

Fix $s \in (0,1)$ *and a torus* $\mathtt{T}^{\mathtt{d}} = \mathtt{T}^{\mathtt{d}}(\mathcal{S}_0, I_{\mathtt{m}})$ *as before.*

Then, for any $\delta \ll 1$ *and* $K \gg 1$ *there exists an orbit of cubic defocusing NLS and a time* T *such that*

- $\mathrm{dist}_{H^s}\left(u(0), \mathtt{T}^{\mathtt{d}}\right) \leq \delta$
- $\|u(T)\|_{H^s} \geq K.$
- $|T| \lesssim e^{\left(\frac{K}{\delta}\right)^{\beta}}$ *for some* $\beta > 1$.

Comments

- Transversal instability: these tori are stable as solutions of 1D NLS but there are 2D solutions arbitrarily close to the torus which attain large Sobolev norms.
- All tori satisfying Maspero-Procesi result (elliptic tori – linearly stable) are unstable after longer time scales.
- $s \in (0,1)$: $u(t)$ has very small mass and very large energy.
- Same result should be true for $s > 1$ but our techniques do not apply.
- The tori seem to be more stable in those topologies.

Why the torus needs to be small? Why $\mathcal{S}_0$ generic?

- To understand the normal behavior at the torus: we reduce the linearized equation to constant coefficients.
- We only know how to do the reduction for small tori (linearized equation is close to constant coefficients $\to$ KAM scheme).
- To perform the reduction we need a generic choice of modes.
- Transversal instability "should be true" for non-small tori

Some ideas about the proof

- Analysis of the 1D NLS: Birkhoff coordinates.
- Reduce linearization of 2D NLS around $\mathbb{T}^d$ to constant coefficients (KAM scheme).
- Birkhoff Normal form around the torus.
- Apply I-team ideas to travel along resonances to achieve growth: drift along resonances.

The I-team approach

- Select (cleverly) a finite set of modes $\Lambda \subset \mathbb{Z}^2$ such that

$$U_\Lambda = \{a_n = 0 : n \notin \Lambda\}$$

  is invariant under certain Birkhoff normal form truncation of NLS.

- Size of Λ depends on the desired growth of Sobolev norms.

- Use combinatorics to choose the modes in Λ such that the dynamics on U_Λ is “easy” to analyze.

The I-team approach for the cubic case

- Choosing well Λ and after several reductions (taking advantage strongly of the particular form of NLS):

  Finite dimensional (toy) model

  $$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right), \; j = 1, \dots N.$$

  gives the dynamics on Λ.

- Each b_j represents "many" modes in Λ.
- Hamiltonian system on a lattice $\mathbb{Z}$ with nearest neighbor interactions.
- Growth of Sobolev norms $\equiv$ Pushing energy from b_1 to b_N.

Dynamics of the cubic toy model

$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right), \; j = 0, \dots N,$$

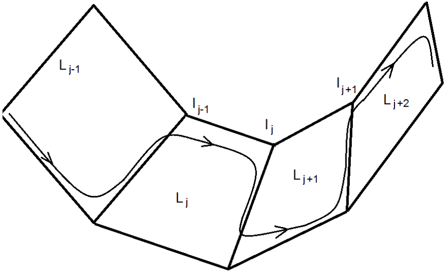
- Each 4-dimensional plane

  $$L_j = \{b_1 = \cdots = b_{j-1} = b_{j+2} = \cdots = b_N = 0\}$$

  is invariant.

- In L_j, $\mathbb{T}_j = \{b_j \neq 0, b_{j+1} = 0\}$ and $\mathbb{T}_{j+1} = \{b_j = 0, b_{j+1} \neq 0\}$ are (partially hyperbolic) periodic orbits.

- They are connected through non-transversal heteroclinic orbits.



- I-team: To travel from close to $\mathbb{T}_1$ to close to $\mathbb{T}_N$, shadow these sequence of heteroclinics.
- M. G.-V. Kaloshin: Shadowing with time estimates.
- Delshams-Simon-Zgliczyński: Shadowing with Shilnikov techniques.
- Conclusion: there are orbits which are localized first at b_1 and after some time at b_N.

Conclusions and open problems

- Cubic NLS in $\mathbb{T}^2$ has orbits undergoing finite large growth of Sobolev norms.
- Families of periodic orbits, invariant tori are unstable in H^s.
- What about other more general invariant objects? infinite dimensional invariant objects (almost periodic tori)?
- Growth of Sobolev norm in other Hamiltonian PDEs? Or NLS in other compact manifolds?
- Unbounded growth of Sobolev norms for Hamiltonian PDEs on compact manifolds?